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RESEARCH ARTICLE

Certainquantum calculus operators associated with the basic analogue of Fox-Wright hypergeometricfunction

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Abstract

The objective of this paper is to derive the relationship that exists between the basic analogue of Fox-Wright hypergeometric function ${}_p\psi_q$ (z;q) and the quantum calculus operators, in particular Riemann-Louville q-integral and q-differential operators. Some special cases have been also discussed.

Mathematics Subject Classification: 33D60, 33D99 and 26A33.

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Introduction

Fractional calculus owes its origin to a question of whether the meaning of a derivative toan integer order n could be extended to still be valid when n is not an integer. This question was first raised by L'Hospital on September 30th, 1695. On that day, in a letter to Leibniz, he posed a question about , Leibniz's notation for the nth derivative . L'Hospital curiously asked what the result would be if n = $\frac{1}{2}$? Leibniz responded prophetically that it would be "an apparent paradox, from which one day useful consequences will be drawn. It took about three hundred years for this prophecy to be true. In recent decades, it has been found useful in various fields like differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and financial mathematics.

On the other side Hypergeometric functions evolved as natural unification of a host of functions discussed by analysts from the seventeenth century to the present day. Functions of this type may also be generalized using the concept of basic number. This was first effected systematically by E. Heine (1898) at the end of nineteenth century in connection with what is known as a basic analogue or q-analogue of Gauss hypergeometric function ${}_2F_1$ and was subsequently extended by F.H. Jackson, L.J. Slater, G.E. Andrews and others up to the present day.More details on this type of calculus can also be found in Kac and Cheung's book [1] entitled "Quantum Calculus" provides for the basics of so called q-calculus.

Let us consider the following expression

$$\frac{f(x)-f(x_0)}{(x-x_0)}$$

Now letting $x \rightarrow x_0$,we get the well - known definition of the derivative $\frac{df}{dx}$ of a function $f(x)$at $x = x_0$. However ever , if we take $x = qx_0$ or $x = x_0$+h , where q is a fixed number different from 1, and h a fixed number different from 0, and don't take the limit , we enter the fascinating world of quantum calculus . The corresponding expressions are the definitions of the q-derivative and h-derivative of $f(x)$as defined in [1 & 2] . The same was latter on introduced by F.H.Jackson in the beginning of the twentieth century . He was the first to develop q- calculus in a systematic way.

The basic analogue of Fox-Wright hypergeometric function denoted $_p\psi_q$ (z;q) for z ϵ C is defined in series form as [1]

$${}_p\Psi_q(z;q) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_q(a_i+kA_i)}{\prod_{j=1}^{q} \Gamma_q(bj+kB_j)} \frac{z^k}{(q;q)_k}, \text{ where } |q|< 1. \tag{1}$$

Where $a_i, b_j \epsilon$ C>0 ; $A_i > 0$, $B_j > 0$; $1+\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq 0$; α ϵR ,for suitably bounded value of |z|. Moreover in view of the relation

$${}_p\Psi_q(z;q) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma_q(a_j+kA_j)}{\prod_{j=1}^{q} \Gamma_q(bj+kB_j)} \frac{z^k}{k!} = H_{p,q+1}^{1,p}\left[-z; q\middle|\begin{matrix}(1-a_j, A_j)_{1,p} \\ (0,1),(1-b_j, B_j)_{1,q}\end{matrix}\right]$$

the function $_p\Psi_q$ (z;q) converges under the convergence conditions of the well-known Fox's H-function which are as follows, the integral convergesif $\mathrm{Re}[\mathrm{slog}(x) - \log\sin\pi s] < 0$, on the contour C, where 0<|q|< 1, logq = -ω = $(\omega_1 + i\omega_2)$.$\omega_1$andω_2 *being real* , verified by Saxena, et al [2] .

Agrawal [3] introduced the basic analogue of the Reimann-Liouville fractional operator as follows.

$$I_{q,x}^{\alpha} f(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)_{\alpha-1} f(t)\, d_q(t) \quad ; \mathrm{Re}(\alpha) > 0. \tag{2}$$

In particular, for f(x) = x^p ; we have

$$I_{q,x}^{\alpha}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+\alpha+1)} x^{p+\alpha}; \mathrm{Re}(\alpha) > 0. \tag{3}$$

Also q-analogue of the Reimann-Liouville fractional derivative defined as [3]

$$D_{q,x}^{\alpha} f(x) = D_q^n (I_{q,x}^{n-\alpha} f)\, x \;; \; \mathrm{Re}(\alpha) < 0, |q|<1 \tag{4}$$

In particular, for f(x) = x^p ; we have

$$D_{q,x}^{\alpha}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p-\alpha+1)} x^{p-\alpha}; \; \mathrm{Re}(\alpha) < 0, |q|<1. \tag{5}$$

Main Results

Theorem(1.1): Let α >0 ,β>0 , γ > 0 and aϵR, let $I_{q,x}^{\alpha}$ be the Riemann- Liouville fractional integral operator, then

$$I_q^{\alpha}\left\{t^{\gamma-1}\, {}_p\Psi_q\left(\begin{matrix}(a_i, A_i)_{1,A} \\ (b_j, B_j)_{1,B}\end{matrix}\middle| at^{\beta}; q\right)\right\} x = x^{\gamma+\alpha-1}\, {}_{p+1}\Psi_{q+1}\left(\begin{matrix}(a_i, A_i)_{1,A};\ (\gamma, \beta) \\ (b_j, B_j)_{1,B};\ (\alpha+\gamma, \beta\end{matrix}\middle| ax^{\beta}; q\right)$$

Proof : To prove theorem(1.1) we apply equations (1) and (2) to the left side oftheorem(1.1) we get

$$I_q^{\alpha}\left\{t^{\gamma-1}\, {}_p\Psi_q\left(\begin{matrix}(a_i, A_i)_{1,A} \\ (b_j, B_j)_{1,B}\end{matrix}\middle| at^{\beta}; q\right)\right\}(x) = I_q^{\alpha}\left\{t^{\gamma-1} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{A} \Gamma_q(a_i+kA_i)}{\prod_{j=1}^{B} \Gamma_q(bj+kB_j)} \frac{(at^{\beta})^k}{(q;q)_k}\right\}(x)$$

$$I_q^{\alpha}\left\{t^{\gamma-1}\, {}_p\Psi_q\left(\begin{matrix}(a_i, A_i)_{1,A} \\ (b_j, B_j)_{1,B}\end{matrix}\middle| at^{\beta}; q\right)\right\}(x) = I_q^{\alpha}\left\{\sum_{k=0}^{\infty} \frac{\prod_{i=1}^{A} \Gamma_q(a_i+kA_i)}{\prod_{j=1}^{B} \Gamma_q(bj+kB_j)} \frac{a^k t^{\beta k+\gamma-1}}{(q;q)_k}\right\}(x)$$

$$I_q^{\alpha}\left\{t^{\gamma-1}\, {}_p\Psi_q\left(\begin{matrix}(a_i, A_i)_{1,A} \\ (b_j, B_j)_{1,B}\end{matrix}\middle| at^{\beta}; q\right)\right\}(x) = \left\{\sum_{k=0}^{\infty} \frac{\prod_{i=1}^{A} \Gamma_q(a_i+kA_i)}{\prod_{j=1}^{B} \Gamma_q(bj+kB_j)} \frac{a^k}{(q;q)_k}\right\} I_q^{\alpha}(t^{\beta k+\gamma-1})(x).$$

Making the use of equation (3) we get $I_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}(x)$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{A}\Gamma_q\,(a_i+kA_i)\Gamma_q(\gamma+k\beta)}{\prod_{j=1}^{B}\Gamma_q\big(bj+kB_j\big)\,\Gamma_q(\gamma+\alpha+k\beta)}\frac{a^k x^{\beta k+\gamma+\alpha-1}}{(q;q)_k}$$

$$I_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}(x)\qquad = x^{\gamma+\alpha-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{A}\Gamma_q\,(a_i+kA_i)\Gamma_q(\gamma+k\beta)}{\prod_{j=1}^{B}\Gamma_q\big(bj+kB_j\big)\,\Gamma_q(\gamma+\alpha+k\beta)}\frac{a^k x^{\beta k}}{(q;q)_k}$$

$$I_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}(x)=\ x^{\gamma+\alpha-1}\ {}_{p+1}\Psi_{q+1}\left(\begin{matrix}(a_i,A_i)_{1,A};\ (\gamma,\beta)\\ (b_j,B_j)_{1,B};\ (\alpha+\gamma,\beta\end{matrix}\middle| ax^{\beta};q\right).$$

This completes proof of the theorem.

Corollary (1.1.1): For α>0 ,β>0,γ,λ >0 , then

$$I_q^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(\delta,1)\\ (\gamma,\beta)\end{matrix}\middle| at^{\beta};q\right)\right\}(x)=\Gamma_q(\delta)\,x^{\gamma+\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta};q) \tag{6}$$

Corollary (1.1.2): By setting δ = 1 in equation(6) we get

$$I_q^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(1,1)\\ (\gamma,\beta)\end{matrix}\middle| at^{\beta};q\right)\right\}(x)=\ x^{\gamma+\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta};q) \tag{7}$$

Theorem (2.1): Let Re(α) < 0 , β> 0, γ > 0 and α ϵ R, let $D_{q,x}^{\alpha}$ be the Riemann- Liouville fractional derivative operator, then there holds following results

$$D_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}x\ =\ x^{\gamma-\alpha-1}\ {}_{p+1}\Psi_{q+1}\left(\begin{matrix}(a_i,A_i)_{1,A};\ (\gamma,\beta)\\ (b_j,B_j)_{1,B};\ (\alpha+\gamma,\beta\end{matrix}\middle| ax^{\beta};q\right)$$

Proof : To prove theorem (2.1) we apply equations (1) and (4) to the left side of theorem (2.1) we get,

$$D_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}(x)=D_q^{\alpha}\left\{t^{\gamma-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{A}\Gamma_q(a_i+kA_i)}{\prod_{j=1}^{B}\Gamma_q(bj+kB_j)}\frac{(at^{\beta})^k}{(q;q)_k}\right\}$$

$$D_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}(x)\ =\ \left\{\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{A}\Gamma_q(a_i+kA_i)}{\prod_{j=1}^{B}\Gamma_q(bj+kB_j)}\frac{a^k}{(q;q)_k}\right\}D_q^{\alpha}(t^{\beta k+\gamma-1})(x)$$

Making the use of equation (5) we

get $D_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}(x)=$

$$\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{A}\Gamma_q(a_i+kA_i)\Gamma_q(\gamma+k\beta)}{\prod_{j=1}^{B}\Gamma_q(bj+kB_j)\Gamma_q(\gamma-\alpha+k\beta)}\frac{a^k x^{\beta k+\gamma-\alpha-1}}{(q;q)_k}.$$

$$D_q^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\ (b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta};q\right)\right\}x\ =\ x^{\gamma-\alpha-1}\ {}_{p+1}\Psi_{q+1}\left(\begin{matrix}(a_i,A_i)_{1,A};\ (\gamma,\beta)\\ (b_j,B_j)_{1,B};\ (\alpha+\gamma,\beta\end{matrix}\middle| ax^{\beta};q\right).$$

Corollary (2.1.1): For Re(α) <0 ,β> 0, and γ> 0 , then

$$D_q^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(\delta,1)\\ (\gamma,\beta)\end{matrix}\middle| at^{\beta};q\right)\right\}(x)=\Gamma_q(\delta)\,x^{\gamma-\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta};q) \tag{8}$$

Corollary (2.1.2): By setting δ = 1 in equation (8) we get

$$\mathrm{D}_q^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta};q\right)\right\}(x) = x^{\gamma-\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta};q) \tag{9}$$

Special cases:

(1) By setting q = 1 in the theorems (1.1) and (2.1) , we get well –known results reported in [4] as follows

$$\mathrm{I}^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\(b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta}\right)\right\}x = x^{\gamma+\alpha-1}\ {}_{p+1}\Psi_{q+1}\left(\begin{matrix}(a_i,A_i)_{1,A};\ (\gamma,\beta)\\(b_j,B_j)_{1,B};\ (\alpha+\gamma,\beta\end{matrix}\middle| ax^{\beta}\right) \tag{10}$$

and

$$\mathrm{D}^{\alpha}\left\{t^{\gamma-1}\ {}_p\Psi_q\left(\begin{matrix}(a_i,A_i)_{1,A}\\(b_j,B_j)_{1,B}\end{matrix}\middle| at^{\beta}\right)\right\}x = x^{\gamma-\alpha-1}\ {}_{p+1}\Psi_{q+1}\left(\begin{matrix}(a_i,A_i)_{1,A};\ (\gamma,\beta)\\(b_j,B_j)_{1,B};\ (\alpha+\gamma,\beta\end{matrix}\middle| ax^{\beta}\right) \tag{11}$$

(2) By setting q = 1 in the equations (6),(7),(8) and (9) we get well-known results established by Saxena and Saigo [5] and [6]

$$\mathrm{I}^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta}\right)\right\}(x) = \Gamma\ (\delta)\ x^{\gamma+\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta}) \tag{12}$$

By setting δ = 1 in equation (12), we get

$$\mathrm{I}^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta}\right)\right\}(x) = x^{\gamma+\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta}) \tag{13}$$

$$\mathrm{D}^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta}\right)\right\}(x) = \Gamma\ (\delta)\ x^{\gamma-\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta}) \tag{14}$$

By setting δ = 1 in equation (14) we get

$$\mathrm{D}^{\alpha}\left\{t^{\gamma-1}\ {}_1\Psi_1\left(\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\middle| at^{\beta}\right)\right\}(x) = x^{\gamma-\alpha-1}\ E_{\beta,\gamma+\alpha}^{\alpha+\gamma-1}(ax^{\beta}). \tag{15}$$

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