ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

The Deficient discrete quartic spline interpolation

Y.P. Dubey[1] and Anil Shukla[2]

1. Department of Mathematics, L.N.C.T. Jabalpur 482 003.
2. Department of Mathematics, Gyan Ganga College of Technogy, Jabalpur.

Manuscript Info

Manuscript History:

Received: 11 November 2013
Final Accepted: 28 November 2013
Published Online: December 2013

Key words:
Discrete, Quartic Spline, Interpolation, Error Bounds, Deficient.

Abstract

In the present paper, we have studied the existence, uniqueness and convergence properties of discrete quartic spline interpolation, which match the given functional values at mesh points, mid points and second derivative at boundary points.

Introduction

Let us consider a mesh on [0, 1] which is defined by $P: 0 = x_0 < x_1 - < x_n = 1$, For i = 1,2,....n-1, P_i shall denote the length of the mesh interval $[x_i, x_{i+1}]$. Let P = max P_i and $P^* = \min i\ p_i$, for uniform mesh $P = P_i$ for all i, throughout, h will represent a given positive real number, consider a real valued function s (x, h) defined over [0, 1] which is such that its restriction s_i on $[x_i, x_{i+1}]$ is Polynomial of degree 4 or less with deficiency 1, if

$$D_h^{\{i\}} s_i(x_i, h) = D_h^{\{j\}} S_{i+1}(x_i, h) \qquad j = 0, 1, 2 \qquad (1)$$

Where the difference operator $D_n^{\{i\}}$ for a function *f* is defined by

$$D_h^{\{0\}} f(x) = f(x),\ D_h^{\{1\}} f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$D_h^{\{2\}} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \text{ and}$$

$$D_h^{\{m+n\}} f(x) = D_h^{\{m\}} D_h^{\{n\}}(x), m, n \geq 0$$

Let the S (4, 1, p, h) denoted the class of all such deficient discrete quartic splines with deficiency 1 satisfying the boundary conditions.

$$D_h^{\{1\}} s(x_0, h) = D_h^{\{1\}} f(x_0, h) \qquad (2)$$

$$D_h^{\{1\}}\, s(x_n,h) = D_h^{\{1\}}\, f(x_n,h).$$

Discrete splines have been introduced by Mangasarian and Schumaker in [7] in connection with certain studies of minimization problems, involving differences. They have a close connection with best summation formula in [8] which is a special case of abstract theory of best approximation of linear functionals. To compute non-linear splines intratively, Malcolm in [6] has used discrete splines. In the direction of some constructive aspects of discrete splines, we refer to Astor and Duris in [1], Jia in [4] and Schumaker in [10], Existence, uniqueness and convergence properties of discrete cubic spline interpolation matching the given function value at intermediate points for uniform mesh have been studied by Dikshit and Powar in [2] (see also in [3]), Rana and Dubey in [9] have obtained an asymtotically precise estimate of the difference between discrete cubic spline interpolant and the function interpolated, which is sometime used to smooth a histrogram. Deficient spline are more useful than usual splines as they require less continuity requirement at mesh points. In this paper, we have investigate convergence properties, existence and uniqueness and also obtain error bounds.

Non writing $\alpha_i = \dfrac{x_1 + x_{i+1}}{2}$, we introduced the following interpolatory conditions for a given function *f*,

$$s(x_i,h) = f(x_i) \qquad i = 0,1,...n \tag{3}$$

$$s(\alpha_i,h) = f(\alpha_i) \qquad i = 0, 1,...n\text{-}1 \tag{4}$$

and pose the following.

Problem 1

A given h > 0, for what restriction on p does there exist a unique $s(x,h) \in S(4,.1,p,h)$ which satisfies the condition (3) and (4).

Existence and uniqueness

Let P(z) be a discrete quartic polynomial on [0, 1]. Then we can show that

$$P(z) = P(0)\,Q_1(z) + P(1)\,Q_2(z) + P(½)\,Q_3(z) \tag{5}$$

$$+ D_h^{\{2\}}\, P(0), Q_4(z) + D_h^{\{2\}}\, P(1)\,Q_5(z)$$

$$Q_1(z) = \frac{1}{6A}\left[6A - (72h^2 + 78)\,z + 48h^2 z^2 + 96z^3 - 48z^4\right]$$

$$Q_2(z) = \frac{1}{6A}\left[(-24h^2 - 18)z + 48h^2 z^2 + 96z^3 - 48z^4\right]$$

$$Q_3(z) = \frac{1}{6A}\left[96(h^2+1)z - 96h^2 z^2 - 192z^3 + 96z^4\right]$$

$$Q_4(z) = \frac{1}{6A}\left[-(2h^2+4)z + 3(5+2h^2)z^2 - (17+4h^2)z^3 + 6z^4\right]$$

$$Q_5(z)=\frac{1}{6A}\left[(2h^2+1)z-6h^2z^2+(4h^2-7)z^3+6z^4\right]$$

Where $A=\frac{1}{\left[5+4h^2\right]}$

Now, we are set to answer problem A in the following.

Theorem 2.1. Suppose h >0 is real, then there exists a unique deficient descrete quartic spline $s(x,h)\in S(4,1,p,h)$ which satisfies the conditions (2) and (3).

***Proof of theorem* 2.1 :** Let $P_i t=(x-x_i),0\le t\le 1$ we can write (5) in the form of the restriction s_i of the quartic spline s (x, h) on $\left[x_i,x_{i+1}\right]$ as follows.

$$s_i(x,h)=f(x_i)Q_1(t)+f(x_{i+1})Q_2(t)+f(\alpha_i)Q_3(t)+P_i^2Q_4(t)D_n^{\{2\}}s_i(x,h)$$

$$+P_i^2Q_5(t)D_n^{\{2\}}s_{i+1}(x,h) \qquad (6)$$

In view of (5), it may be seen that $s_i(x,h)$ is quartic on $[x_i,x_{i+1}]$ for i = 0,1,...n-1 and satisfies (2) and (3). Let $G_i(a,b)=ap_i^2+bh^2$ and $g(c,d)=ch^2+d$ where a, b, c, and d are real number. Now applying the continuity condition of first difference of $s_i(x,h)$ at x_i, given by (1), we get the following system of equations :

$$p_i\left\{G_{i-1}(g(2,1),g(4,-7))\right\}D_n^{\{2\}}s_{i-1}(x,h)$$

$$-D_n^{\{2\}}s_i(x,h)[p_i\{G_{i-1}(g(2,4),g(4,17))+p_{i-1}\{G_i(g(2,4),g(4,17))]+$$

$$p_{i-1}\{G_i(g(2,1),g(4,-7))\}D_n^{\{2\}}s_{i+1}(x,h)=\frac{p_{i-1}}{P_i^2}\{G_{i-1}(g(24,18),$$

$$g(0,-96))f(x_{i-1})+G_{i-1}(g(72,78),g(0,-96))f(x_i)$$

$$+\{G_{i-1}(g(-96,-96),g(0,192))f(\alpha_{i-1})$$

$$+\frac{p_i}{p_{i-1}^2}[\{G_i(g(72,78),g(0,-96))\}]+f(x_i)$$

$$+\{G_i(g(24,18),g(0,-96))f(x_{i+1})$$

$$+\{G_i(g(-96,-96),g(0,192)\}f(\alpha_i)]$$

$$=F_i \qquad i = 1,2.....n \qquad \text{(Say)} \qquad (7)$$

Write $D_n^{\{2\}} s(x_i, h) = M_i(h) = M_i$ (say) for all we can easily see that excess of the absolute value of the coefficient of M_i over the sum of the absolute value of coefficients of M_{i-1} and M_{i+1} in (7) under the condition of theorem 2.1 is given by.

$$\frac{1}{6A}\left[P_i G_{i-1}\left(g(0,3), g(0,24)\right) + P_{i-1} G_i\left(g(0,3), (0,24)\right)\right]$$

which is clearly positive, therefore, the coefficient matrix of the system of equation (7) is diagonally dominant and hence invertible. Thus the system of equations (7) has a unique solution. This complete the proof of the theorem 2.1.

Error bounds

We assume in this section 1 = Nh where N is positive integer; Let $x_i \in [0,1]_h$ for i = 1,2,...n. Where the discrete interval $[0,1]_h$ is the set of points {0, h, 2h, ...Nh}. For a function f and two distinct points x_1, x_2 in its domain, the first divided difference is defined by

$$[x_1, x_2] f = \frac{(f(x_1) - f(x_2))}{(x_1 - x_2)}$$

For convenience we write $f^{\{2\}}$ for $D_h^{\{2\}} f$, $f_i^{\{2\}}$ for $D_h^{\{2\}} f(x_i)$ and w (f, p) is the modulus of continuity of f, the discrete norm of a function *f* over the interval [0,1)h is defined by

$$\| f \| = \max_{[0,1]_h} | f(x) |$$

We shall obtain error bounds for the error function $e(x) = s(x,h) - f(x)$ over the discrete interval $[0,1]_h$.

***Theorem* 3.1 :** Suppose s(x,h) is the deficient discrete quartic spline interpolant of theorem 2.1 then

$$\| e_i^{\{2\}} \| \le C_1(h) K(p,h) w(f,p) \tag{8}$$

$$\| e_i^{\{1\}} \| \le C_2(h) K_1(p,h) w(f,p) \tag{9}$$

and $$\| e(x) \| \le P^2 K^*(P,h) w(f,p) \tag{10}$$

where K(P,h), K_1 (P,h) & K* (P, h) are positive constant.

***Proof of Theorem* 3.1 :** To obtain the error estimate (8) first we replace

$M_i(h)$ by $e^{\{2\}}(x_i) = D_n^{\{2\}} s(x_i, h) - f_i^{\{2\}}$ in (7) and get

$$A(h)(e^{\{2\}}(x_i) = F_i(h) - A(h) f_i^{\{2\}} = (L_i) \quad \text{(Say)} \tag{11}$$

To estimate the row max norm of the matrix (L_i) in (11). We shall need the following Lemma due to Lyche [5].

***Lemma* 3.1 :** Let $\{a_i\}_{i=1}^{m}$ and $\{b_j\}_{j=1}^{n}$, be given sequences of non negativereal numbers such that $\sum a_i = \sum b_j$ then for any real value function *f* defined over discrete interval $[0,1]_h$ we have

$$\left|\sum_{i=1}^{m} a_i \left[x_{i0,}\ x_{i1}.....x_{ik}\right]_f - \sum_{j=1}^{n} b_j \left[y_{j0}, y_{j1},....y_{jk}\right]_f\right| \tag{12}$$

$$\leq w(f^{\{k\}}, 1 - kh|) \sum a_i / k!$$

Where $x_{ik}, y_{jk} \in [0,1]_h$ for relevant values of i, j and k.

It may be observed that the i[th] row of the right hand side of (11) is written as

$$|(L_i)| = |\sum_{i=1}^{7} a_i \left[x_{i0}, x_{i1}\right] f - \sum_{j=1}^{7} b_j \left[y_{j0}, y_{j1}\right]_f | \tag{13}$$

$$a_1 = \frac{P_i}{P_{i-1}} G_{i-1}\left(g(12,9), g(0,-48)\right) = b_1$$

$$a_2 = P_i\, G_{i-1}\left(g(2,1), g(4,-7)\right) = b_2$$

$$a_3 = P_i\, G_{i-1}\left(g(2,4), g(4,17)\right) = b_3$$

$$a_4 = P_i\, P_{i-1}\, G_{i-1}(g(24,30)\ g(0,0)) = b_4$$

$$a_5 = \frac{P_{i-1}}{P_i} G_i\left(g(12,9), g(0,-48)\right) = b_5$$

$$a_6 = P_{i-1}\, G_i\left(g(2,1), g(4,-7)\right) = b_6$$

$$a_7 = P_{i-1}\, G_i\left(g(2,4), g(4,17)\right) = b_7$$

and $x_{10} = \alpha_{i-1} = y_{10} = x_{40}$

$$x_{11} = x_i = x_{30} = y_{40} = y_{60} = y_{31} = y_{41} = x_{41} = y_{51} = x_{61}$$

$$y_{11} = x_{i-1} = y_{20} = x_{21}, \quad x_{20} = x_{i-1} - h, \quad y_{21} = x_{i-1} + h$$

$$y_{30} = x_i + h = y_{61}, \quad x_{31} = x_i - h = x_{60}, \quad x_{50} = \alpha_i = y_{50} = y_{40}$$

$$x_{51} = x_{i+1} = x_{70}, \quad x_{71} = x_{i+1} + h, \quad y_{70} = x_{i+1} - h, \quad y_{71} = x_{i+1}$$

Clearly $\sum_{i=1}^{7} a_i = \sum_{i=1}^{7} b_j$

Thus applying Lemma 3.1, for i=j=7 and k=1, we get

$$|(L_i)| \le N(p,h)\, w(f^{\{1\}}, |1-p|) \tag{14}$$

Now, using the equation (14) in (11), we have

$$\| e^{\{2\}}(x_i) \| \le C_1(h)\, k(p,h)\, w(f^{\{1\}}, p) \tag{15}$$

Where k(p, h) is some positive function of P and h.

We need proceed to obtain an upper bound for e(x), Replacing $M_i(h)$ by $e_i^{\{2\}}$ in equation (7), we obtain

$$e(x_i,h) = P_i^2\left[Q_4(t)\,e_i^{\{2\}} + Q_5(t)\,e_{i+1}^{\{2\}}\right] + M_i(f) \tag{16}$$

Now, we write the expression of $M_i(f)$ used in the right hand side of (16) in terms of the divided difference as follows :-

$$M_i(f) = \sum_{i=1}^{m} u_i[x_{i0}, x_{i1}] - \sum_{j=1}^{n} v_j[y_{j0}, y_{j1}]\, f \tag{17}$$

Where $u_1 = v_1 = p_i \dfrac{g(24,30)}{6A}$

$$u_2 = v_2 = p_i \frac{g(12,9)}{6A}$$

$$u_3 = 24z^2\left(\frac{1+2z-z^2}{6A\; g(2,4)}\right) p_i = v_3$$

$$u_4 = P_i^2[-g(2,4)z + g(6,15)z^2 - g(4,17)z^3 + 6z^4)] = v_4$$

$$u_5 = \frac{p_i^2}{6A(2h)}\left[g(2,1)z - 6h^2z^2 + g(4,-7)z^3 + 6z^4\right] = v_5$$

and $x_{10} = x_i = y_{10} = y_{20} = y_{31} = y_{41} = x_{40}, y_{11} = x_{20} = y_{21} = y_{30} = \alpha_i = x_{30}$

$x_{11} = x$, $x_{31} = x_{i+1}$, $x_{21} = x_{i+1} = y_{51} = x_{50}$, $x_{41} = x_i + h$

$y_{40} = x_i - h$, $x_{i+1}, x_{51} = x_{i+1} + h$, $y_{50} = x_{i+1} - h,$

Clearly $\sum_{i=1}^{5} u_i = \sum_{j=1}^{5} v_j$

We again apply Lemma 3.1 in (14) ;for i = j = 5 and k=1 to see that

$$| M_i(f) | \leq P\, N^*(P,h)\, w(f^{\{1\}}, P) \qquad (18)$$

Where $N^*(P,h) = \frac{1}{6A}[g(36,39)\, p_i$

$$+24z^2(1+2z-z^2)p_i + P_i^2\{-3z+15z^2-24z^3+12z^4\}],$$

where $A = g(4,5)$.

Thus, using (3.8) and (3.11) in (3.9) we get the following :-

$$\| e(x) \| \leq P^2 K^*(P,h)\, w(f^{\{1\}}, P] \qquad (19)$$

Where K* (P,h) is a positive constant of P and h. This is the inequality (10) of theorem 3.1.

We now proceed to obtain an upper bound of $e_i^{\{1\}}$, from equation (6) we get

$$s^{\{1\}}_i(x,h) = f_i Q_i^{\{1\}}(t) + f_{i+1} Q_2^{\{1\}}(t) + f_{\alpha_i} Q_3^{\{1\}}(t)$$

$$+P_i^{\{2\}} s_i^{\{2\}}(x,h) Q_4^{\{1\}}(t) + P_i^2 s_{i+1}^{\{2\}}(x,h) Q_5^{\{1\}}(t) \qquad (20)$$

Thus

$$6Ae_i^{\{1\}}(x,h) = P_i^2[e_i^{\{2\}} Q_4^{\{1\}}(t) + e_{i+1}^{\{2\}} Q_i^{\{1\}}(t)] + U_i(f) \qquad (21)$$

Where $U_i(f) = f_i Q_1^{\{1\}}(t) + f_{i+1} Q_2^{\{1\}}(t) + f_{\alpha_i} Q_3^{\{1\}}(t)$

$$+P_i^2\left[f_i^{\{2\}} Q_4^{\{1\}}(t) + f_{i+1}^{\{2\}} Q_5^{\{1\}}(t)\right] - 6A\, f_i^{\{1\}}(x,h)$$

By using Lemma 3.1 and first and second divided difference in U_i (f) as follows:-

$$|U_i(f)| \leq W\left(f^{\{1\}}, P\right) \sum_{i=1}^{4} a_i = \sum_{j=1}^{4} b_j \qquad (22)$$

$$= p_i\left[g(36,39) - 48h^2 Z - 48g(3,1) + 96Z(Z^2+h^2)\right]$$

$$+ p_i^2\left[-3 + 30Z - 24(3Z^2+h^2) + 48Z(Z^2+h^2)\right]$$

Where $a_1 = p_i\left[g(12,9) - 48^2 Z - 48(3Z^2+h^2) + 96Z(Z^2+h^2)\right] = b_1$

$$a_2 = p_i\, g(24,30) = b_2$$

$$a_3 = p_i^2\left[-g(2,4) + g(12,30)Z - g(4,17)(3Z^2+h^2) + 24Z(Z^2+h^2)\right] = b_3$$

$$a_4 = p_i^2\left[g(2,1) - 12h^2 Z + g(4,-7)(3Z^2 + h^2 + 24Z(Z^2 + h^2)\right] = b_4$$

and $x_{10} = x_i = x_{20} = x_{30} = y_{31}, x_{11} = \alpha_i = x_{21} = y_{11}$

$$y_{10} = x_{i+1} = x_{40} = y_{41},\ y_{20} = x,$$

$$y_{21} = x + h, x_{31} = x_i + h, y_{30} = x_i - h$$

$$x_{41} = x_{i+1} + h, y_{40} = x_{i+1} - h$$

From equation (15) put value of $e_i^{\{2\}}$ in (21) we get upper bound of $e_i^{\{1\}}$. This is inequality (9) of theorem 3.1.

Conclusion

We have constructed deficient discrete quartic spline interpolation and obtained existence uniqueness and error bounds.

References

1. P.H. Astor and C.S. Duris, Discrete Splines, Numer Math 22 (1974), pp 303-402.
2. H.P. Dikshit and P.L. Powar. Discrete Cubic Spline interpolation, Numer. Math 40 (1982), pp. 71-78.
3. H.P. Dikshit and S.S. Rana. Discrete Cubic Spline interpolation over a non uniform mesh. Rocky Mountain J. Math 17 (1987), pp. 709-718.
4. Rong, Quing Jia. Total positivity of the discrete spline Collocation matrix, J. Approx. theory 39 (1983), 11-23.
5. T. Lyche. Discrete cubic spline interpolation Report (RRI 5), University of Oslo 1975.
6. M.A. Malcolm, Non Linear Spline functions. Report Stan (S-73-372) Stenford University, 1973.
7. O.L. Mangasarian and L.L. Schumaker. Discrete spline via mathematical programming, SIAMJ Control 9 (1971), 174-183.
8. O.L. Mangasarian and L.L. Schumaker. Best summation formula and discrete splines, SIAM. Numer Anal. 10 (1973), 448-459.
9. S.S. Rana and Y.P. Dubey, Local Behaviour of the deficient discrete cubic spline interpolation J. Approx. Theory 86 (1996), pp 120-127.
10. L.L. Schumaker, Constructive aspects of discrete polynomial spline functions in Approximation theory (G.G. Lorentz, ed.) Academic Press, New York 1973.