ON $R^*$-CONTINUOUS AND $R^*$-IRRESOLUTE MAPS IN TOPOLOGICAL SPACES.

Basavaraj M. Ittanagi$^1$ and Raghavendra K$^2$.

1. Department of Mathematics, Siddaganga Institute of Technology, Tumakuru-03, Affiliated to VTU, Belagavi, Karnataka state, India.
2. Department of Mathematics, ACS College of Engineering, Bengaluru-74, Affiliated to VTU, Belagavi, Karnataka state, India.

Abstract

In this paper, a new class of continuous functions called $R^*$-continuous maps in topological spaces are introduced and studied. Also some of their properties have been investigated. We also introduce $R^*$-irresolute maps, strongly $R^*$-continuous maps, perfectly $R^*$-continuous maps and discussed some of their properties.

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Introduction:

In general topology continuous functions play a very vital role. The regular continuous and completely continuous functions are introduced and studied by Arya S P [2]. Later, R S Walli et al [33] introduced and investigated $\alpha r w$-continuous functions in topological space. Recently, Basavaraj M Ittanagi et al [5] introduced and studied the basic properties of $R^*$-closed sets in topological space. The aim of this paper is to introduce $R^*$-continuous and irresolute maps in topological space.

Preliminaries:

In this paper X or $(X, \tau)$ and Y or $(Y, \sigma)$ denote topological spaces on which no separation axioms are assumed. For a subset A of a topological space X, $\text{cl}(A)$, $\text{int}(A)$, $X-A$ or $A^c$ represent closure of A, interior of A and complement of A in X respectively.

Definition 2.1: A subset A of a topological space $(X, \tau)$ is called a

i. Regular open set [26] if $A=\text{int}(\text{cl}(A))$ and regular closed if $A=\text{cl}(\text{int}(A))$
ii. Regular semi open set [9] if there exists a regular open set $U$ such that $U \subseteq A \subseteq \text{cl}(U)$
iii. Generalized closed set ($g$-closed) [18] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
iv. $R^*$-closed set [5] if $\text{gcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $R^*$ open in $(X, \tau)$.

The complement of the closed sets mentioned above are their open sets respectively and vice versa.

Corresponding Author:- Basavaraj M. Ittanagi.
Address:- Department of Mathematics, Siddaganga Institute of Technology, Tumakuru-03, Affiliated to VTU, Belagavi, Karnataka state, India.
Definition 2.2: A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called a
i. Continuous if \( f^{-1}(V) \) is closed in \( X \) for every closed subset \( V \) of \( Y \).
ii. Regular continuous \([2]\) if \( f^{-1}(V) \) is \( r \)-closed in \( X \) for every closed subset \( V \) of \( Y \).
iii. Completely continuous \([2]\) if \( f^{-1}(V) \) is regular closed in \( X \) for every closed subset \( V \) of \( Y \).
iv. \( \alpha \)-continuous \([14]\) if \( f^{-1}(V) \) is \( \alpha \)-closed in \( X \) for every closed subset \( V \) of \( Y \).
v. Semi continuous \([15]\) if \( f^{-1}(V) \) is semi closed in \( X \) for every closed subset \( V \) of \( Y \).
vi. Semi pre continuous \([1]\) if \( f^{-1}(V) \) is semi pre closed in \( X \) for every closed subset \( V \) of \( Y \).
vii. Strongly Continuous \([24]\) if \( f^{-1}(V) \) is clopen in \( X \) for every subset \( V \) of \( Y \).
viii. \( g \)-continuous \([4]\) if \( f^{-1}(V) \) is \( g \) closed in \( X \) for every closed subset \( V \) of \( Y \).
ix. \( w \)-continuous \([28]\) if \( f^{-1}(V) \) is \( w \) closed in \( X \) for every closed subset \( V \) of \( Y \).
x. \( gr \)-continuous \([22]\) if \( f^{-1}(V) \) is \( gr \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xi. \( g^* \)-continuous \([30]\) if \( f^{-1}(V) \) is \( g^* \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xii. \( swg^* \)-continuous \([19]\) if \( f^{-1}(V) \) is \( swg^* \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xiii. \( \beta wg^* \)-continuous \([11]\) if \( f^{-1}(V) \) is \( \beta wg^* \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xiv. \( r^g \)-continuous \([21]\) if \( f^{-1}(V) \) is \( r^g \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xv. \( rwg \)-continuous \([20]\) if \( f^{-1}(V) \) is \( rwg \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xvi. \( \beta wg^{**} \)-continuous \([25]\) if \( f^{-1}(V) \) is \( \beta wg^{**} \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xvii. \( g \alpha \)-continuous \([10]\) if \( f^{-1}(V) \) is \( g \alpha \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xviii. \( swg \)-continuous \([20]\) if \( f^{-1}(V) \) is \( swg \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xix. \( ag \)-continuous \([17]\) if \( f^{-1}(V) \) is \( ag \) closed in \( X \) for every closed subset \( V \) of \( Y \).
x. \( g \alpha \)-continuous \([18]\) if \( f^{-1}(V) \) is \( g \alpha \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xii. \( w \)-continuous \([28]\) if \( f^{-1}(V) \) is \( w \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xiii. \( w \alpha \)-continuous \([7]\) if \( f^{-1}(V) \) is \( w \alpha \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xiv. \( \alpha \alpha \)-continuous \([31]\) if \( f^{-1}(V) \) is \( \alpha \alpha \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xv. \( \rho \)-continuous \([9]\) if \( f^{-1}(V) \) is \( \rho \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xvi. \( cg \)-continuous \([26]\) if \( f^{-1}(V) \) is \( cg \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xvii. \( g \alpha \)-continuous \([3]\) if \( f^{-1}(V) \) is \( g \alpha \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xviii. \( rps \)-continuous \([23]\) if \( f^{-1}(V) \) is \( rps \) closed in \( X \) for every closed subset \( V \) of \( Y \).
xix. \( gsp \)-continuous \([12]\) if \( f^{-1}(V) \) is \( gsp \) closed in \( X \) for every closed subset \( V \) of \( Y \).

Definition 2.3:
A map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called a
i. Irresolute if \( f^{-1}(V) \) is semi closed in \( X \) for every semi closed subset \( V \) of \( Y \).
ii. \( w \)-Irresolute \([28]\) if \( f^{-1}(V) \) is \( w \)-closed in \( X \) for every \( w \)-closed subset \( V \) of \( Y \).
iii. \( gc \)-Irresolute \([27]\) if \( f^{-1}(V) \) is \( gc \) closed in \( X \) for every \( gc \)-closed subset \( V \) of \( Y \).
iv. Contra \( w \)-Irresolute \([28]\) if \( f^{-1}(V) \) is \( w \) open in \( X \) for every \( w \)-closed subset \( V \) of \( Y \).
v. Contra Irresolute \([14]\) if \( f^{-1}(V) \) is semi open in \( X \) for every semi closed subset \( V \) of \( Y \).
vi. Contra \( r \)-irresolute \([2]\) if \( f^{-1}(V) \) is regular open in \( X \) for every regular closed subset \( V \) of \( Y \).
vii. Contra continuous \([13]\) if \( f^{-1}(V) \) is open in \( X \) for every closed subset \( V \) of \( Y \).

Results 2.4[5]:
i. Every closed (respectively regular closed, \( g \)-closed, \( w \)-closed, \( \bar{g} \)-closed set) set is \( R^8 \)-closed set in \( X \).
ii. Every \( R^8 \)-closed set in \( X \) is \( rg \)-closed (respectively \( gpr \)-closed, \( rwg \)-closed, \( gspr \)-closed, \( r^g \)-closed, \( rg \beta \)-closed) set in \( X \).

Results 2.5[5]:
Let \( A \) be a subset of a topological space \((X, \tau)\)
i. If \( A \) is regular open and \( rg \)-closed set in \((X, \tau)\) then \( A \) is \( R^8 \)-closed set in \((X, \tau)\).
ii. If \( A \) is \( g \)-open and \( rg \)-closed set in \((X, \tau)\) then \( A \) is \( R^8 \)-closed set in \((X, \tau)\).
iii. If \( A \) is a regular-open and \( rwg \)-closed set in \((X, \tau)\) then \( A \) is \( R^8 \)-closed in \((X, \tau)\).
iv. If \( A \) is a regular-open and \( gpr \)-closed set in \((X, \tau)\) then \( A \) is \( R^8 \)-closed in \((X, \tau)\).
v. If \( A \) is regular open and \( r^g \)-closed set in \((X, \tau)\) then \( A \) is \( R^8 \)-closed in \((X, \tau)\).
vi. If \( A \) is regular open and \( \beta wg^{**} \)-closed set in \((X, \tau)\) then \( A \) is \( R^8 \)-closed set in \((X, \tau)\).

\( R^8 \)-Continuous Functions:-
Definition 3.1:
A function $f$ from a topological space $X$ to a topological space $Y$ is called a $R^k$-continuous if inverse image of every closed set in $Y$ is a $R^k$-closed set in $X$.

Example 3.2: Let $X=Y=[a,b,c]$. Let $\tau=\{\emptyset,X,\{a\},\{b\},\{a, b\},\{a, c\}\}$ be a topology on $X$ and $\sigma=\{\emptyset,Y,\{a\},\{b\},\{a, b\}\}$ be a topology on $Y$. $R^kC(X)=(X,\emptyset,\{a\},\{b\},\{a, b\},\{a, c\},\{b, c\})$ and closed set of $Y$ are $\sigma=\{Y,\emptyset,\{c\},\{a, c\},\{b, c\}\}$. Let $f:X\rightarrow Y$ be defined by $f(a)=a, f(b)=c, f(c)=c$ is $R^k$-continuous.

Theorem 3.3: Every continuous function is $R^k$-continuous but not conversely.

Proof: Let $f:X\rightarrow Y$ be continuous and $F$ be any closed set in $Y$. Then $f^{-1}(F)$ is closed set in $X$. Since every closed set in $X$ is $R^k$-closed then $f^{-1}(F)$ is $R^k$-closed set in $X$. Therefore $f$ is $R^k$-continuous.

Example 3.4: Let $X=\{a, b, c\}$. Let $\tau=\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ be a topology on $X$ and $\sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$ be a topology on $Y$, closed set of $X$ are $\tau=\{X, \emptyset, \{a\}, \{b\}\}$, closed set of $Y$ are $\sigma=\{Y, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$. $R^kC(X)=(X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\})$. Let $f:X\rightarrow Y$ defined by $f(a)=a, f(b)=c, f(c)=c$ is $R^k$-continuous.

Theorem 3.5:

i. Every $g$-continuous is $R^k$-continuous but not conversely.

ii. Every $w$-continuous is $R^k$-continuous but not conversely.

iii. Every $\tilde{g}$-continuous is $R^k$-continuous but not conversely.

iv. Every $r$-continuous is $R^k$-continuous but not conversely.

Proof: The proof follows from the fact that every $g$-closed (resp. $w$-closed, $\tilde{g}$-closed and $r$-closed) set is $R^k$-closed set.

Similarly we can prove ii, iii, iv.

Example 3.6: Let $X=Y=\{a, b, c, d\}$, let $\tau=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ be a topology on $X$ and $\sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$ be a topology on $Y$. Closed sets of $X=\{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$, closed sets of $Y=\{Y, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. Let $f:X\rightarrow Y$ defined by $f(a)=a, f(b)=d, f(c)=d, f(d)=b$ is $R^k$-continuous function, as the closed set $\{d\}$ in $Y$, then $f^{-1}(\{d\})=\{a, b, c\}$ is not g-closed set in $X$.

Theorem 3.7: Every $R^k$-continuous function is $rg$-continuous but not conversely.

Proof: Let $f:X\rightarrow Y$ be $R^k$-continuous and $F$ be a closed set in $Y$, by definition $f^{-1}(F)$ is $R^k$-closed set in $X$. Since every $R^k$-closed set is $rg$-closed, then $f^{-1}(F)$ is $rg$ closed in $X$. Hence $f$ is $rg$-continuous.

Theorem 3.8:

i. Every $R^k$-continuous function is $f\tilde{g}$ continuous but not conversely.

ii. Every $R^k$-continuous function is $gspr$ continuous but not conversely.

iii. Every $R^k$-continuous function is $gp$ continuous but not conversely.

iv. Every $R^k$-continuous function is $rg$ continuous but not conversely.

v. Every $R^k$-continuous function is $rwg$ continuous but not conversely.

vi. Every $R^k$-continuous function is $wgr$ continuous but not conversely.

Proof: The proof follows from the fact that every $R^k$-closed set is $f\tilde{g}$-closed (resp. $gspr$-closed, $gpr$-closed, $rg$-closed, $rwg$-closed, $wgr$-closed) set in $X$.

Similarly we can prove (i), (iii), (iv), (v) and (vi).

Example 3.9: Let $X=Y=\{a, b, c\}$. Let $\tau=\{\emptyset, X, \{a\}, \{b\}\}$ be a topology on $X$ and $\sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, c\}\}$ be a topology on $Y$. Closed sets of $X=\{X, \emptyset, \{a\}, \{b\}, \{a, c\}\}$, Y=\{Y, \emptyset, \{c\}\}. $R^kC(X)=(X, \emptyset, \{a\}, \{b\}, \{a, c\})$. Let $f:X\rightarrow Y$ defined by $f(a)=a, f(b)=a, f(c)=b$ is $rg$ continuous, $r^*g$-continuous, $gspr$-continuous, $gp$-continuous, $g^*p$-continuous, $wgr$-continuous, $pgpr$-continuous, $wpsp$-continuous and $gr^{**}$-continuous in $X$, as $f^{-1}(d)=\{a, b, c\}$ is not a $rs$-closed set.
gs-closed set, \(\alpha\)-closed set, gp-closed set, \(g^*\)-closed set, \(g^*p\)-closed set, \(w\alpha\)-closed set, pgpr-closed set, rps-closed set and \(g\alpha^*\)-closed set in \(X\).

Example 3.12: Let \(X=\{a, b, c, d\}\), let \(\tau=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\) be a topology on \(X\) and \(\sigma=\{\emptyset, Y, \{a\}, \{a, b\}, \{a, b, c\}\}\) be a topology on \(Y\). Closeds sets of \(X=\{X, \emptyset, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}\) and closed sets of \(Y=\{Y, \emptyset, \{d\}, \{c, d\}, \{b, c, d\}\}\). \(R^\#\)-C(X)=\{X, \emptyset, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}. Let \(f: X\to Y\) defined by \(f(a)=d, f(b)=d, f(c)=b, f(d)=d\) is pre continuous. Semi continuous, sp continuous, b-continuous, swg-continuous, g\(\alpha\)-continuous, sgb-continuous, rg\(^*\)b-continuous, wrg-continuous, g\(\alpha^*\)-continuous, g\(s^*\)-continuous and \(#g\alpha\)-continuous, but not \(R^\#\)-continuous in \(X\) as \(f^{-1}(a, b) = \{c\}\) is pre closed set, Semi pre closed set, sp pre closed set, b pre closed set, swg- pre closed set, g\(\alpha\)-pre closed set, sgb- pre closed set, rg\(^*\)b- pre closed set, wrg- pre closed set, g\(\alpha^*\)-pre closed set, g\(s^*\)-pre closed set and \(#g\alpha\)-closed set in \(X\) but not \(R^\#\)-closed set in \(X\).

Remark 3.13: From the above discussions and known facts, the relation between \(R^\#\)-continuous and some existing continuous functions in topological space is shown in the following figure.

![Diagram showing the relation between \(R^\#\)-continuous and other continuous functions.]

Theorem 3.14: Let \(f: X\to Y\) be a map. Then the following statements are equivalent
i. \(f\) is \(R^\#\)-continuous
ii. The inverse image of each open set in \(Y\) is \(R^\#\)-open in \(X\).

Proof:

i. Let \(f: X\to Y\) be a \(R^\#\)-continuous, let \(U\) be an open set in \(Y\). Since \(f\) is \(R^\#\)-continuous, \(f^{-1}(U)\) is \(R^\#\)-closed in \(X\). But \(f^{-1}(U^c) = X - f^{-1}(U)\). Thus \(f^{-1}(\emptyset)\) is \(R^\#\)-open set in \(X\).

ii. Suppose that inverse image of each open set in \(Y\) is \(R^\#\)-open in \(X\). Let \(V\) be any closed set in \(Y\). By assumption \(f^{-1}(V^c)\) is \(R^\#\)-open set in \(X\). But \(f^{-1}(V^c) = X - f^{-1}(V)\). Thus \(f^{-1}(V^c)\) is \(R^\#\)-closed in \(X\). Thus \(f\) is \(R^\#\)-continuous. Hence the proof.

Theorem 3.15: If \(f: (X, \tau) \to (Y, \sigma)\) is a map then the following holds
i. If \(f\) is contra \(r\)-irresolute and \(r\)-continuous map then \(f\) is \(R^\#\)-continuous.
Theorem 3.16: If \( f : X \rightarrow Y \) is \( R^k \)-continuous then \( f(R^k(cl(A)) \subseteq cl(f(A)) \) for every subset \( A \) of \( X \).

Proof: Let \( f: X \rightarrow Y \) be \( R^k \)-continuous. Let \( A \) be a subset of \( X \). Then \( cl(f(A)) \) is closed in \( Y \), this implies \( f^{-1}[cl(f(A))] \) is \( R^k \)-closed in \( X \). Also \( f(A) \subseteq cl(f(A)) \) and \( A \subseteq f^{-1}[cl(f(A))] \). Hence \( R^k(cl(A)) \subseteq f^{-1}[cl(f(A))] \). Therefore \( f(R^k(cl(A)) \subseteq cl(f(A)) \).

Theorem 3.17: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. Then the following statements are equivalent

i. For each point \( x \in X \) and each open set \( V \) in \( Y \) with \( f(x) \in V \), there is a \( R^k \)-open set \( U \) in \( X \) such that \( x \in U \) and \( f(U) \subseteq V \).

ii. For each subset \( A \) of \( X \), \( f(R^k(cl(A)) \subseteq cl(f(A)) \)

iii. For each subset \( B \) of \( Y \), \( R^k(cl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \)

Proof:

(i) \rightarrow (ii): Suppose (i) holds and let \( y \in f(R^k(cl(A))) \) and \( V \) be an open set containing \( y \). From (i), there exists \( x \in R^k(cl(A)) \) such that \( f(x)=y \) and \( R^k \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \) and \( x \in R^k(cl(A)) \). Then we know that for a subset \( A \) of a topological space \( X \), \( cl(R^k(cl(A))) \) and only if \( U \cap A \neq \emptyset \) for every \( R^k \)-open set \( A \) containing \( x \). That is \( \emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq f(U) \). Therefore \( f(R^k(cl(A)) \subseteq cl(f(A)) \).

(ii) \rightarrow (i): Suppose (ii) holds and let \( A \) be an open set in \( Y \) containing \( f(x) \). Let \( A \subseteq f^{-1}(V) \). This implies that \( x \in A \). Since \( f(R^k(cl(A)) \subseteq cl(f(A)) \subseteq V \). This implies that \( R^k(cl(A)) \subseteq f^{-1}(V)=A \). Since \( x \in A \) implies that \( x \in R^k(cl(A)) \) and we know that for a subset \( A \) of a topological space \( X \), \( x \in R^k(cl(A)) \) if and only if \( U \cap A \neq \emptyset \) for every \( R^k \)-open set \( A \) containing \( x \). Therefore \( f(R^k(cl(A)) \subseteq cl(f(A)) \).

Definition 3.18: Let \( (X, \tau) \) be a topological space and \( \tau^{g}_{g}=\{V \subseteq X \cap R^k(cl(V))=V \} \) is a topology on \( X \).

Definition 3.19: A topological space \( (X, \tau) \) is called a \( R^k \)-space if every \( R^k \)-closed is closed.

Definition 3.20: A topological space \( (X, \tau) \) is called a \( R^k \)-space if every \( R^k \)-closed is g-closed in \( X \).

Remark 3.21: The composition of two \( R^k \)-continuous maps need not be continuous.

Example 3.22: Let \( X=\mathbb{Z}=\{a, b, c, d \} \) and \( \tau=\{\emptyset, X, \{a, b, c, d \}, \{a, b \}, \{a, c \}, \{a, d \}, \{b, c \}, \{b, d \}, \{c, d \} \} \). Then \( f:X \rightarrow Y \) is \( R^k \)-continuous and \( g : Y \rightarrow Z \) is \( R^k \)-continuous but \( gof:X \rightarrow Z \) is not \( R^k \)-continuous.

Theorem 23.23: Let \( f: X \rightarrow Y \) be \( R^k \)-continuous and \( g: Y \rightarrow Z \) be \( R^k \)-continuous then \( (gof): X \rightarrow Z \) is \( R^k \)-continuous.

Theorem 23.24: Let \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) be \( R^k \)-continuous and \( Y \) be \( T^g_{g} \)-space then \( g: X \rightarrow Z \) is \( R^k \)-continuous.

Proof: Let \( V \) be an open set in \( Z \) since \( g \) is continuous, \( g^{-1}(V) \) is open in \( Y \). Since \( f \) is \( R^k \)-continuous, \( f^{-1}(g^{-1}(V))=gof^{-1}(V) \) is \( R^k \)-open in \( X \). Hence \( g \) is \( R^k \)-continuous.

Definition 3.25: A function \( f: X \rightarrow Y \) is called a perfectly \( R^k \)-continuous if \( f^{-1}(V) \) is clopen (open and closed) set in \( X \) for every \( R^k \)-open set \( V \) in \( Y \).

Theorem 3.26: If \( f: X \rightarrow Y \) is continuous then the following holds.

i. If \( f \) is perfectly \( R^k \)-continuous then it is \( R^k \)-continuous
ii. If \( f \) is perfectly \( R^\delta \)-continuous then it is rg-continuous (resp. \( r^\bullet g \)-continuous, gpr-continuous, gspr-continuous, rg\( \beta \)-continuous, rwg-continuous, wgr\( \alpha \)-continuous)

**Proof:**

i. Let \( U \) be open set in \( Y \). Since \( f \) is perfectly continuous then \( f^{-1}(U) \) is both open and closed in \( X \). Since every open is \( R^\delta \)-open, \( f^{-1}(U) \) is \( R^\delta \)-open in \( X \). Hence \( f \) is \( R^\delta \)-continuous.

ii. Let \( U \) be open set in \( Y \). Since \( f \) is perfectly continuous then \( f^{-1}(U) \) is both open and closed in \( X \). Since every open is \( rg \)-open (resp. \( r^\bullet g \)-open, gpr-open, gspr-open, \( rg\beta \)-open, rwg-open, wgr\( \alpha \)-open) set in \( X \). Hence \( f \) is rg-continuous (resp. \( r^\bullet g \)-continuous, gpr-continuous, gspr-continuous, \( rg\beta \)-continuous, rwg-continuous, wgr\( \alpha \)-continuous).

**Definition 3.27:** A function \( f: X \rightarrow Y \) is called \( R^\delta \)-continuous if \( f^{-1}(V) \) is \( R^\delta \)-closed set in \( X \) for every \( g \)-closed set \( V \) in \( Y \).

**Theorem 3.28:** If \( f: X \rightarrow Y \) is \( R^\delta \)-continuous then it is \( R^\delta \)-continuous but converse is not true.

**Proof:** Let \( f: X \rightarrow Y \) be \( R^\delta \)-continuous. Let \( F \) be any closed set in \( Y \). Since \( f \) is \( R^\delta \)-continuous, \( f^{-1}(F) \) is \( R^\delta \)-closed set in \( X \). Since every closed set in \( Y \) is \( g \)-closed set in \( Y \), then \( f^{-1}(F) \) is \( R^\delta \)-closed set in \( X \). Hence \( f \) is \( R^\delta \)-continuous.

**Example 3.29:** Let \( X=Y=\{a, b, c\} \), \( \tau=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) be a topology on \( X \) and \( \sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\} \) be a topology on \( Y \). \( R^\delta \)-C(\( X=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), \( R^\delta \)-C(\( Y=\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(a)=a \), \( f(b)=b \), \( f(c)=c \). Thus \( f \) is \( R^\delta \)-continuous but not a \( R^\delta \)-continuous function as the \( g \)-closed set \( F=\{a\} \) in \( Y \), \( f^{-1}(F)=\{a\} \) is not a \( R^\delta \)-closed set in \( X \).

**R\( ^\delta \)-Irresolute and strongly \( R^\delta \)-continuous functions**

**Definition 3.30:** A function \( f: X \rightarrow Y \) is called \( R^\delta \)-irresolute map if \( f^{-1}(V) \) is \( R^\delta \)-closed set in \( X \) for every \( R^\delta \)-closed set \( V \) in \( Y \).

**Definition 3.31:** A function \( f: X \rightarrow Y \) is called a strongly \( R^\delta \)-irresolute map if \( f^{-1}(V) \) is closed set in \( X \) for every \( R^\delta \)-closed set \( V \) in \( Y \).

**Theorem 3.32:** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( R^\delta \)-irresolute then it is \( R^\delta \)-continuous but not conversely.

**Proof:** Let \( f: X \rightarrow Y \) be \( R^\delta \)-irresolute. Let \( F \) be any closed set in \( Y \) and hence \( R^\delta \)-closed in \( Y \). Since \( f \) is \( R^\delta \)-irresolute, \( f^{-1}(F) \) is \( R^\delta \)-closed set in \( X \). Therefore \( f \) is \( R^\delta \)-continuous.

**Example 3.33:** Let \( X=Y=\{a, b, c\} \), \( \tau=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) be a topology on \( X \) and \( \sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\} \) be a topology on \( Y \). \( R^\delta \)-C(\( X=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), \( R^\delta \)-C(\( Y=\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(a)=a \), \( f(b)=b \), \( f(c)=c \). Thus \( f \) is \( R^\delta \)-irresolute map as the \( R^\delta \)-closed set \( F=\{a\} \) in \( Y \), \( f^{-1}(F)=\{a\} \) is not a \( R^\delta \)-closed set in \( X \).

**Theorem 3.34:** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( R^\delta \)-irresolute if and only if \( f^{-1}(V) \) is \( R^\delta \)-open set in \( X \) for every open set \( V \) in \( Y \).

**Proof:** Suppose that \( f: X \rightarrow Y \) is \( R^\delta \)-irresolute and \( U \) be \( R^\delta \)-open set in \( Y \). Then \( U^c \) is \( R^\delta \)-closed in \( Y \). By the definition of \( R^\delta \)-irresolute, \( f^{-1}(U) \) is \( R^\delta \)-closed in \( X \). But \( f^{-1}(U^c)=X-f^{-1}(U) \). Thus \( f(U) \) is \( R^\delta \)-open in \( X \).

Conversely, suppose that \( f^{-1}(F) \) is \( R^\delta \)-open set in \( X \) for every \( R^\delta \)-open set \( F \) in \( Y \). Let \( F \) be any \( R^\delta \)-closed set in \( Y \). By the definition, \( f^{-1}(F) \) is \( R^\delta \)-open in \( X \). But \( f^{-1}(F)=X-f^{-1}(F) \). Thus \( f \) is \( R^\delta \)-irresolute in \( X \) and hence \( f^{-1}(F) \) is \( R^\delta \)-closed in \( X \). Therefore \( f \) is \( R^\delta \)-irresolute.

**Theorem 3.35:** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( R^\delta \)-irresolute then it is \( R^\delta \)-continuous but not conversely.

**Proof:** Let \( f: X \rightarrow Y \) be \( R^\delta \)-irresolute. Let \( F \) be any \( g \)-closed set in \( Y \) and hence \( f \) is \( R^\delta \)-closed in \( Y \). By the definition of \( R^\delta \)-irresolute, \( f^{-1}(F) \) is \( R^\delta \)-closed set in \( X \). Therefore \( f \) is \( R^\delta \)-continuous.

**Example 3.36:** Let \( X=Y=\{a, b, c\} \), \( \tau=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \), \( \sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\} \). \( R^\delta \)-C(\( X=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), \( R^\delta \)-C(\( Y=\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(a)=a \), \( f(b)=b \), \( f(c)=c \). Thus \( f \) is \( R^\delta \)-irresolute map as the \( R^\delta \)-closed set \( F=\{a\} \) in \( Y \), \( f^{-1}(F)=\{a\} \) is not a \( R^\delta \)-closed set in \( X \).

**Theorem 3.37:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be \( R^\delta \)-irresolute then \( f(R^\delta cl(A)) \subseteq gcl(f(A)) \) for every subset \( A \) of \( X \).

**Proof:** Let \( A \subseteq X \) and \( gcl(f(A)) \) is \( R^\delta \)-closed in \( Y \). Since \( f \) is \( R^\delta \)-irresolute, \( f^{-1}(R^\delta cl(A)) \) is \( R^\delta \)-closed in \( X \). Further \( A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(gcl(f(A))) \). By the definition of the \( R^\delta \)-closure, \( R^\delta cl(A) \subseteq f^{-1}(gcl(A)) \). Hence \( f(R^\delta cl(A)) \subseteq gcl(f(A)) \).
Theorem 3.38: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be any two functions. Then

i. \( \text{gof}: (X, \tau) \rightarrow (Z, \eta) \) is \( R^k \)-irresolute if \( g \) is \( R^k \)-irresolute and \( f \) is \( R^k \)-irresolute

ii. \( \text{gof}: (X, \tau) \rightarrow (Z, \eta) \) is \( R^k \)-continuous if \( g \) is \( R^k \)-continuous and \( f \) is \( R^k \)-continuous.

Proof: (i) Let \( F \) be any \( R^k \)-closed set in \((Z, \eta)\). Since \( g \) is \( R^k \)-irresolute then \( g^{-1}(F) \) is \( R^k \)-closed in \((Y, \sigma)\). Since \( f \) is \( R^k \)-irresolute \( f^{-1}(g^{-1}(F)) \) is \( R^k \)-closed in \((X, \tau)\). But \((gof)^{-1}(F)=f^{-1}(g^{-1}(F))\) and hence gof is \( R^k \)-irresolute.

(ii) Let \( F \) be any \( R^k \)-closed set in \((Z, \eta)\). Since \( g \) is \( R^k \)-continuous then \( g^{-1}(F) \) is \( R^k \)-closed in \((Y, \sigma)\). Since \( f \) is \( R^k \)-irresolute \( f^{-1}(g^{-1}(F)) \) is \( R^k \)-closed in \((X, \tau)\). But \((gof)^{-1}(F)=f^{-1}(g^{-1}(F))\) and hence gof is \( R^k \)-continuous.

Theorem 3.39: If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( R^k \)-continuous then \( f \) is continuous but converse is not true.

Proof: Let \( f: X \rightarrow Y \) be strongly \( R^k \)-continuous. Let \( F \) be any closed set in \( Y \). Since every closed set is \( R^k \)-closed and hence \( F \) is \( R^k \)-closed set in \( Y \). Since \( f \) is strongly \( R^k \)-continuous then \( f^{-1}(F) \) is closed set in \( X \). Therefore \( f \) is continuous.

Example 3.40: Let \( X=Y=\{a, b, c\} \). Let \( \tau=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) be the topology on \( X \), and \( \sigma=\{\emptyset, Y, \{a\}, \{b\}, \{c\}\} \) be the topology on \( Y \). Closed sets of \( X=\{\emptyset, X, \{a\}, \{b\}, \{b, c\}\} \). \( R^k \)-continuous functions in \( Y \) are \( \{\emptyset, \{a\}, \{b\}, \{c\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(a)=a, f(b)=b, f(c)=c \) is continuous but not strongly \( R^k \)-continuous as the \( R^k \)-closed set \( F=\{a\} \) in \( Y \), \( f^{-1}(F)=\{a\} \) is not a closed set in \( X \).

Theorem 3.41: Every strongly \( R^k \)-continuous is strongly \( g \)-continuous but not conversely.

Proof: Let \( f: X \rightarrow Y \) be strongly \( R^k \)-continuous. Let \( F \) be any \( g \)-closed set in \( Y \). Since every \( g \)-closed set is \( R^k \)-closed and hence \( F \) is \( R^k \)-closed set in \( Y \). Since \( f \) is strongly \( R^k \)-continuous then \( f^{-1}(F) \) is closed set in \( X \) and hence \( g \)-closed set in \( X \). Therefore \( f \) is \( g \)-continuous.

Example 3.42: In example 3.40, \( f \) is strongly \( g \)-continuous but not a strongly \( R^k \)-continuous as the \( R^k \)-closed set \( F=\{a\} \) in \( Y \), \( f^{-1}(F)=\{a\} \) is not a closed set in \( X \).

Theorem 3.43: If a mapping \( f: (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( R^k \)-continuous if and only if \( f^{-1}(U) \) is open set in \( X \) for every \( R^k \)-open set \( U \) in \( Y \).

Proof: Suppose that \( f: X \rightarrow Y \) is strongly \( R^k \)-continuous. Let \( U \) be any \( R^k \)-open set in \( Y \) and hence \( U^c \) is \( R^k \)-closed set in \( Y \). Since \( f \) is strongly \( R^k \)-continuous, \( f^{-1}(U^c) \) is closed set in \( X \). But \( f^{-1}(U^c)=X- f^{-1}(U) \). Thus \( f^{-1}(U) \) is open in \( X \).

Conversely, suppose that \( f^{-1}(U) \) is open set in \( X \) for every \( R^k \)-open set \( U \) in \( Y \). Let \( F \) be any \( R^k \)-closed set in \( Y \) and hence \( F^c \) is \( R^k \)-open in \( Y \). But \( f^{-1}(F^c)=X- f^{-1}(F) \). Thus \( X- f^{-1}(F) \) is open in \( X \) and so \( f^{-1}(F) \) is closed in \( X \). Therefore \( f \) is strongly \( R^k \)-continuous.

Theorem 3.44: Every strongly continuous is strongly \( R^k \)-continuous but not conversely.

Proof: Let \( f: X \rightarrow Y \) be strongly continuous. Let \( G \) be any \( R^k \)-open set in \( Y \) and also any subset of \( Y \). Since \( f \) is strongly continuous then \( f^{-1}(G) \) is both open and closed in \( X \), say \( f^{-1}(G) \) is open in \( X \). Therefore \( f \) is strongly \( R^k \)-continuous.

Example 3.45: Let \( X=Y=\{a, b, c\} \). Let \( \tau=\{\emptyset, X, \{a\}, \{b\}, \{c\}\} \) and \( \sigma=\{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(a)=a, f(b)=b, f(c)=c \) is strongly \( R^k \)-continuous but not a strongly continuous as the set \( F=\{c\} \) in \( Y \), \( f^{-1}(F)=\{c\} \) is not a clopen set in \( X \).

Theorem 3.46: Every strongly \( R^k \)-continuous is \( R^k \)-continuous but not conversely.

Proof: Let \( f: X \rightarrow Y \) be strongly \( R^k \)-continuous. Let \( F \) be any closed set in \( Y \) and hence \( R^k \)-closed in \( Y \). Since \( f \) is strongly \( R^k \)-continuous, then \( f^{-1}(F) \) is closed set in \( X \) and hence \( R^k \)-closed set in \( X \). Therefore \( f \) is \( R^k \)-continuous.

Example 3.47: In example 3.40, \( f \) is \( R^k \)-continuous but not strongly \( R^k \)-continuous as the \( R^k \)-closed set \( F=\{a\} \) in \( Y \), \( f^{-1}(F)=\{a\} \) is not a closed set in \( X \).

Theorem 3.48: In discrete topological space, every strongly \( R^k \)-continuous is strongly continuous.

Proof: Let \( f: X \rightarrow Y \) be strongly \( R^k \)-continuous in a discrete topological space. Let \( F \) be any subset of \( Y \). Since \( F \) is both open and closed subset of \( Y \) in discrete space. We have the following two cases.
Case (i): Let $F$ be any closed subset of $Y$ and hence $R^a$-closed in $Y$. Since $f$ is strongly $R^a$-continuous then $f^{-1}(F)$ is closed in $X$.

Case (ii): Let $F$ be any open subset of $Y$ and hence $R^a$-open in $Y$. Since $f$ is strongly $R^a$-continuous then $f^{-1}(F)$ is open in $X$.

Therefore $f^{-1}(F)$ is both open and closed in $X$. Hence $f$ is strongly continuous.

**Theorem 3.49** Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be any two functions. Then

i. $\text{gof}: X \rightarrow Z$ is strongly $R^a$-continuous if both $f$ and $g$ are $R^a$-continuous.

ii. $\text{gof}: X \rightarrow Z$ is strongly $R^a$-continuous if $g$ is strongly $R^a$-continuous and $f$ is continuous.

iii. $\text{gof}: X \rightarrow Z$ is irresolute if $g$ is strongly $R^a$-continuous and $f$ is $R^a$-continuous.

iv. $\text{gof}: X \rightarrow Z$ is continuous if $g$ is $R^a$-continuous and $f$ is strongly $R^a$-continuous.

**Proof:**

i. Let $G$ be $R^a$-closed set in $(Z, \eta)$. Since $g$ is strongly $R^a$-continuous then $g^{-1}(G)$ is closed set in $(Y, \sigma)$ and hence $R^a$-closed set in $(Y, \sigma)$. Since $f$ is also strongly $R^a$-continuous then $f^{-1}(g^{-1}(G))$ closed set in $(X, \tau)$. But $(gof)^{-1}(G)=f^{-1}(g^{-1}(G))$ and hence $\text{gof}$ is strongly $R^a$-continuous.

ii. Let $G$ be $R^a$-closed set in $(Z, \eta)$. Since $g$ is strongly $R^a$-continuous then $g^{-1}(G)$ is closed set in $(Y, \sigma)$. Since $f$ is continuous then $f^{-1}(g^{-1}(G))$ closed set in $(X, \tau)$. But $(gof)^{-1}(G)=f^{-1}(g^{-1}(G))$ and hence $\text{gof}$ is strongly $R^a$-continuous.

iii. Let $G$ be any $R^a$-closed set in $(Z, \eta)$. Since $g$ is strongly $R^a$-continuous then $g^{-1}(G)$ is closed set in $(Y, \sigma)$. Since $f$ is $R^a$-continuous then $f^{-1}(g^{-1}(G))$ is $R^a$-closed set in $(X, \tau)$. But $(gof)^{-1}(G)=f^{-1}(g^{-1}(G))$. Hence $\text{gof}$ is irresolute.

iv. Let $G$ be any closed set in $(Z, \eta)$. Since $g$ is $R^a$-continuous then $g^{-1}(G)$ is $R^a$-closed set in $(Y, \sigma)$. Since $f$ is strongly $R^a$-continuous then $f^{-1}(g^{-1}(G))$ closed set in $(X, \tau)$. But $(gof)^{-1}(G)=f^{-1}(g^{-1}(G))$. Hence $\text{gof}$ is continuous.

**Theorem 3.50:** Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be any two functions. Then

i. $\text{gof}: X \rightarrow Z$ is strongly $R^a$-continuous if $g$ is perfectly $R^a$-continuous and $f$ is continuous.

**Proof:**

i. Let $G$ be any $R^a$-open set in $(Z, \eta)$. Since $g$ is perfectly $R^a$-continuous then $g^{-1}(G)$ is clopen set in $(Y, \sigma)$, say $g^{-1}(G)$ is open set in $(Y, \sigma)$. Since $f$ is continuous then $f^{-1}(g^{-1}(G))$ open set in $(X, \tau)$. Thus $(gof)^{-1}(G)=f^{-1}(g^{-1}(G))$. Hence $\text{gof}$ is strongly $R^a$-continuous.

**Theorem 3.51:** Let $(X, \tau)$ be a discrete topological space and $(Y, \sigma)$ be any topological space. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following statements are equivalent.

i. $f$ is strongly $R^a$-continuous

ii. $f$ is perfectly $R^a$-continuous.

**Proof:**

(i) $\rightarrow$ (ii): Let $G$ be any open set in $(Y, \sigma)$. Since $f$ is strongly $R^a$-continuous then $f^{-1}(G)$ is open set in $(X, \tau)$. But in discrete space, $f^{-1}(G)$ is closed set in $(X, \tau)$. Thus $f^{-1}(G)$ is both open and closed in $(X, \tau)$. Hence $f$ is perfectly $R^a$-continuous.

(ii) $\rightarrow$ (i): Let $U$ be any $R^a$-open set in $(Y, \sigma)$. Since $f$ is perfectly continuous then $f^{-1}(G)$ is both open and closed in $(X, \tau)$. Hence $f$ is strongly $R^a$-continuous.

**Theorem 3.52:** Let $(X, \tau)$ be any topological space and $(Y, \sigma)$ be $T_{R^a}$ space and $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map. Then the following are equivalent.

i. $f$ is strongly $R^a$-continuous

ii. $f$ is continuous

**Proof:**

(i) $\rightarrow$ (ii): Let $F$ be any closed set in $(Y, \sigma)$. Since every closed set is $R^a$-closed and hence $F$ is $R^a$-closed in $(Y, \sigma)$. Since $f$ is strongly $R^a$-continuous then $f^{-1}(F)$ is closed set in $(X, \tau)$. Hence $f$ is continuous.
(i) $\Rightarrow$ (ii): Let $G$ be any $R^*_s$-closed set in $(Y,\sigma)$. Since $(Y,\sigma)$ is $T_{R^*_s}$ space, $F$ is closed set in $(Y,\sigma)$. Since $f$ is continuous then $f^{-1}(F)$ is closed set in $(X,\tau)$. Hence $f$ is strongly $R^*_s$-continuous.

**Theorem 3.53:** Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a map. Both $(X,\tau)$ and $(Y,\sigma)$ are $T_{R^*_s}$ space. Then the following are equivalent.

i. $f$ is $R^*_s$-irresolute

ii. $f$ is strongly $R^*_s$-continuous

iii. $f$ is continuous

iv. $f$ is $R^*_s$-continuous

The proof is obvious.

**Theorem 3.54:** Let $X$ and $Y$ be $R^*_s$ spaces. Then for the function $f : (X,\tau) \rightarrow (Y,\sigma)$ the following are equivalent.

i) $f$ is gc-irresolute

ii) $f$ is $R^*_s$-irresolute

**Proof:**

(i) $\Rightarrow$ (ii): Let $f: X \rightarrow Y$ be gc-irresolute. Let $F$ be a g-closed set in $Y$ and hence $R^*_s$-closed in $Y$. Since $f$ is gc-irresolute then $f^{-1}(F)$ is g-closed set in $X$ and hence $R^*_s$-closed in $X$. Therefore $f$ is $R^*_s$-irresolute.

(i) $\Rightarrow$ (ii): Let $f : X \rightarrow Y$ be $R^*_s$-irresolute. Let $F$ be a g-closed set in $Y$ and hence $R^*_s$-closed in $Y$. Since $f$ is $R^*_s$-irresolute then $f^{-1}(F)$ is $R^*_s$-closed in $X$. But $X$ is $R^*_s$ space and hence $f^{-1}(F)$ is g-closed set in $X$. Therefore $f$ is gc-irresolute.

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