Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

ISSN NO. 2320-5407

RESEARCH ARTICLE

Ideals Purity On Smooth Semigroups

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Manuscript Info

Manuscript History:

Received: 11 December 2013
Final Accepted: 25 January 2014
Published Online: February 2014

Key words:
Complete, globally, Indempotent, purity of ideals, smooth groups.

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Abstract

Our aim in this paper is to investigate the purity of the ideals on smooth semigroups. Smooth groups were due to Warrack B.D (2006). We prove every ideal of a semigroup S is globally idempotent iff it is complete. Clifford. A. H (1999) observed some basic necessary conditions and Szas S.B(2001) indicated an analogous notion of pure subgroup of an abelian group. Further we proved A is $\Im$-pure iff $A \cap [x]S = [x]A$, $A \cap S[x] = A[x]\ \forall\ x \in S$ and generalized the theorem of Kuroki. N (1998) to obtain some important results which are useful in poroscopy of a finger prints.

Introduction

Let S be a semigroup. A non-empty subset A of S is said to be ideal of S if $SA \subseteqq A$ and $AS \subseteqq A$.

Let $\Im(S)$ be the set of all ideals of a semigroup S. We define a binary operation on $\Im(S)$ as follows for $X, Y \in \Im(S)$,

$XY = \{xy : x \in X \text{ and } y \in Y\}$.

then it is easily seen that $\Im(S)$ is a semigroup and is called the ideal-semigroup of S. An ideal A of a semigroup S is said to be globally idempotent if $A^2 = A$. An ideal A of S is said to be complete if $SA = AS = A$. The present paper is concerned about the kind of semigroups which gives the proof of converse of the proposition 1, therefore while observing same we introduce the notion of purity of ideals of semigroups, which is an analogous notion of pure subgroup of an abelian group while discussing about all these facts we obtain the following results.

PROPOSITION 1. If every ideal of a semigroup S is globally idempotent then every ideal of S is complete.

Proof. Let A be any globally idempotent ideal of S . Then

$$A = AA \subseteqq AS \subseteqq A \text{ and } A = AA \subseteqq SA \subseteqq A$$

and so

$$A = AS = SA.$$

This means that A is a complete ideal of S.

Let A be any ideal of a semigroup S, then we shall say that A is $\Im - pure$ if

$A \cap XS = XA$ and $A \cap SX = AX$

holds for all $X \in \Im(S)$. A semigroup S is said to be $\Im^* - pure$ if every ideal of S is $\Im - pure$.

It is shown that if a semigroup S is $\Im^* - pure$ then the converse of Propostition 1 is valid.

We give some properties of $\Im-pure$ ideals. By the definition of purity, the semigroup S itself is a trivial example of an $\Im-pure$ ideal of S. if $S=S^o$ then $\{O\}$ is an $\Im-pure$ ideal of S. From these a simple semigroup and a $O-$simple semigroup and $\Im^*-pure$, we denote by [x] the principal ideal of a semigroup S generated by x in S, so the following theorem.

THEOREM 1. For any ideal A of a semigroup S the following statements are equivalent:

(1) A is $\Im-pure$

(2) $A\cap XS\subseteq XA$ and $A\cap SX\subseteq AX$ for all $X\in\Im(S)$.

(3) $A\cap[x]S=[x]A$ and $A\cap S[x]=A[x]$ for all $x\in S$.

(4) $A\cap[x]S\subseteq[x]A$ and $A\cap S[x]\subseteq A[x]$ for all $x\in S$.

Proof. Since the inclusions

$$XA\subseteq A\cap XS \text{ and } AX\subseteq A\cap SX$$

are true for any ideal A of S and for all $X\in\Im(S)$, it follows that (1) and (2) are equivalent. Similarly, we showed (3) and (4) are equivalent. It is clear that (2) implies (4). We assume that (4) holds. Let X be any ideal of S, then for any element

$$a=xs\in(a\in A, x\in X, s\in S) \text{ of } A\cap XS \text{, we have}$$

$$a=xs\in A\cap[x]S\subseteq[x]A\subseteq XA$$

and so we get

$$A\cap XS\subseteq XA$$

for all $X\in\Im(S)$. In a similar way we show that

$$A\cap SX\subseteq AX$$

for all $X\in\Im(S)$, therefore we obtain (4) implies (2). This completes the proof of Theorem 1.
The present property is the immediate consequence of Theorem 1 and so we obtain corollary 1 and theorem 2.

COROLLARY 1. For any element a of a semigroup S the following statements are equivalent:

(1) $[a]$ is $X\in\Im(S)$.

(2) $[a]\cap XS\subseteq X[a]$ and $[a]\cap SX\subseteq[a]X$ for all $X\in\Im(S)$;

(3) $[a]\cap[x]S=[x][a]$ and $[a]\cap S[x]=[a][x]$ for all $X\in\Im(S)$;

(4) $[a]\cap[x]S\subseteq[x][a]$ and $[a]\cap S[x]\subseteq[a][x]$ for all $X\in\Im(S)$;

THEOREM 2. For a semigroup S the following statements are equivalent:

(1) S is $\Im^*-pure$;

(2) For every ideal A of S, $A\cap XS\subseteq XA$ and $A\cap SX\subseteq AX$ for all $X\in\Im(S)$;

(3) For every ideal A of S, $A\cap[x]S=[x]A$ and $A\cap S[x]=A[x]$ for all $x\in S$;

(4) For every ideal A of S, $A\cap[x]S\subseteq[x]A$ and $A\cap S[x]\subseteq A[x]$ for all $x\in S$;

(5) Every principal ideal of S is $\Im-pure$;

(6) For every element a of S, $[a]\cap XS\subseteq X[a]$ and $[a]\cap SX\subseteq[a]X$ for all $X\in\Im(S)$;

(7) For every element a of S, $[a]\cap[x]S=[x][a]$ and $[a]\cap S[x]=[a][x]$ for all $x\in S$;

(8) For every element a of S, $[a]\cap[x]S\subseteq[x][a]$ and $[a]\cap S[x]\subseteq[a][x]$ for all $x\in S$;

Proof. It follows from Theorem 1 that (1)-(4) are equivalent with each other, and follows from Corollary 1 that (5)-(8) are equivalent with each other. It is clear that (2) implies (6). We suppose that (6) holds. Let A and X be any ideals of S and let $a=xs\left(a\in A, x\in X, s\in S\right)$ be any element of $A\cap XS$,then we obtain

$$a=xs\in[a]\cap XS\subseteq X[a]\subseteq XA$$

and so we get

$$A\cap XS\subseteq XA$$

for all $X\in\Im(S)$. In a similar way we show that

$$A\cap SX\subseteq AX$$

for all $X\in\Im(S)$ and we obtain (6) implies (2). This complete the proof of our theorem 2. In addition we obtain some properties and obtain the following result as residuated groups S.K. Panday (2008).

LEMMA 1. For a semigroup S the following statements are equivalent:

(1) $\Im(S)$ is idempotent;

(2) Every principal ideal of S is globally idempotent;

(3) $X\cap Y=XY$ for all $X,Y\in\Im(S)$.

The following is an immediate consequence of Lemma 1.

LEMMA 2. An idempotent ideal-semigroup $\Im(S)$ of a semigroup S is commutative.

LEMMA 3. Any globally idempotent ideal of a semigroup is complete.

THEOREM 3. Any complete $\Im-pure$ ideal of a semigroup is globally idempotent.

Proof. Let A be any complete $\Im-pure$ ideal of a semigroup S. Then, for all $X\in\Im(S)$, we have

$$XA=A\cap XS.$$

This holds for $X=A$, that is,

$$AA=A\cap AS.$$

Since A is complete, we have

$$AA=A\cap A=A.$$

This means that A is globally idempotent. This completes the proof of Theorem 3.
From Theorem 3 and Proposition 1 we obtain our main theorem 4 and theorem 5.

THEOREM 4. For any $\Im^{*}-pure$ semigroup S the following statements are equivalent.

(1) $\Im(S)$ is idempotent;

(2) Every ideal of S is complete.

THEOREM 5. Let S be a semigroup such that the ideal-semigroup $\Im(S)$ of is idempotent then S is $\Im^{*}-pure$.

Proof. Let A be any ideal of S. Since $\Im(S)$ is idempotent, it follows from Lemma 1 that

$$XS=X$$

for all $X\in\Im(S)$. Then by Lemma 2 we have

$$A\cap XS=A(XS)=AX.$$

Then it follows from Lemma 3 that

$$A\cap XS=XA \text{ and } A\cap SX=AX$$

for all $X \in \Im(S)$, that is, A is $\Im - pure$. Since A is any ideal of S, we obtain that S is $\Im^* - pure$. This completes the proof of our theorem 5.

The following theorem 6 can be obtained from Theorems 4 and 5.

THEOREM 6. Let S be a semigroup such that every ideal of S is compete. Then the following statements are equivalent:

(1) S is $\Im^* - pure$;

(2) $\Im(S)$ is idempotent.

Further we construct some examples for the theorem 6 and preposition 1.

Example 1. Let $S = \{a,b,c\}$ be a semigroup with the following multiplication table.

	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

Then it can be shown that

$$\{a,b\} = S\{a,b\} = \{a,b\}$$

That is, $\{a,b\}$ is a complete ideal of S. And it can be shown that

$$\{a,b\},\{a,b\} = \{a\} \neq \{a,b\},$$

That is, $\{a,b\}$ is not a globally idempotent ideal of S. Therefore this example gives the converse of Proposition 1 does not hold in general.

Example 2. Let $S = \{a,b,c\}$ be a semigroup with the following multiplication table.

	a	b	c
a	a	a	a
b	a	a	b
c	a	a	c

It can be shown that $\{a,b\}$ is an $\Im - pure$ ideal of S, and that it is not a complete ideal of S, or not a globally idempotent ideal of S. Therefore this example gives the converse of Theorem 6 does not hold in general and some termination conditions can be seen by Relaxing one of the condition. It is observed that the generalization results of the Kuroki's . N (1998) theorem by using the method of theorem 2 and S of example 2 are useful in the poroscopy.

Result and Discussion

In the present paper we discuss the study of application problems which have been widely emerged in the growth and response of the areas such as Information systems, cryptography and network connectivity. The theoretical results obtains here are not only of interest in mathematics but also useful for better understanding the problems of modeling in medicine and the fuzzy semigroups of Zadeh. L.A and Guttwad. G (1997).

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