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RESEARCH ARTICLE

Bifurcation and Stability Analysis in a Discrete Three Species Prey-Predator Interactions

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Abstract

This paper investigates an eco-system with two prey species and a predator. The model equations constitute a set of three first order non-linear coupled difference equations. All possible positive equilibrium points of the system are computed and criteria for the stability of all equilibrium states are established. Time series plots and phase portraits are obtained for different sets of parameter values. Bifurcation diagrams are provided for selected range of growth parameter.

INTRODUCTION

Mathematical properties and ecological meaning of continuous and discrete models have been investigated qualitatively and numerically in order to explain mutual interactions between populations. Prey-predator models are of great interest to both ecologists and mathematicians, because the problem attempts to model the complex relationship in the populations of different species that share the same environment. In recent decades, many researchers [2, 6, 8, 11] have focused on the ecological models with three and more species to understand complex dynamical behaviors of ecological systems in the real world. The models have demonstrated very complex dynamic nature of the models, including cycles, periodic doubling and chaos. The discrete time models produce much richer patterns [1, 4, 7, 5, 9]. Discrete time models are ideally suited to describe the population dynamics of species, which are characterized by discrete generations.

Mathematical Model

In this paper, we consider the discrete-time prey-predator system describing the interactions among three species by the following system of difference equations:

$$\begin{aligned} x(n+1) &= x(n)\left(1-rx(n)\right) - a\,x(n)\,y(n) - bx(n)z(n) \\ y(n+1) &= sy(n)\left(1-y(n)\right) - c\,x(n)\,y(n) - d\,y(n)z(n) \\ z(n+1) &= \left(1-e\right)z(n) + f\,x(n)z(n) + g\,y(n)z(n) \end{aligned} \qquad (1)$$

where the densities of prey species are denoted by x(n), y(n) and z(n) represents predator population density and all parameters are non - negative. It is assumed that all the species in the model grow logistically. Also the prey species help each other against predator. This is a discrete version of a model discussed in [3].

Existence of Equilibrium

The equilibrium points of (1) are the solutions of the equations

$x = x(1-rx) - axy - bxz;\ y = sy(1-y) - cxy - dyz;\ z = (1-e)z + fxz + gyz.$

The equilibrium points are

$E_0 = (0,0,0)$, $E_1 = \left(0, \frac{s-1}{s} 0\right)$, $E_2 = \left(0, \frac{e}{g}, \frac{g(s-1)-es}{gd}\right)$, $E_3 = \left(\frac{a(s-1)}{ca-sr}, \frac{r(s-1)}{sr-ca}, 0\right)$ and $E_4 = \left(x^*, y^*, z^*\right)$. Where,

$x^* = \frac{e(ad-bs)+bg(s-1)}{f(ad-bs)+g(cb-rd)}$, $y^* = \frac{e(bc-rd)-fb(s-1)}{f(ad-bs)+g(cb-rd)}$ and $z^* = \frac{e(rs-ca)+(s-1)(fa-rg)}{f(ad-bs)+g(cb-rd)}$.

Interior equilibrium point E_4 corresponds to the coexistence of all species.

Dynamical Behavior of the Model

In this section, we investigate the local behavior of the system (1) around each equilibrium point. The local stability analysis of the system (1) can be studied by computing the variation matrix corresponding to each equilibrium point. The variation matrix for the system (1) is

$$J(x,y,z) = \begin{bmatrix} 1-2rx-ay-bz & -ax & -bx \\ -cy & s(1-2y)-cx-dz & -dy \\ fz & gz & 1-e+fx+gy \end{bmatrix} \quad (2)$$

Since we are interested in the nontrivial equilibrium points, we neglect E_0.

Theorem 1: The equilibrium point E_1 is locally asymptotically stable if $1 < s < \frac{a}{a-2}$, $1 < s < 3$ and $\frac{g}{g-e+2} < s < \frac{g}{g-e}$, otherwise unstable equilibrium point.

Proof: The Jacobian matrix *J* for the system evaluated at the equilibrium point E_1 is given by

$$J(E_1) = \begin{pmatrix} 1+\frac{a(1-s)}{s} & 0 & 0 \\ \frac{c(1-s)}{s} & 2-s & \frac{d(1-s)}{s} \\ 0 & 0 & 1-e-\frac{g(1-s)}{s} \end{pmatrix}.$$

Hence the eigenvalues of the matrix $J(E_1)$ are $\lambda_1 = 1+\frac{a(1-s)}{s}$, $\lambda_2 = 2-s$ and $\lambda_3 = 1-e-\frac{g(1-s)}{s}$. Hence E_1 is locally asymptotically stable when $1 < s < \frac{a}{a-2}$, $1 < s < 3$ and $\frac{g}{g-e+2} < s < \frac{g}{g-e}$, and unstable when $s > \frac{a}{a-2}$, $s > 3$ and $\frac{g}{g-e} < s < \frac{g}{g-e+2}$.

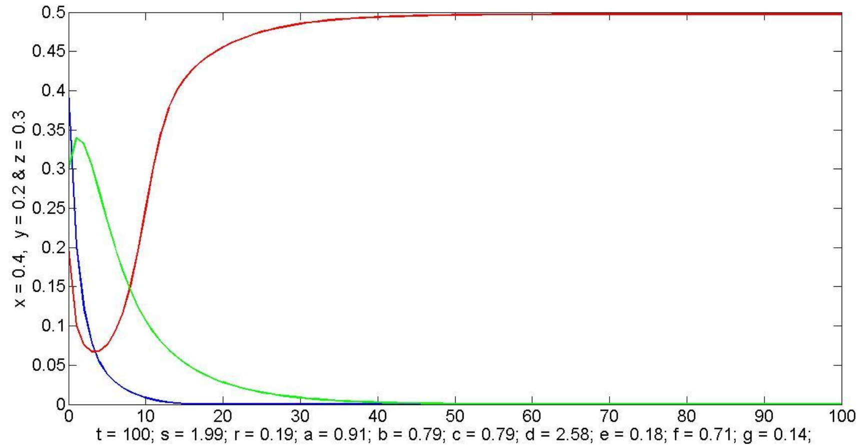


Figure 1: Time Series Plot at E_1

Theorem 2: The equilibrium point E_2 is locally asymptotically stable if $s < \dfrac{adc - bg}{b(e-g)}$ and $\beta < 0$ where $\beta = \alpha - es$ such that $\alpha = \sqrt{e^2 s(s+4g) + 4eg^2(1-s)}$, otherwise unstable equilibrium point.

Proof: The Jacobian matrix evaluated at E_2 is given by

$$J(E_2) = \begin{pmatrix} 1 - \dfrac{ae}{g} + \dfrac{bes - bg(s-1)}{gd} & 0 & 0 \\ -\dfrac{ce}{g} & s\left(1 - \dfrac{2e}{g}\right) + \dfrac{es - g(s-1)}{g} & -\dfrac{ed}{g} \\ \dfrac{gf(s-1) - fes}{gd} & \dfrac{g(s-1) - es}{d} & 1 \end{pmatrix}.$$

Hence the eigenvalues of the matrix $J(E_2)$ are $\lambda_1 = 1 - \dfrac{ae}{g} + \dfrac{bes - bg(s-1)}{gd}$ and $\lambda_{2,3} = \dfrac{2g - es \pm \alpha}{2g}$. Hence E_2 is locally asymptotically stable when $s < \dfrac{ade - bg}{b(e-g)}$ and $\beta < 0$, and unstable when $s > \dfrac{ade - bg}{b(e-g)}$ and $\beta > 0$.

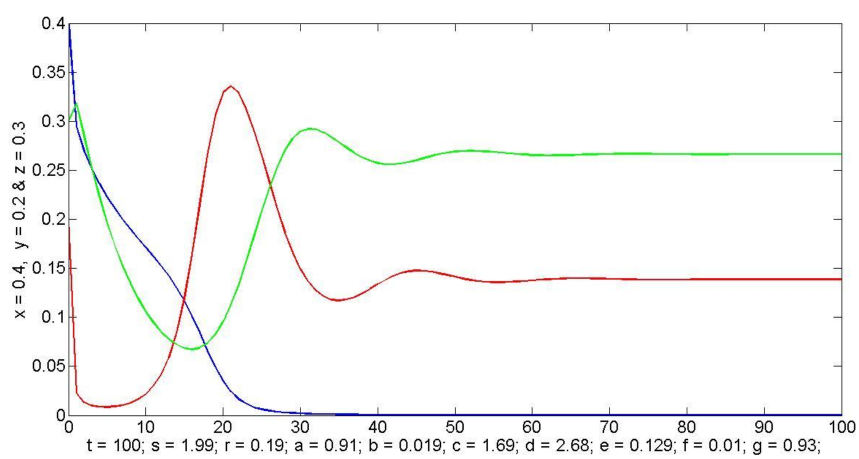


Figure 2: Time Series Plot at E_2

Theorem 3: The equilibrium point E_3 is locally asymptotically stable if $a_{33} < 1$ and $\gamma < 0$ where $\gamma = a_{11} + a_{22} + \mu - 2$ such that $\mu = \sqrt{a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}}$, otherwise unstable equilibrium point.

Proof: The Jacobian matrix evaluated at E_3 is given by

$$J(E_3) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Where, $a_{11} = 1 + \frac{ar(s-1)}{sr - ca}$, $a_{12} = \frac{a^2(s-1)}{sr - ca}$, $a_{13} = \frac{ab(s-1)}{sr - ca}$, $a_{21} = \frac{cr(1-s)}{sr - ca}$ $a_{22} = s + \frac{(s-1)(ac-2rs)}{sr - ca}$, $a_{23} = \frac{dr(1-s)}{sr - ca}$ and $a_{33} = 1 - e + \frac{(s-1)(gr - fa)}{sr - ca}$. Hence the eigenvalues of the matrix $J(E_3)$ are $\lambda_1 = a_{33}$ and $\lambda_{2,3} = \frac{a_{11} + a_{22} \pm \mu}{2}$. Hence E_3 is locally asymptotically stable when $a_{33} < 1$ and $\gamma < 0$, and unstable when $a_{33} > 1$ and $\gamma > 0$.

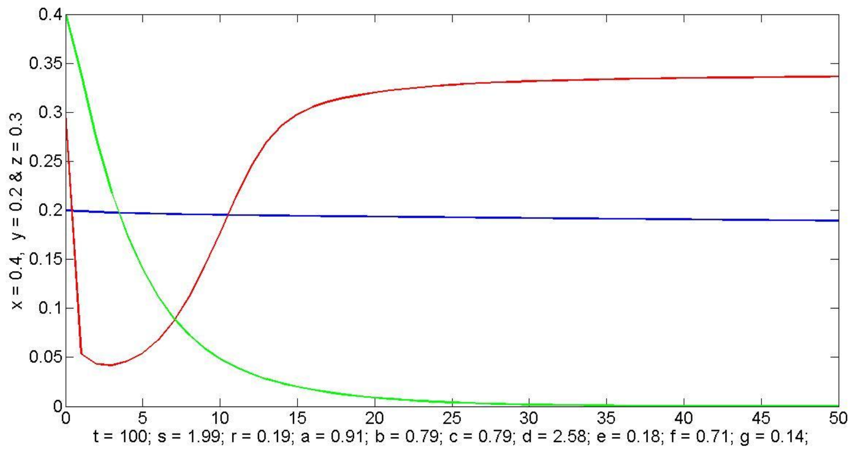


Figure 3: Time Series Plot at E_3

Local Stability and Dynamical Behavior around Interior Fixed Point E_4

We now investigate the local stability and bifurcation of interior fixed point E_4. The Jacobian matrix *J* at E_4 has the form

$$J(E_4) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}. \quad (3)$$

Where, $b_{11} = 1 - 2rx^* - ay^* - bz^*$, $b_{12} = -ax^*$, $b_{13} = -bx^*$, $b_{21} = -cy^*$, $b_{22} = s(1-2y^*) - cx^* - dz^*$, $b_{23} = -dy^*$, $b_{31} = fz^*$, $b_{32} = gz^*$ and $b_{33} = 1 - e + fx^* + gy^*$.

Its characteristic equation is

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (4)$$

with $A = -(b_{11} + b_{22} + b_{33})$, $B = b_{11}b_{33} + b_{22}b_{33} + b_{11}b_{22} - b_{12}b_{21} - b_{13}b_{31} - b_{23}b_{32}$ and $C = (b_{23}b_{32} - b_{22}b_{33})b_{11} + (b_{21}b_{33} - b_{23}b_{31})b_{12} + (b_{22}b_{31} - b_{21}b_{32})b_{13}$.

By the Routh-Hurwitz criterion, E_4 is locally asymptotically stable if and only if *A, C,* and *AB*-C are positive.

Phase-Plane & Bifurcation Diagram Analysis

In this section, we provide the phase-plane diagrams for the prey system and also we present the bifurcation diagrams of the model (1) that have been obtained with data from 500 iterations with time-step of 0.001 units. The bifurcation diagrams are presented with the presence of predator and the plots have been generated using MATLAB 7 [10].

In Figure (4) and (5) shows that the Trajectories of the solutions around the positive equilibrium point for all the systems, they showed the stabilizing and destabilizing effect of the discrete prey predator system. We now assume the parameter values $t = 500, s = 3.66,\ r = 0.03,\ a = 0.01, b = 0.02, c = 2.99,\ d = 2.34,\ f = 0.19$ and $g = 1.99$ are taken together with the initial values for prey and predator population $x = 0.2, y = 0.3$ and $z = 0.4$. Hence the system (1) is unstable.

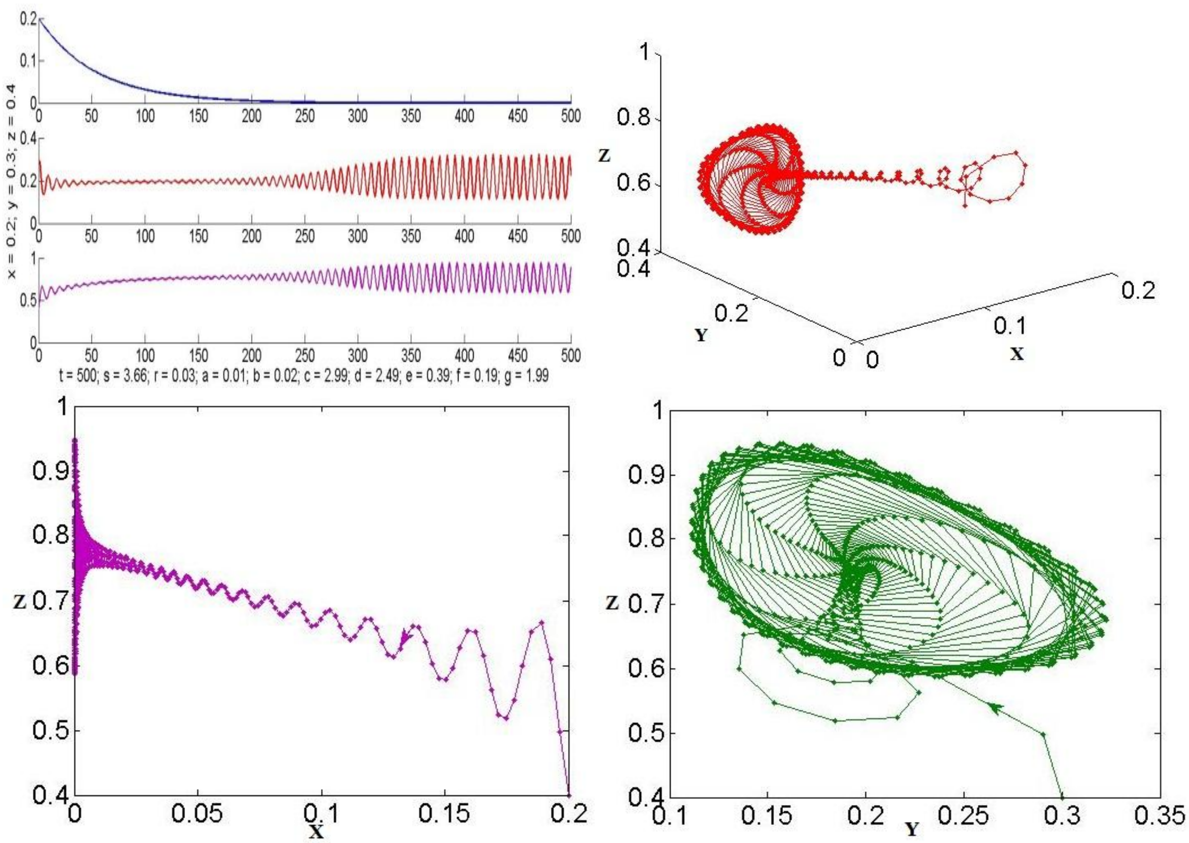


Figure 4: Time series plot and Phase Portrait are Instability at E_4

While with $t = 300,\ s = 3.56,\ r = 0.003, a = 0.001,\ b = 0.002,\ d = 2.39,\ f = 0.89$ and keeping all other parameter same, the phase portrait shows a sink and the trajectory spirals towards an interior equilibrium point. We observe the system (1) is stable (see Figure - 5).

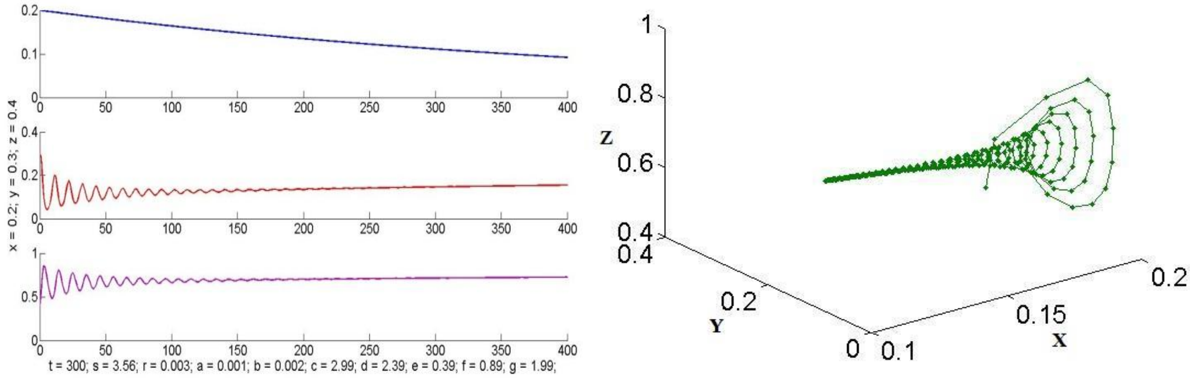
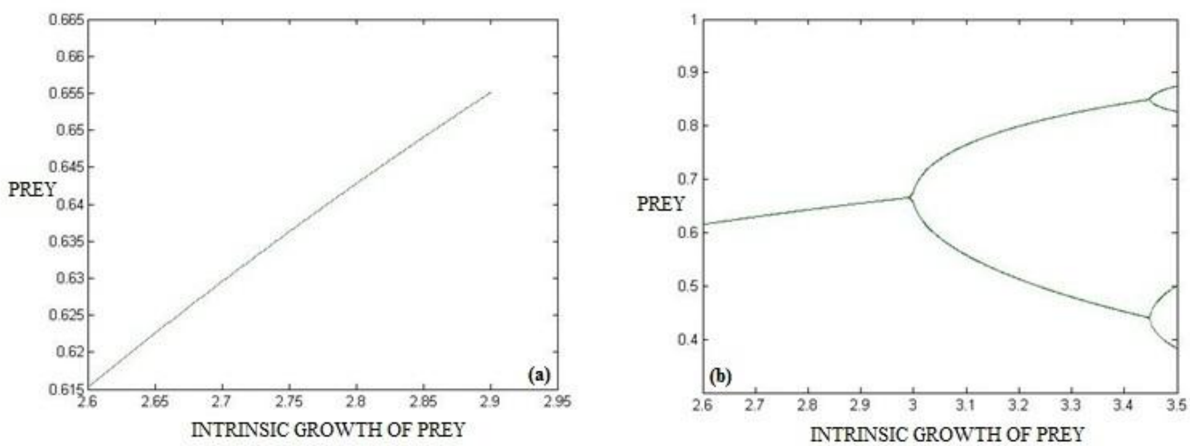


Figure 5: Time series plot and Phase Portrait are Stability at E_4

Finally in Figure 6 indicates, the bifurcation diagrams for predator densities of the system (1) with initial conditions $x = 0.4, y = 0.3$ and $z = 0.5$ as above and we consider the parameter values $r = 1.91,$ $a = 0.69, b = 1.39, c = 0.69,$ $d = 0.29,$ $e = 0.83,$ $f = 0.31,$ $g = 0.29$ and $s = 2.6 - 4$. It can be observed from Figure 6 (a) shows that there is no period-doubling bifurcation for intrinsic growth rate $s = 2.6$ to 2.9, with only one predator, (b) shows the bifurcation that bifurcates 2 cycles when the intrinsic growth rate = 3 with one predator and the prey population bifurcates 4 cycles at 3.5 and (c) shows when intrinsic growth rate = 2.6 to 4 with one predator, the prey population bifurcate 2 cycle at $s = 3$ and bifurcate 4 cycles at $s = 3.5$ and chaos after $s = 3.5$ that is increasing the parameters effectively makes the bounds on the system tighter and pushes it from stability towards unstable behavior. This unstability manifests itself as a period-doubling bifurcation as a result of which the single equilibrium level of the population splits into two and the population starts oscillating between two levels which are quite different in their relative magnitudes.



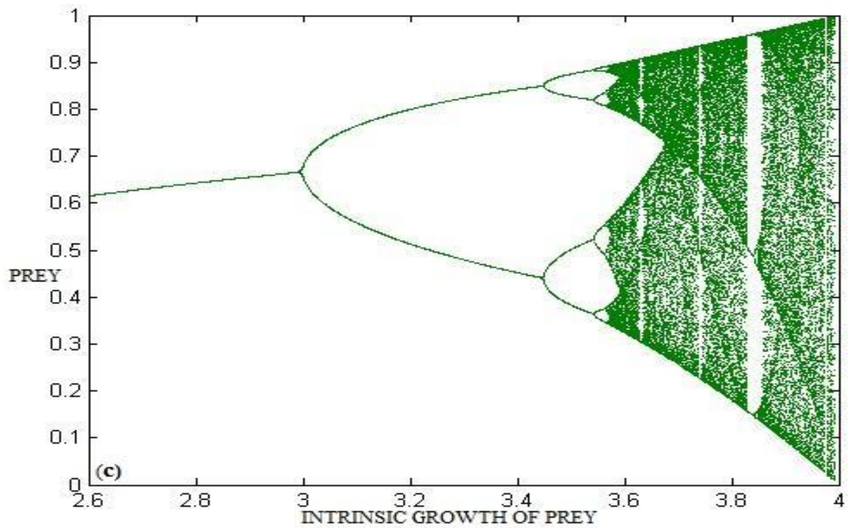


Figure 6: Bifurcation diagram for the prey system with intrinsic growth rate *s*, in the presence of predator (a) when s = 2.6 to 2.9; (b) when s = 2.6 to 3.5; (c) when s = 2.6 to 4.

As we keep on increasing the parameters, these levels individually split up more and more frequently, until all order is lost and we found an infinite number of possible equilibrium states visited by the population. At this point, the population behavior seems to lose any stability. This appearance of non-periodic behavior from equilibrium population levels may be referred to as the “period-doubling route to chaos”, the non-periodic dynamics being described as *chaotic* [Figure 6 (c)].

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