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RESEARCH ARTICLE

ON FIXED POINT THEOREM IN WEAK CONTRACTION PRINCIPLE.

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Abstract

The study of Fixed Point Theorem has been widely done in many fields. The Banach Fixed Point Theorem plays important role in this theory. It becomes milestone in the various paths in this field.
In this paper we have discussed existence and uniqueness of fixed point in more general conditions. The concept of weak contraction mapping over contractive metric space is discussed. In general, for a function $f: X \to X$ to have a fixed point, weak contraction is not a sufficient condition for function.
Additionally function needs to be a compact to have a fixed point. Banach contraction principle is one of the directive theorems in the analysis of the result.

1. Introduction:-

The concept of weak contraction principle was firstly noted by Alber and Guerre – Delabriere [3] but in Banach Space. Later Rhoades [5] generalised the result of Banach Contraction Principle as contraction as a special case.

The concept of weak contraction then further developed very interestingly by Boyd – Wong [6] in complete metric space.

2. Preliminaries:-

Definition 2.1: Metric Space:

Let $d: X \to X$ be the mapping then d is called metric on X if

i) $d(x,y) \geq 0$
ii) $d(x,y) = d(y,x)$
iii) $d(x,y) = 0$, iff $x = y$
iv) $d(x,y) \leq d(x,z) + d(z,y)$, for $\forall x, y \in X$

Theorem 2.1: Banach Contraction Theorem:

Let (X,d) be the metric space then $f: X \to X$ is said to be a Lipschitz continuous if there exists $\lambda \geq 0$, such that

$$d\big(f(x_1), f(x_2)\big) \leq \lambda d(x_1, x_2), \forall\, x_1, x_2 \in X$$

Where λ is called Lipschitz constant
If $\lambda \leq 1$ then f is called non expansive,

$$\lambda < 1 \text{ then } f \text{ } is\ called\ contraction\ mapping$$

Let f be a contraction mapping on complete metric space X, then f has unique fixed point $x \in X$

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Definition 2.2: Complete Metric Space:
If every Cauchy's sequence in X converges to a limit x in X, then such metric space is called Complete Metric Space

Definition 2.3: Identity Map:
Let $f: X \to X$ and $n \in N$, then f^n is the n^{th} iterate of f (n times)then f^n is called identity map.

Theorem 2.2:
Let (X, d) be a metric space and $f: X \to X$ be a mapping. If f^n is a contraction mapping then f has unique fixed point $\bar{x} \in X$, foe some $n \geq 1$.

Theorem 2.3: Weak contraction Principle:
Let (X,d) be a metric space. Then $f: X \to X$, is a weak contraction if,

$$d\big(f(x_1), f(x_2)\big) \leq d(x_1, x_2), \forall\, x_1 \neq x_2 \text{ and } x_1, x_2 \in X$$

Theorem 2.4: Boyed –Wong theorem:
Let (X, d) be a complete metric space and $f: X \to X$. Define $\psi: [0, \infty) \to [0, \infty)$ as right continuous map such that

$$d\big(f(x_1), f(x_2)\big) \leq \psi[d(x_1, x_2)], \forall\, x_1, x_2 \in X$$

Then f has unique fixed point $\bar{x} \in X$.
Moreover for any $x_0 \in X$, the sequence $f^n(x_0)$converges to $\bar{x}$

3. Main Result:-

Theorem 3.1:
Let (X,d) be a compact metric space and f be a weak contraction on X, . Then f has unique fixed point $x \in X$ and $f^n(x_0)$ converges to x for some $x_0 \in X$

Proof: Let (X,d) be a compact metric space, then by Banach contraction principle
We have
$f(\bar{x}) = \lim f(x_n) = \lim x_{n+1} = \bar{x}$
If not, i.e.
$f(\bar{x}) \neq \bar{x}$, then

$$\begin{aligned} d\big(\bar{x}, f(\bar{x})\big) &= \min\big(x, f(x)\big) \\ &\leq d(f(\bar{x}), f(f(\bar{x}))) \\ &< d\big(\bar{x}, f(\bar{x})\big), \quad \text{which contradicts to the assumption.} \end{aligned}$$

$$\therefore \bar{x} \text{ is unique fixed point of f and so of } f^n(x)$$

Now let $\bar{x} \neq x_0$ and define $d_n = d(f^n(x_0), \bar{x})$
Then $d_{n+1} = d(f^{n+1}(x_0), f(\bar{x}))$
$< d\,(f^n(x_0), \bar{x}) = d_n$
Which implies that d_n is strictly decreasing. And now let $y_k(x_0)$be a subsequence of $f^n(x_0)$
Which converges to $y \in X$ then

$$\begin{aligned} \text{Define } r = d(y, \bar{x}) &= \lim_{k\to\infty} d_{n_k} \\ &= \lim_{k\to\infty} d_{n_{k+1}} \\ &= \lim_{k\to\infty}[\, f(y_k(x_0)), \bar{x}]\ (\\ &= d(f(y), \bar{x}), \qquad \text{but } y = \bar{x} = f(\bar{x}) \\ &= d(f(y), f(\bar{x})) \\ &< d(y, \bar{x}) \end{aligned}$$

$$\therefore \text{subsequence of } f^n \text{ has a limit } \bar{x}$$

Since X is compact, it implies that $f^n(x_0)$ converges to $\bar{x}$
Which prove that imposing the condition of compactness of X, being a weak contraction of f becomes sufficient condition in order to have fixed point.

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