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RESEARCH ARTICLE

Data Dependence Result And Stability Of Picard-S Iteration Scheme For Approximating Fixed Point Of Almost Contraction Mappings.

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Abstract

We study the stability of a Picard-S iteration method for the almost contraction mappings. Furthermore, we prove a data dependence result for fixed point of the almost contraction mappings with the help of the Picard-S iteration method.

Introduction:-

Fixed point theory is concerned with investigating a wide variety of issues such as the existence (and uniqueness) of fixed points, the construction of fixed points, etc. One of these themes is data dependency of fixed points. Data dependency of fixed points has been the subject of research in fixed point theory for some time now, and data dependence research is an important theme in its own right. Several authors had made contributions to the study of data dependence of fixed points such as Berinde [2] and others.

In the study of iterations, it is also important to examine their stability. The concept of stability was introduced by Harder [6], Harder and Hicks [7], [8] and roughly speaking of a fixed point iteration procedure is numerically stable if small modification in the initial data involved in the computation process will produce a small influence on the computed value of the fixed point.

In this paper, we establish data dependence result of Picard-S iterative scheme[5]. Also, we prove the stability of this iteration.

Throughout this paper the set of all positive integers and zero is shown by $\mathbb{N}$. Let B be a Banach space, C be a nonempty closed convex subset of B and T a self-map of C. An element u^* of C is called a fixed point of T if and only if $Tu^* = u^*$ [4]. The set of all fixed point of T is denoted by F_T.

Data Dependence:-

In this section we will prove data dependence results for the Picard-S iterative scheme [5] when applied to the almost contraction operator. We define the Picard-S iteration as follows:

$$\begin{cases} x_0 \in C \\ x_{n+1} = Ty_n \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{cases} \qquad (1)$$

where $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ under the following kind of mappings.

The following definitions and lemmas will be needed in obtaining the main results of this article.

Definition1.1[3]:-

Let $(B,\|.\|)$ be a Banach space. A map T:B→B is called an almost contraction mapping if there exists a constant δ∈(0,1) and some L∈[0,∞) such that

$\|Tx-Ty\| \leq \delta\|x-y\| + L\|y-Tx\|$ for all x,y∈B (2)

Definition1.2 [3]:-

Let T,$\tilde{T}$:B→B be two operators. We say that $\tilde{T}$ is an approximate operator of T if for all x∈B and for a fixed ε>0 we have

$\|Tx-Ty\| \leq \varepsilon$ (3)

Definition1.3 [1]:-

Let $(B,\|.\|)$ be a Banach space. A map T:B→B is said to satisfy condition (B) if there exist 0<δ<1 and L≥0 such that for all x,y∈B we have

$$\|Tx-Ty\| \leq \delta\|x-y\| + Lmin\{\|x-Tx\|,\|y-Ty\|,\|x-Ty\|,\|y-Tx\|\}$$

Notation.

We will abbreviate the set $\{\|x-Tx\|,\|y-Ty\|,\|x-Ty\|,\|y-Tx\|\}$ by $N_{x,y}$.

Lemma1.4 [9]:-

Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists n_0∈ℕ, such that for all n≥n_0 one has satisfied the inequality

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\rho_n$$

where λ_n∈(0,1), for all n∈ℕ, $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\rho_n \geq 0$, for all n∈ℕ. Then the following inequality holds

$$\lim_{n\to\infty} sup a_n \leq \lim_{n\to\infty} sup \rho_n$$

Theorem1.5:-

Let T:C→C be an almost contraction map satisfying condition (B) with $u^* \in F_T$ and $\{x_n\}_{n=0}^{\infty}$ an iterative sequence defined by (1) such that $x_n \to u^*$ as n→∞ and $\tilde{T}$ an approximate operator of T. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1) for T and define an iterative sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ as follows

$$\begin{cases} \tilde{x}_0 \in C \\ \tilde{x}_{n+1} = \tilde{T}\tilde{y}_n \\ \tilde{y}_n = (1-\beta_n)\tilde{T}x_n + \beta_n\tilde{T}\tilde{z}_n \\ \tilde{z}_n = (1-\gamma_n)\tilde{x}_n + \gamma_n\tilde{T}\tilde{x}_n, n \in \mathbb{N} \end{cases} \quad (4)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ be real sequences in [0,1] satisfying

1. $\frac{1}{2} < \beta_n\gamma_n$ for all $n \in \mathbb{N}$.
2. $\sum_{n=0}^{\infty}\beta_n\gamma_n = \infty$.

If $\tilde{T}\tilde{u}^* = \tilde{u}^*$ such that $\tilde{x}_n \to \tilde{u}^*$ as $n \to \infty$, then we have

$$\|u^* - \tilde{u}^*\| \leq \frac{5\varepsilon}{1-\delta}$$

Where $\varepsilon > 0$ is a fixed number.

Proof:-

It follows from (1), (2), (3), (4) and condition (B) that

$$\begin{aligned} \|z_n - \tilde{z}_n\| &\leq (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n\|Tx_n - \tilde{T}\tilde{x}_n\| \\ &\leq (1-\gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n\|Tx_n - T\tilde{x}_n\| + \gamma_n\|T\tilde{x}_n - \tilde{T}\tilde{x}_n\| \\ &\leq [1-\gamma_n(1-\delta)]\|x_n - \tilde{x}_n\| + \gamma_n LminN_{x_n,\tilde{x}_n} + \gamma_n\varepsilon \quad (5) \end{aligned}$$

Also

$$\begin{aligned} \|y_n - \tilde{y}_n\| &= \|(1-\beta_n)Tx_n + \beta_n Tz_n - (1-\beta_n)\tilde{T}x_n + \beta_n\tilde{T}\tilde{z}_n\| \\ &\leq (1-\beta_n)\|Tx_n - \tilde{T}\tilde{x}_n\| + \beta_n\|Tz_n - \tilde{T}\tilde{z}_n\| \\ &\leq (1-\beta_n)\|Tx_n - T\tilde{x}_n\| + (1-\beta_n)\|T\tilde{x}_n - \tilde{T}\tilde{x}_n\| + \beta_n\|Tz_n - T\tilde{z}_n\| + \beta_n\|T\tilde{z}_n - \tilde{T}\tilde{z}_n\| \end{aligned}$$

$$\leq (1-\beta_n)\delta\|x_n - \tilde{x}_n\| + \beta_n\delta\|z_n - \tilde{z}_n\| + (1-\beta_n)\, LminN_{x_n,\tilde{x}_n} + \beta_n LminN_{z_n,\tilde{z}_n} + \varepsilon \quad (6)$$

Therefore

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &= \|Ty_n - \tilde{T}\tilde{y}_n\| \\ &\leq \|Ty_n - T\tilde{y}_n\| + \|T\tilde{y}_n - \tilde{T}\tilde{y}_n\| \end{aligned}$$

$\leq \delta\|y_n - \tilde{y}_n\|$
$+LminN_{y_n,\tilde{y}_n} + \varepsilon$ (7)

From (5), (6) and (7), we get:

$\|x_{n+1} - \tilde{x}_{n+1}\| \leq \delta^2[1 - \beta_n\gamma_n(1-\delta)]\|x_n - \tilde{x}_n\| +$
$[\beta_n\gamma_n\delta^2 L + (1-\beta_n)\delta L]minN_{x_n,\tilde{x}_n} + LminN_{y_n,\tilde{y}_n} + \beta_n\delta\, LminN_{z_n,\tilde{z}_n} +$
$\beta_n\gamma_n\delta^2\varepsilon + (1-\beta_n)\delta\varepsilon + \beta_n\delta\varepsilon + \varepsilon$ (8)

Since $\beta_n, \gamma_n \in [0,1]$ and $\frac{1}{2} < \beta_n\gamma_n$ for all $n \in \mathbb{N}$.

$1 - \beta_n\gamma_n \leq \beta_n\gamma_n$ (9)
$1 - \beta_n \leq 1 - \beta_n\gamma_n \leq \beta_n\gamma_n$ (10)
$1 \leq 2\beta_n\gamma_n$ (11)

Use of the facts δ, $\delta^2 \in (0,1)$, (9), (10) and (11) in (8) yields:

$\|x_{n+1} - \tilde{x}_{n+1}\| \leq [1 - \beta_n\gamma_n(1-\delta)]\|x_n - \tilde{x}_n\| +$
$\beta_n\gamma_n(1-\delta)\left[\frac{L\delta(1+\delta)min\, N_{x_n,\tilde{x}_n} + 2Lmin\, N_{y_n,\tilde{y}_n} + 2\delta Lmin\, N_{z_n,\tilde{z}_n} + 5\varepsilon}{(1-\delta)}\right]$ (12)

Define

$a_n = \|x_n - \tilde{x}_n\|$
$\lambda_n = \beta_n\gamma_n(1-\delta) \in (0,1)$
$\rho_n = \dfrac{L\delta(1+\delta)minN_{x_n,\tilde{x}_n} + 2LminN_{y_n,\tilde{y}_n} + 2\delta LminN_{z_n,\tilde{z}_n} + 5\varepsilon}{(1-\delta)}$

Hence, the inequality (12) perform all assumptions in lemma (1.4) and thus an application of lemma (1.4) to (12) yields

$0 \leq \lim_{n\to\infty} sup\|x_n - \tilde{x}_n\|$ (13)
$\leq \lim_{n\to\infty} sup \frac{L\delta(1+\delta)min\, N_{x_n,\tilde{x}_n} + 2Lmin\, N_{y_n,\tilde{y}_n} + 2\delta Lmin\, N_{z_n,\tilde{z}_n} + 5\varepsilon}{(1-\delta)}$

Since $\lim_{n\to\infty} x_n = u^*$ and $Tu^* = u^*$, then
$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|y_n - Ty_n\| = \lim_{n\to\infty}\|z_n - Tz_n\| = 0$,
Therefore, the inequality (13) becomes:

$$\|u^* - \tilde{u}^*\| \leq \frac{5\varepsilon}{1-\delta}$$

Stability:-

In this section we shall present in the following the stability result of the Picard-S iterative scheme (1) for the mapping given by (2) satisfying condition (B).

Definition 2.1 [5]:-

Let $\{q_n\}_{n=0}^{\infty}$ be an arbitrary sequence in C. Then an iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T, if for
$\varepsilon_n = \|z_{n+1} - f(T, z_{n+1})\|$
we have
$\lim_{n\to\infty} \varepsilon_n = 0$ if and only if $\lim_{n\to\infty} z_n = u^*$

Lemma2.2 [10]:-

Let $\{\tau_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:
$\tau_{n+1} \leq (1-\lambda_n)\tau_n + \rho_n$,
where $\lambda_n \in (0,1)$ for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \tau_n = 0$.

Theorem2.3:-

Let C be a nonempty closed convex subset of a Banach space B and $T: C \to C$ be an almost contraction map satisfying condition (2). Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by (1) with $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ real sequences in $[0,1]$ such that $\sum_{k=0}^{n} a_k b_k = \infty$. Then the iterative scheme is T-stable.

Proof:-

By definition to prove an iterative is a stable with respect to a map T, let $\{q_n\}_{n=0}^{\infty}$ be an arbitrary sequence in C.

$\|q_{n+1} - u^*\| \leq \|q_{n+1} - Tr_n\| + \|Tr_n - u^*\|$
$\leq \delta\|r_n - u^*\| + LminN_{r_n,u^*} + \varepsilon_n$

$$
\begin{aligned}
&= \delta\|r_n - u^*\| + \varepsilon_n \\
&= \delta\|(1-b_n)Tq_n + b_nTs_n - u^*\| + \varepsilon_n \\
&\le \delta(1-b_n)\|Tq_n - Tu^*\| + \delta b_n\|Ts_n - Tu^*\| + \varepsilon_n \\
&\le \delta^2(1-b_n)\|q_n - u^*\| + \delta^2 b_n\|s_n - u^*\| + \varepsilon_n + \delta(1-b_n)LminN_{q_n,u^*} + \delta b_n LminN_{s_n,u^*} \\
&= \delta^2(1-b_n)\|q_n - u^*\| + \delta^2 b_n\|(1-c_n)q_n + c_nTq_n - u^*\| + \varepsilon_n \\
&\le \delta^2(1-b_n)\|q_n - u^*\| + \delta^2 b_n(1-c_n)\|q_n - u^*\| + \delta^2 b_nc_n\|Tq_n - u^*\| + \varepsilon_n \\
&\le \delta^2(1-b_nc_n)\|q_n - u^*\| + \delta^3 b_nc_n\|q_n - u^*\| + \varepsilon_n + \delta^2 b_nc_n LminN_{q_n,u^*}
\end{aligned}
$$

$$\|q_{n+1} - u^*\| \le \delta^2[1 - b_nc_n(1-\delta)]\|q_n - u^*\| + \varepsilon_n$$

By hypothesis we have $\lim_{n\to\infty}\varepsilon_n = 0$ and $b_n, c_n, \delta \in (0,1)$ then using Lemma (2.2) we get $\lim_{n\to\infty}\|q_n - u^*\| = 0$. Hence, we get $\lim_{n\to\infty}q_n = u^*$.

Now suppose that $\lim_{n\to\infty}q_n = u^*$ and we have to show that $\lim_{n\to\infty}\varepsilon_n = 0$.

We have that

$$\|q_{n+1} - Tr_n\| \le \|q_{n+1} - u^*\| + \|u^* - Tr_n\|$$

$$
\begin{aligned}
\|q_{n+1} - Tr_n\| &\le \|q_{n+1} - u^*\| + \delta\|u^* - r_n\| + LminN_{r_n,u^*} \\
&= \|q_{n+1} - u^*\| + \delta\|u^* - (1-b_n)Tq_n - b_nTs_n\| \\
&\le \|q_{n+1} - u^*\| + \delta(1-b_n)\|u^* - Tq_n\| + \delta b_n\|u^* - Ts_n\| \\
&\le \|q_{n+1} - u^*\| + \delta^2(1-b_n)\|q_n - u^*\| + \\
&\quad \delta^2 b_n\|s_n - u^*\| + \delta(1-b_n)LminN_{q_n,u^*} + \delta b_n LminN_{s_n,u^*} \\
&= \|q_{n+1} - u^*\| + \delta^2(1-b_n)\|q_n - u^*\| + \\
&\quad \delta^2 b_n\|(1-c_n)q_n + c_nTq_n - u^*\| \\
&\le \|q_{n+1} - u^*\| + \delta^2(1-b_n)\|q_n - u^*\| + \\
&\quad \delta^2 b_n(1-c_n)\|q_n - u^*\| + \delta^2 b_nc_n\|Tq_n - u^*\| \\
&\le \|q_{n+1} - u^*\| + \delta^2(1-b_nc_n)\|q_n - u^*\| + \\
&\quad \delta^3 b_nc_n\|q_n - u^*\| + \delta^2 b_nc_n\, LminN_{q_n,u^*} \\
&= \|q_{n+1} - u^*\| + \delta^2[1 - b_nc_n(1-\delta)]\|q_n - u^*\|
\end{aligned}
$$

By taking n goes to infinity we get:

$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\|q_{n+1} - Tr_n\| = 0.$

Then, (1) is stable with respect to T.

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