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RESEARCH ARTICLE

EXPONENTIATED MOMENT EXPONENTIAL DISTRIBUTION AND POWER SERIES DISTRIBUTION WITH APPLICATIONS: A NEW COMPOUND FAMILY.

Zafar Iqbal[1], Muhammad Wasim[1] and Naureen Riaz[2].

1. Department of Statistics, Govt. College Satellite Town Gujranwala.
2. Department of Statistics, Garrison University, Lahore.

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Abstract

This article introduces a new family of lifetime distributions called the exponentiated moment exponential power series (EMEPS) which generalizes the moment exponential power series (MEPS) proposed by Sadaf (2013). This new family is obtained by compounding the exponentiated moment exponential and truncated power series distributions, where the compounding procedure follows same way that was previously carried out by Adamidis and Loukas (1998). The new family contains some new distributions such as exponentiated moment exponential geometric distribution, exponentiated moment exponential Poisson distribution, exponentiated moment exponential logarithmic distribution and exponentiated moment exponential binomial distribution. Some former works derived by Sadaf 2014 such as moment exponential geometric and moment exponential Poisson distributions are special cases of the new EMEPS family. We obtain several properties of EMEPS family, among them; quantile function, order statistics, moments and entropy. Some special models in the exponentiated moment exponential power series family of distributions are provided. Maximum likelihood (ML) method is applied to obtain parameter estimates of the EMEPS family. A simulation study is carried out to check the consistency of the ML estimators of the parameters. Two real data sets are used to validate the distributions and the results demonstrate that the sub-models from the family can be considered as suitable models under several real situations.

Introduction:-

The problem of finding a suitable model for the real life data has been studied extensively in the literature, however, there are many situations where existing models are not suitable or less representative of real data, therefore, as a result to resolve this situation one needs to develop a general model. In many practical situations, most of the classical distributions do not produce a good fit to real data. To overcome these difficulties, various distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions with discrete distribution. Compounding lifetime distributions have been obtained by mixing up the distribution when the lifetime can be expressed as the minimum of a sequence of independent and identically

Corresponding Author:- Zafar Iqbal.
Address:- Department of Statistics, Govt. College Satellite Town Gujranwala.

distributed (iid) random variables with a discrete random variable. This idea was first pioneered by Adamidis and Loukas (1998) by compounding the exponential random variable simultaneously with a geometric random variable. Several authors introduced new lifetime distributions (see for example; Kus (2007), Barreto-Souza et al. (2011), and Lu and Shi (2012)).

In recent years, a great effort has been made to define new compounding families of distributions by mixing some useful lifetime and power series distributions. The new families extend some compound distributions and yield more flexibility in modeling several practical data. Some authors defined new families of lifetime distributions (see for example; exponential-power series family (Chahkandi and Ganjali (2009)), Weibull-power series distributions (Morais and Barreto-Souza (2011)), generalized exponential power series family (Mahmoudi and Jafari (2012)), extended Weibull power series distributions (Silva et al.(2013)), Burr XII power series(Silva and Corderio (2015)), generalized inverse Weibull power series (Hassan et al. (2016)).

The moment exponential (or length biased) distribution was proposed by Dara (2012) and discussed hazard and reversed hazard rate function with the next probability density function (pdf):

$$f(x;\beta)=\beta^2 xe^{-\beta x}, \quad x,\beta>0. \tag{1}$$

Properties, as well as extensions and applications of the moment exponential distribution mentioned in the context of reliability analysis by Dara (2012).

Hasnain et al.(2015) studied the properties of exponentiated moment exponential distribution with real life data applications. Its PDF is defined as

$$g(x;\beta,\gamma)=\gamma\beta^2 xe^{-\beta x}\left(1-\left(1+\beta x\right)e^{-\beta x}\right)^{\gamma-1}, \quad x,\beta>0. \tag{2}$$

Also, a discrete random variable, Z is a member of power series distributions (truncated at zero) when its probability mass function (pmf) is given by:

$$P(Z=z;\theta)=\frac{a_z\theta^z}{K(\theta)},\ z=1,2,3..., \tag{3}$$

where, $\theta>0$ is the scale parameter. The coefficients a_z's depend only on z, $K(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$ is finite, $K'(.)$ and $K''(.)$ denote its first and second derivatives, respectively. The expression "power series distribution" is generally credited to Noack (1950). This family of distributions includes many of the most popular distributions, which are the binomial, Poisson, geometric, negative binomial, and logarithmic distributions.

In this article, a new family of lifetime distributions is obtained by compounding the exponentiated moment exponential and truncated power series distributions, where the compounding procedure follows same way that was previously carried out by Adamidis and Loukas (1998).

The paper is organized as follows. In Section 2, we define EMEPS distributions and present its density function as well as, cumulative, survival and hazard rate functions of the new family. In Section 3, some mathematical properties are derived such as quantile, moments, entropy and order statistics. In Section 4, some special sub-models also some of its mathematical properties for two new sub-models are discussed. In Section 5, maximum likelihood estimator for the unknown parameters on the basis of the family is obtained. In Section 6, applications to real data sets are given to show the pliability and potentiality of the proposed family of distributions. Finally, concluding remarks are mentioned in Section 7.

New Family Of Distributions:-

In this section, the EMEPS family of distributions is proposed. This new family is derived by compounding the exponentiated moment exponential (EME) distribution and power series distributions. The cumulative distribution function, reliability and hazard rate function are also derived.

Let $X_1, X_2, ..., X_z$ be iid random variables having *EME* distribution with pdf (1) and the following cumulative distribution function (cdf):

$$G(x;\beta,\gamma) = \left(1-(1+\beta x)\ e^{-\beta x}\right)^{\gamma} \qquad (4)$$

Suppose that Z has a zero truncated power series distribution with the pmf (3). Let $X_{(1)} = \min\{X_1, X_2, ..., X_z\}$ independent of X 's, then the conditional probability density function of $X_{(1)} \mid Z$ is obtained as follows

$$f_{X_{(1)}|Z}(x \mid z;\beta,\gamma) = z\gamma\beta^2 x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)^{z-1}.$$

The joint probability density function of $X_{(1)}$ and Z is obtained as follows

$$f_{X_{(1)},Z}(x,z;\beta) = \frac{z\gamma\beta^2 a_z \theta^z x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}}{K(\theta)}\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)^{z-1}.$$

The probability density function of a EME power series family of distributions can be defined by the marginal density of X, that is,

$$f(x;\psi) = \gamma\beta^2\theta x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\frac{K'\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}, x,\beta,\theta,>0. \qquad (5)$$

where $\psi \equiv (\beta,\gamma,\theta)$ is a set of parameters. A random variable X with density function (5) is denoted by$X$~*EMEPS* (β,γ,θ).

Furthermore, the cumulative distribution function of *EMEPS* family of distributions corresponding to (5) is obtained as follows

$$F(x;\psi) = 1-\frac{K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}. \qquad (6)$$

In addition, the reliability and hazard rate functions for E*MEPS* family of distributions, respectively, take the following forms

$$R(x;\psi) = \frac{K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}, \qquad (7)$$ and,

$$h(x;\psi) = \frac{\beta^2\gamma\theta x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}K'\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}. \qquad (8)$$

Some Mathematical Properties:-

In this section, some mathematical properties of the *EMEPS* family including, expansion for pdf (5), quantile function, rth moment, Re'nyi entropy and distribution of order statistics are obtained.

Useful Expansion:-

In this subsection, two important propositions are provided. The first proposition indicates that the new family has the *EME* distribution as a limiting case while the second proposition provides useful expansion for the pdf of *EMEPS* distribution.

Proposition (1)

The *EME* distribution with parameters β *and* γ is a limiting special case of *EMEPS* family of distributions when $\theta \to 0^{+}$.

Proof: By applying $f(\theta) = \sum_{z=1}^{\infty} a_z \theta^z$, for $x > 0$ in cdf (6), then we obtain

$$\underset{\theta\to 0^{+}}{Lim} F(x;\psi) = 1 - \underset{\theta\to 0^{+}}{Lim} \frac{\sum_{z=1}^{\infty} a_z \left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z}}{\sum_{z=1}^{\infty} a_z \theta^z}.$$

By using L'Hˆopital's rule, we have

$$\underset{\theta\to 0^{+}}{Lim} F(x;\psi) = 1 - \frac{\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)[1 + a_1^{-1} \underset{\theta\to 0^{+}}{Lim} \sum_{z=2}^{\infty} z a_z \left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}]}{1 + a_1^{-1} \underset{\theta\to 0^{+}}{Lim} \sum_{z=2}^{\infty} z a_z \theta^{z-1}}.$$

Hence,

$$\underset{\theta\to 0^{+}}{Lim} F(x;\psi) = \left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma},$$

which is the distribution function of the *EME* distribution.

Proposition (2)

The density function of *EMEPS* family can be expressed as a linear combination of the density of $X_{(1)} = \min\{X_1, X_2, \ldots X_z\}$

Proof.

Since $f'(\theta) = \sum_{z=1}^{\infty} z a_z \theta^{z-1}$, then the pdf (5) can be expressed as follows

$$f(x;\psi) = \sum_{z=1}^{\infty} P(Z = z;\theta) g_{x_{(1)}}(x;z), \tag{9}$$

where $g_{x_{(1)}}(x;z)$ is the pdf of $X_{(1)} = \min\{X_1, X_2, \ldots, X_z\}$ given by:

$$g_{X_{(1)}}(x,\beta,\gamma;z) = z\beta^2 x(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma})^{z-1} e^{-\beta x}.$$

The Lambert W function:-

The Lambert W function has attracted a great deal of attention beginning with Lambert in 1758 and Euler in 1779.The name "Lambert W function" has become a standard after its implementation in the computer algebra system Maple in the 1980s and subsequent publication by Corless et al. (1996) of a comprehensive survey of the history, theory and applications of this function. The Lambert W function is a multivalued complex function defined as the solution of the equation

$$W(z)\exp\left(W(z)\right) = z \tag{10}$$

where z is a complex number. If z is a real number such that $z \geq -1/e$ then W(z) becomes a real function and there are two possible real branches. The real branch taking on values in $(-\infty,-1]$ is called the negative branch and denoted by W_{-1}. The real branch taking on values in $[-1,\infty)$ is called the principal branch and denoted by W_0

Lemma 1 Let $a, b \text{ and } c$ complex numbers, the solution of the equation

$z + ab^z = c$ with respect to $z \in C$ is

$$z = c - \frac{1}{\log(b)} W\left(ab^c \log(b)\right) \tag{11}$$

where W denotes the lambert W function.

Quantile function:-

In this subsection, the quantile function of the EMEPS distribution is derived. The quantile function, denoted by,$Q(p)$, defined by $Q(p) = p$, is the root of the following equation.

$$1 - \frac{K\left(\theta\left(1 - \left(1 - (1 + \beta(Q(p)))e^{-\beta(Q(p))}\right)^{\gamma}\right)\right)}{K(\theta)} = p, \ 0 < p < 1.$$

Let $D(p) = -(1 + \beta(Q(p)))$. Then,

$$1 - \left(1 + D(p)e^{D(p)+1}\right)^{\gamma} = -\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}.$$

$$D(p)e^{D(p)} = \frac{1}{e}\left(-1 + \left(1 + \left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right).$$

Then solution for $D(p)$ is

$$D(p) = W\left[\frac{1}{e}\left(-1 + \left(1 + \left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

where $W(.)$ is the negative branch of the Lambert W function(see Corless et al. (1996)). Consequently, the quantile function of the *MEPS* family is given by solving the following equation for $Q(p)$.

$$Q(p) = -\frac{1}{\beta} - \frac{1}{\beta} W\left[\frac{1}{e}\left(-1 + \left(1 + \left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right], \tag{12}$$

Moments and moment generating function:-

The rth moment of a random variable X from the *EMEPS* distribution, is given by using pdf (7) as the following

$$\mu_r' = \sum_{z=1}^{\infty} P(Z = z; \theta) \int_0^{\infty} x^r g_{X_{(1)}}(x; z) dx.$$

Then,

$$\mu_r' = \sum_{z=1}^{\infty} P(Z = z; \theta) \int_0^{\infty} z\beta^2 x^{r+1} (1 - \left(1 - (1 + \beta x)e^{-\beta x}\right)^{\gamma})^{z-1} e^{-\beta x} dx.$$

Let

$\beta x = u \rightarrow \beta\, dx = du$ then

$$\mu_r' = \sum_{z=1}^{\infty} P(Z = z; \theta) \int_0^{\infty} z\beta \left(\frac{u}{\beta}\right)^{r+1} (1 - \left(1 - (1 + u)e^{-u}\right)^{\gamma})^{z-1} e^{-u} du.$$

By using binomial series more than two times, then

$$\mu_r' = \frac{1}{\beta^r}\sum_{z=1}^{\infty}\sum_{i=1}^{z-1}\sum_{j=1}^{\infty}(-1)^{i+j}\binom{z-1}{i}\binom{i\gamma}{j} z.P(Z=z;\theta)\int_0^{\infty} u^{r+1}(1+u)^j e^{-ju}e^{-u}du.$$

After some simplifications, it takes the following form

$$\mu_r' = \frac{1}{\beta^r}\sum_{z=0}^{\infty}\sum_{i=0}^{z-1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+j}}{(j+1)^{m+r+2}}\binom{z-1}{i}\binom{i\gamma}{j}\binom{j}{m} z.P(Z=z;\theta)\int_0^{\infty} v^{m+r+2-1}e^{-v}du.$$

$$\mu_r' = \frac{1}{\beta^r}\sum_{z=0}^{\infty}\sum_{i=0}^{z-1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+j}}{(j+1)^{m+r+2}}\binom{z-1}{i}\binom{i\gamma}{j}\binom{j}{m} z.P(Z=z;\theta)\Gamma(m+r+2).$$

Based on the first four moments of the *MEPS* family, the measures of skewness (SK) and kurtosis (K) can be obtained from following relations respectively

$$SK = \frac{\mu'_3 - 3\mu'_2\,\mu'_1 + 2\mu'^3_1}{(\mu'_2 - \mu'^2_1)^{\frac{3}{2}}}, \qquad K = \frac{\mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,\mu'^2_1 - 3\mu'^4_1}{(\mu'_2 - \mu'^2_1)^2},$$

where, μ_1', μ_2', μ_3' and μ_4' can be obtained from (9), by substituting $r = 1,2,3,4.$

Also, it is easy to show that, the moment generating $M_X(t)$ function can be written as follows

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r',$$

where, μ_r' is the rth moment. Then by using (9), the moment generating function of *MEPS* can be written as follows:

$$M_X(t) = \sum_{r=0}^{\infty}\frac{1}{\beta^r}\sum_{z=0}^{\infty}\sum_{i=0}^{z-1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+j}}{(j+1)^{m+r+2}}\binom{z-1}{i}\binom{i\gamma}{j}\binom{j}{m} z.P(Z=z;\theta)\Gamma(m+r+2)\frac{t^r}{r!}, \quad r = 1,2....$$

Order statistics:-

In this subsection, an expression for the pdf of the ith order statistics from the *EMEPS* distribution is derived. In addition, the distributions of the smallest and largest order statistics are obtained.

Let $X_1, X_2, ...X_n$ be a simple random sample from a *EMEPS* family with pdf (3) and cdf (4). Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ denote the corresponding order statistics from the sample. The pdf of $X_{i:n}$, *where* $i = 1,...n$ is given by

$$f_{i:n}(x;\psi) = \frac{1}{\mathrm{B}(i,n-i+1)} f(x;\psi)[F(x;\psi)]^{i-1}[1-F(x;\psi)]^{n-i}, \tag{10}$$

where, $\mathrm{B}(.,.)$ is the beta function. By using cdf (4) and applying the binomial expansion in (10), then we get

$$f_{i:n}(x;\psi) = \frac{f(x;\psi)}{\mathrm{B}(i,n-i+1)}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\left(\frac{K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}\right)^{n+j-i}.$$

Now, since an expansion for $\left(K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}$ can be written as follows

$$\left(K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i} = \left(\sum_{z=1}^{\infty}a_z\theta^z\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^z\right)^{n+j-i},$$

$$\left(K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}=\left(a_1\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}\times$$
$$\left[1+\frac{a_2}{a_1}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)+\frac{a_3}{a_2}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^2+\ldots\right]^{n+j-i}.$$

Hence,

$$\left(K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}=a_1^{\,n+j-i}\times$$
$$\left(\sum_{m=0}^{\infty}\ell_m\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^m\right)^{n+j-i},\ \ell_m=\frac{a_{m+1}}{a_1},\ m=1,2,\ldots \tag{11}$$

According to Gradshteyn and Ryzhik (2000) for a positive integer, we have the following relation

$$\left(\sum_{m=0}^{\infty}\ell_m Y^m\right)^{n+j-i}=\sum_{m=0}^{\infty}d_{n+j-i,m}Y^m.$$

Then (11) can be written as follows

$$\left(K\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}=(a_1)^{n+j-i}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m}, \tag{12}$$

where, $d_{n+j-i,0}=1$ and the coefficients $d_{n+j-i,m}$ are easily determined from the following recurrence equation

$$d_{n+j-i,t}=t^{-1}\sum_{m=1}^{t}[m(n+j-i+1)-t]\ell_m d_{n+j-i,t-m},t\geq 1.$$

In addition,

$$K'\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)=\sum_{z=1}^{\infty}z\,a_z\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}.$$

Let $k=z-1$, then the previous equation can be expressed as

$$K'\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)=\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{k},\ \ell_k=\frac{a_{k+1}}{a_1} \tag{13}$$

Then, the pdf of the ith order statistic from *EMEPS* family of distributions is obtained by substituting expansions (12) and (13) in pdf (10) as follows

$$f_{i:n}(x;\psi)=\frac{\beta^2\gamma\theta xe^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{k}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}$$
$$\times\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j a_1^{n+j-i+1}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m}.$$

Thus, the pdf of the ith order statistics can be formed as follows

$$f_{i:n}(x;\psi)=\frac{\beta^2\gamma xe^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}}{\mathrm{B}(i,n-i+j)}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}(-1)^j\binom{i-1}{j}\ell_k(k+1)$$
$$\times\frac{d_{n+j-i,m}a_1^{n+j-i+1}\theta^{n+j-i+m+k+1}}{(K(\theta))^{n+j-i+1}}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m+k},\ \ x>0.$$

Or it can be written as follows

$$f_{i:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\tau_{j,k,m}\gamma\beta x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m+k},\quad \text{where,}$$

$$\tau_{j,k,m}=(-1)^{j}\binom{i-1}{j}\frac{\beta\ell_{k}(k+1)\theta^{n+j-i+m+k+1}a_{1}^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}.$$

Another form can be written by using binomial expansion as follows:

$$f_{i:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\tau_{j,k,m}\gamma\beta x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m+k},$$

$$f_{i:n}(x;\psi)=\beta\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\sum_{l=0}^{\infty}(-1)^{h+l}\theta^{n+j-i+m+k}\binom{n+j-i+m+k}{h}\xi_{j,k,m,h}\ \gamma x e^{-\beta x}\left((1+\beta x)e^{-\beta x}\right)^{l},$$

$$f_{i:n}(x;\psi)=\beta\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\sum_{l=0}^{\infty}\sum_{s=0}^{l}(-1)^{h+l}\theta^{n+j-i+m+k}\binom{l}{s}\binom{n+j-i+m+k}{h}\xi_{j,k,m,h}\ \beta^{s}\gamma x^{s+1}e^{-(l+\beta)x},\ (14)$$

where,

$$\xi_{j,k,m,h}=(-1)^{j}\binom{i-1}{j}\binom{m+n+j-i+k}{h}\frac{\beta^{h+1}\theta^{n+j-i+m+k+1}\ell_{k}(k+1)a_{1}^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}.$$

In particular, the pdf of the smallest and the largest order statistics of the *EMEPS* distribution is obtained by substituting $i=1,n$, in (14), respectively, as follows

$$f_{1:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\eta_{k,m,h}\beta x^{(h+1)}e^{-(n+m+k)\beta x},$$

where $\eta_{k,m,h}=\binom{m+n-1+k}{h}\dfrac{n\beta^{h+1}\ell_{k}(k+1)\theta^{n+m+k}a_{1}^{n}d_{n-1,m}}{(K(\theta))^{n}}.$

and,

$$f_{n:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\sum_{m=0}^{\infty}\sum_{h=0}^{j+m+k}\varsigma_{j,k,m,h}\beta x^{(h+1)}e^{-(j+m+k+1)\beta x},$$

where,

$$\varsigma_{k,m,h}=\binom{m+j+k}{h}\binom{n-1}{j}(-1)^{j}\frac{n\beta^{h+1}\ell_{k}(k+1)\theta^{j+m+k+1}a_{1}^{j+1}d_{j,m}}{(K(\theta))^{j+1}}.$$

Re'nyi Entropy:-

Entropy has been used in various situations in science and engineering. The entropy of a random variable X is a measure of variation of the uncertainty. If X is a random variable which distributed as *MEPS*, then the Re'nyi entropy, for $\rho>0,$ and $\rho\neq 1,$ is defined as

$$I_{R}(x)=(1-\rho)^{-1}\log_{b}\left(\int_{0}^{\infty}\left(f\ (x;\psi)\right)^{\rho}dx\right).$$

Let, $IP=\int_{0}^{\infty}\left(f\ (x;\psi)\right)^{\rho}dx$, then IP can be written as follows:

$$IP=\int_0^{\infty}\left(\beta^2\gamma\theta x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\right)^{\rho}\left\{\frac{\sum_{z=1}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}}{K(\theta)}\right\}^{\rho}dx.$$

But

$$\left(\sum_{z=1}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right)^{\rho}=a_1^{\rho}\left(\sum_{m=0}^{\infty}\delta_m\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}\right)^{\rho},\delta_m=\frac{a_{m+1}}{a_1},\ m=1,2,...$$

Using the same rule as provided by Gradshteyn and Ryzhik (2000), then we obtain

$$\left(\sum_{z=1}^{\infty}\delta_m\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}\right)^{\rho}=\sum_{m=0}^{\infty}d_{\rho,m}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}.$$

Therefore,

$$\left(\sum_{z=1}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right)^{\rho}=a_1^{\rho}\sum_{z=1}^{\infty}d_{\rho,m}\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}. \qquad (15)$$

The coefficients for $t>1$ are computed from the following recurrence equation:

$$d_{\rho,t}=t^{-1}\sum_{m=1}^{t}[m(\rho+1)-t]\delta_m d_{\rho,t-m},d_{\rho,0}=1$$

Using binomial expansion for $\left(1+\lambda x\right)^{m}$, then (15) can be written as follows:

$$\left(\sum_{z=1}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right)^{\rho}=a_1^{\rho}\sum_{z=1}^{\infty}\sum_{k=0}^{m}\binom{m}{k}d_{\rho,m}\theta^{m}e^{-m\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma k}$$

$$=a_1^{\rho}\sum_{z=1}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^{j+k}\beta^{j}\binom{m}{k}\binom{\gamma k}{j}\binom{j}{i}d_{\rho,m}\theta^{m}e^{-m\beta x}x^{j}e^{-\beta jx}$$

Then the IP can be rewritten as follows

$$IP=\left(\beta^2\gamma\theta a_1\right)^{\rho}\sum_{z=1}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^{j+k}\beta^{j}\binom{m}{k}\binom{\gamma(k+\rho)-\rho}{j}\binom{j}{i}d_{\rho,m}\theta^{m}\int_0^{\infty}e^{-(m+j+\rho)\beta x}x^{j}dx$$

After some simplification, then the Re'nyi entropy takes the following form

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\left(\beta^2\gamma\theta a_1\right)^{\rho}\sum_{z=1}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{(-1)^{j+k}\beta^{j}\Gamma(j+1)\binom{m}{k}\binom{\gamma(k+\rho)-\rho}{j}\binom{j}{i}d_{\rho,m}\theta^{m}}{\left((m+j+\rho)\beta\right)^{j+1}}\right]. \qquad (16)$$

Special Models Of The Family:-

Some sub-models from *EMEPS* family of distributions for selected values of the parameters are presented in this section. Also, some sub-models; which are the exponentiated moment exponential Poisson and exponentiated moment exponential Poisson distributions are discussed in more details.

The sub models are considered as follows:

For $K(\theta)=e^{\theta}-1$, then the *EMEPS* distribution reduces to exponentiated moment exponential Poisson *(EMEP)* distribution with the following cdf:

$$F(x;\psi)=\frac{e^{\theta}-\exp\left[\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right]}{e^{\theta}-1}, \qquad x,\theta,\beta>0.$$

For $K(\theta)=e^{\theta}-1$ & $\gamma=1$, then the *EMEPS* distribution reduces to moment exponential Poisson *(MEP)* distribution with the following cdf:

$$F(x;\psi)=\frac{e^{\theta}-\exp\left[\theta\left((1+\beta x)e^{-\beta x}\right)\right]}{e^{\theta}-1}, \qquad x,\theta,\beta>0.$$ Sadaf (2014)

For $K(\theta)=-\ln(1-\theta)$ then the *EMEPS* distribution reduces to exponentiated moment exponential logarithmic (*EMEL*) distribution with the following cdf:

$$F(x;\psi)=1-\frac{\ln\left[1-\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right]}{\ln(1-\theta)}, \qquad x,\beta>0, \quad 0<\theta<1.$$

For $K(\theta)=-\ln(1-\theta)$ & $\gamma=1$ then the *EMEPS* distribution reduces to moment exponential logarithmic (*MEL*) distribution with the following cdf:

$$F(x;\psi)=1-\frac{\ln\left[1-\theta\left((1+\beta x)e^{-\beta x}\right)\right]}{\ln(1-\theta)}, \qquad x,\beta>0, \quad 0<\theta<1.$$ Sadaf (2014)

For $K(\theta)=\theta(1-\theta)^{-1}$, then the *EMEPS* distribution reduces to exponentiated moment exponential geometric (*EMEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}}{1-\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)}, \qquad x,\beta>0, \quad 0<\theta<1.$$

For $K(\theta)=\theta(1-\theta)^{-1}$ & $\gamma=1$ then the *EMEPS* distribution reduces to moment exponential geometric (*MEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{\left(1-(1+\beta x)e^{-\beta x}\right)}{1-\theta\left((1+\beta x)e^{-\beta x}\right)}, \qquad x,\beta>0, \quad 0<\theta<1.$$

For $K(\theta)=(1-\theta)^{m}-1$, then the *EMEPS* distribution reduces to exponentiated moment exponential binomial (*EMEB*) distribution with the following cdf:

$$F(x;\psi)=\frac{(1-\theta)^{m}-\left[1-\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right]^{m}}{(1-\theta)^{m}-1}, \qquad x,\beta>0, \quad 0<\theta<1.$$

$K(\theta)=(1+\theta)^{m}-1$ & $\gamma=1$ then the *EMEPS* distribution reduces to moment exponential binomial (*MEB*) distribution with the following cdf:

$$F(x;\psi)=\frac{(1-\theta)^{m}-\left[1-\theta(1+\beta x)e^{-\beta x}\right]^{m}}{(1-\theta)^{m}-1}, \qquad x,\beta>0, \quad 0<\theta<1.$$

Exponentiated Moment Exponential Poisson Distribution:-

As mentioned above the *EMEP* distribution is obtained from *EMEPS* family distribution as a special case. The pdf of the *EMEP* distribution corresponding to (17) takes the following form

$$f(x;\psi)=\frac{\beta^2\gamma x e^{-\beta x}\theta\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\exp\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{(e^{\theta}-1)} \quad ,x,\beta,\gamma,\theta>0.$$

In addition, the reliability and hazard rate function take the following form respectively:

$$R(x;\psi)=\frac{\exp\left[\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right]-1}{e^{\theta}-1},$$

and,

$$h(x;\psi)=\frac{\beta^2\gamma\theta x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\exp\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{\left[\exp\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)-1\right]}.$$

Figure 1:- gives plots of the pdf of the *EMEP* distribution for some selected values of parameters exhibiting the behavior of density.

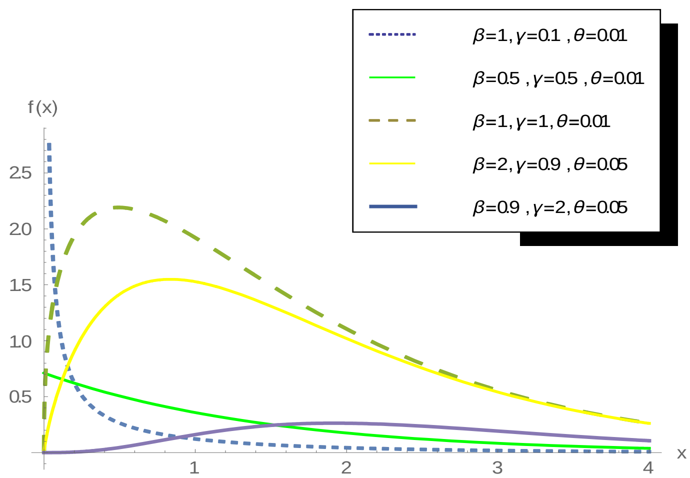

Figure 1.The pdf plots of the *EMEP* distribution

The following figure gives the hazard rate function plots for *EMEP* distribution for some selected values of parameters.

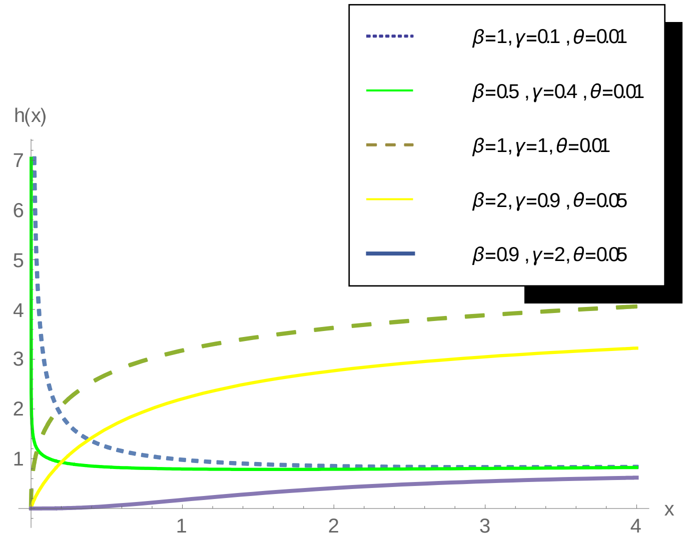

Figure 2:- The hazard rate plots for the *EMEP* distribution

It is clear from Figure 2 that the *EMEP* distribution has increasing, decreasing and constant failure rates.

The quantile function for the *EMEP* distribution is obtained directly from expression (8) with $K(\theta)=e^{\theta}-1$, and $K^{-1}(\theta)=\ln(1+\theta)$ as follows:

$$(Q(p))=-\frac{1}{\beta}-W[-\frac{\ln(p+(1-p)e^{\theta})}{\theta e^{1}}].$$

Solving this equation for $Q(p)$, the quantile function of *EMEP* is obtained.

Furthermore, the rth moment of the *EMEP* distribution about the origin is given by substituting the following pmf of truncated Poisson in (9) as follows

$$P(Z=z;\theta)=\frac{e^{-\theta}\theta^{z}}{z!(1-e^{-\theta})},\; z=1,2,...$$

$$\mu_r'=\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z}\Gamma(r+i+1)}{z!(e^{\theta}-1)z^{r+i}\lambda^{r}},$$

$$r=1,2....$$

Additionally the Re'nyi entropy is obtained by substituting $K(\theta)=e^{\theta}-1$, in (16) as follows

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m+\rho}a_1^{\rho}\Gamma(1+k+h)}{(e^{\theta}-1)^{\rho}(m+\rho)^{1+k+h}}\right].$$

Exponentiated Moment exponential geometric distribution:-

The moment exponential geometric distribution is discussed as the second special model from *EMEPS* family. The pdf of the *EMEG* distribution corresponding to (18) takes the following form

$$f(x;\psi)=\frac{\beta^{2}\gamma x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}(1-\theta)}{\left[1-\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)\right]^{2}},\qquad x>0, 0<\theta<1, \beta>0.$$

In addition, the reliability and hazard rate function take the following form:

$$R(x;\psi)=\frac{(1-\theta)\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)}{1-\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)},$$

and,

$$h(x;\psi)=\frac{\beta^{2}\gamma x e^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}}{\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\left[1-\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right]}.$$

Figures 3 and 4 represent probability density and hazard rate functions plots for *EMEG* distribution for some selected values of parameters.

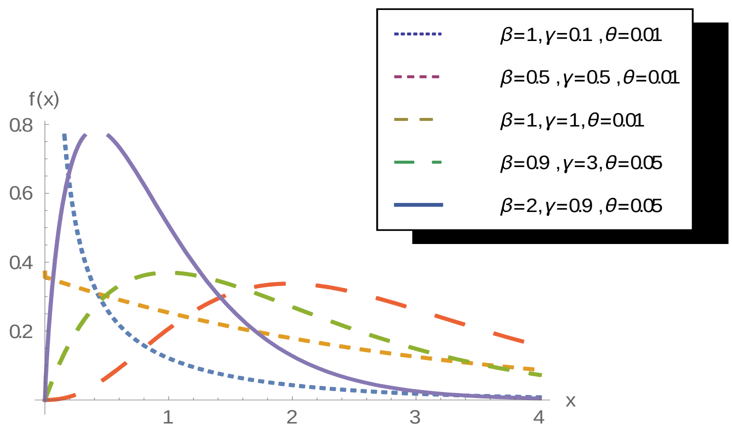

Figure. 3:- The pdf plots of the *EMEG* distribution

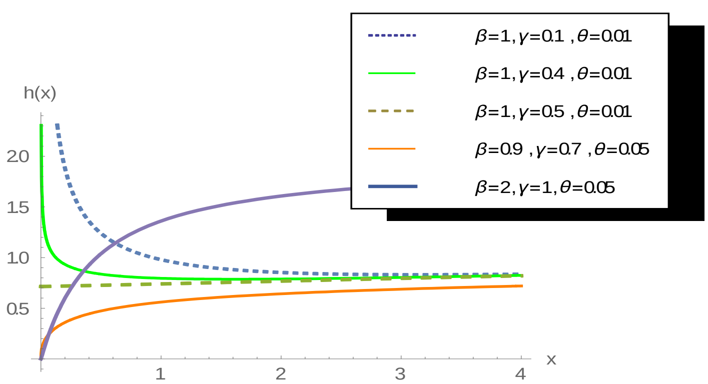

Figure. 4:- The hazard rate plots of the *EMEG* distribution

From this figure, it is observed that the shapes of the hazard rate are increasing at some selected values. For some choices of parameters; the distribution has increasing, decreasing and constant patterns of cumulative instantaneous failure.

The quantile function for the *MEG* distribution is obtained directly from expression (8) with $K(\theta)=\theta(1-\theta)^{-1}$, and $K^{-1}(\theta)=\theta(1+\theta)^{-1}$ as follows

$$(Q(p))=-\frac{1}{\lambda}-W[-\frac{(1-p)}{(1-\theta p)e^{1}}].$$

Solving this equation for $Q(p)$, the quantile function *MEG* is obtained.

Additionally, the rth moment of the *MEG* distribution about the origin is given by substituting the following pmf of truncated geometric

$P(Z=z;\theta)=(1-\theta)\theta^{z-1},\ z=1,2,...$ in (9) as follows

$$\mu_r{}'=\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z-1}(1-\theta)\Gamma\left(r+i+1\right)}{z^{r+i}\beta^{r}},\ \ r=1,2....$$

Further, the Re'nyi entropy is obtained by substituting $K(\theta)=\theta(1-\theta)^{-1}$, in (16) as follows

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m}\lambda^{\rho+h+k}a_1^{\rho}\Gamma(1+k+h)}{(1-\theta)^{-\rho}(m+\rho)^{1+k+h}}\right].$$

Parameter Estimation Of The Family:-

In this section estimation of the model parameters of *EMEPS* family of distributions is obtained by using the maximum likelihood method.

Let $X_1,X_2,...X_n$ be a simple random sample from the *EMEPS* family with set of parameters $\psi\equiv(\beta,\theta)$. The log-likelihood function based on the observed random sample of size n is given by:

$$f(x;\psi)=\beta^{2}\gamma\theta xe^{-\beta x}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\frac{K'\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)},x,\beta,\theta,>0.$$

$$L(x;\psi)=\beta^{2n}\gamma^{n}\left(\prod_{i=1}^{n}x\right)e^{-\beta\sum_{i=1}^{n}x}\prod_{i=1}^{n}\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}\frac{\prod_{i=1}^{n}K'\left(\theta\left(1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}\right)\right)}{\left(K(\theta)\right)^{n}}$$

$$\ln L(x;\psi)=2n\ln\beta+\sum_{i=1}^{n}x_i-\beta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\ln\left(K'(\theta S_i)\right)-n\ln(K(\theta)).$$

where, $\ln L=\ln L(x;\psi)$ and $S_i=1-\left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma}$.

The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by:

$$\frac{\partial\ln L}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}x_i+\theta\sum_{i=1}^{n}\frac{K''(\theta S_i)}{K'(\theta S_i)}\frac{\partial S_i}{\partial\beta},$$

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\left[\frac{K''(\theta S_i)}{K'(\theta S_i)}\right]S_i-\frac{nK'(\theta)}{K(\theta)},$$

where,

$$\frac{\partial S_i}{\partial \beta} = -\gamma\beta x^2 \left(1-(1+\beta x)e^{-\beta x}\right)^{\gamma-1}.$$

The maximum likelihood estimators of the model parameters are determined by solving the non-linear equations $\frac{\ln L}{\partial \beta} = 0, \frac{\ln L}{\partial \theta} = 0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative technique.

A Simulation Studies:-

We adopt the Monte Carlo simulation study to access the performance of the MLE's of $\Theta = (\alpha, \beta, \theta)$ through Mathematica 10.2 version. We generate different n sample observation from the quantile function in equation (20) above of the model EMEG distribution. The parameters are estimated by maximum likelihood method. We considered different sample size =20,30, 50, 100, 300 and 500 and the number of repetition is 10000. The true parameters value as α, β and θ with three different sets of values, in Tables 1 and 2, of below shows the bias and mean squared error (MSE) of the estimate parameters at different parameter values. We observed that, when we increase sample sizes "n" the bias and Mean square error for the EMEP and EMEG models given below as: (α, β, θ) decreases with respect to the best estimation.

Table 1:- The Bias and MSE on Monte Carlo simulation for parameters values of EMEP distribution.

Parameter	True value	Sample size n	Mean	Bias	MSE
α	2	$n=20$	2.2531	0.2531	1.1341
		$n=30$	2.2401	0.2401	1.0914
		$n=50$	2.2032	0.2032	0.9912
		$n=100$	2.1352	0.1352	0.9355
		$n=300$	2.0917	0.0917	0.6225
		$n=500$	2.0039	0.0039	0.4015
β	3	$n=20$	3.2641	0.2641	0.9845
		$n=30$	3.2324	0.2324	0.8434
		$n=50$	3.2131	0.2131	0.7694
		$n=100$	3.2015	0.2015	0.7215
		$n=300$	3.0636	0.0636	0.6319
		$n=500$	3.0419	0.0419	0.2726
θ	0.5	$n=20$	0.6413	0.1413	0.4636
		$n=30$	0.6321	0.132	0.4098
		$n=50$	0.6521	0.1521	0.3257
		$n=100$	0.5527	0.0527	0.1929
		$n=300$	0.5176	0.0176	0.1672
		$n=500$	0.5069	0.0069	0.0189

Table 2:- The Bias and MSE on Monte Carlo simulation for parameters values for EMEG distribution

Parameter	True value	Sample size n	Mean	Bias	MSE
α	2	$n=20$	2.2885	0.2885	0.9212
		$n=30$	2.2532	0.2532	0.8734
		$n=50$	2.2475	0. 2475	0.8578
		$n=100$	2.1238	0.1238	0.7296
		$n=300$	2.0832	0.0832	0.3657
		$n=500$	2.0105	0.0105	0.1747
β	3	$n=20$	3.3184	0.3184	1.0413
		$n=30$	3.2701	0.2701	0.9131
		$n=50$	3.2268	0.2268	0.8264
		$n=100$	3.1993	0.1993	0.7462
		$n=300$	3.1234	0.1234	0.4319
		$n=500$	2.9826	-0.0174	0.1135
θ	0.5	$n=20$	0.6821	0.1821	0.3764
		$n=30$	0.6674	0.1674	0.3426
		$n=50$	0.6521	0.1521	0.3215
		$n=100$	0.5523	0.0523	0.1269
		$n=300$	0.5176	0.0176	0.1145
		$n=500$	0.5069	0.0069	0.0285

Given first three sample moments, the corresponding $\Theta=(\alpha,\beta,\theta)$ values are estimated from the actual theoretical first three population moments derived from (The sampling distributions of estimated $\Theta=(\alpha,\beta,\theta)$ are given in Table 3 based on various sample sizes. For small samples, the percentage of estimates falling in the indicated interval increases with larger sample size. Using this range, we estimate Θ by the method of moments. If we include omitted data, we expect larger Mean Square Error (MSE). This MSE, however, decreases with increasing sample size

Table 3:- Percentage of sample estimates of $\Theta=(\alpha,\beta,\theta)$ through method of moments (MM) for the EMEP model.

N	% estimated values of parameter in indicated interval with $\alpha=2$	% estimated values of parameter in indicated interval with $\beta=3$	% estimated values of parameter in indicated interval with $\theta=0.5$
	$1.4<\hat{\alpha}<2.6$	$2.5<\hat{\beta}<3.5$	$0.3<\hat{\theta}<0.7$
30	88.68%	85.28%	80.52%
50	92.64%	91.26%	86.52%
100	97.45%	94.94%	89.71%
250	98.02%	97.62%	94.76%
500	99.64%	99.23%	96.89%

Table 4:- Percentage of sample estimates of $\Theta = (\alpha, \beta, \theta)$ through method of moments (MM) for the EMEG model.-

N	% estimated values of parameter in indicated interval with $\alpha = 2$	% estimated values of parameter in indicated interval with $\beta = 3$	% estimated values of parameter in indicated interval with $\gamma = 3$	% estimated values of parameter in indicated interval with $\theta = 0.5$
	$1.4 < \hat{\alpha} < 2.6$	$2.5 < \hat{\beta} < 3.5$	$2.5 < \hat{\gamma} < 3.5$	$0.3 < \hat{\theta} < 0.7$
30	89.67%	88.38%	84.31%	83.12%
50	95.32%	93.45%	87.62%	86.34%
100	98.67%	95.14%	93.43%	89.67%
250	99.12%	98.62%	98.24%	97.25%
500	99.89%	99.61%	99.12%	98.37%

Applications:-
In this section, the flexibility of some special models of *EMEPS* family is examined using three real data sets. We illustrate the superiority of new selected distribution as compared with some sub-models.

Based on the maximum-likelihood method, the unknown parameters of each distribution are estimated. Some selected measures as; Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*), the correct Akaike information criterion (*CAIC*) and the Kolmogorov-Smirnov (*k-s*) are obtained to compare the fitted models (as seen in Table 1). The mathematical form of these measures is as follows:

$$AIC = 2k - 2\ln L, \qquad CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$
$$BIC = k\ln(n) - 2\ln L,$$

where k is the number of models parameter, n is the sample size and $\ln L$ is the maximized value of the log-likelihood function under the fitted models.

Also, $k - s = \sup_y [F_n(y) - F(y)]$, where $F_n(y) = \frac{1}{n}$ (number of observation $\leq y$), and $F(y)$ denotes the cdf. The best distribution is the distribution corresponding to the lower values of, *AIC*, *AICC*, *BIC*, and *k-s* statistics. The results for mentioned measures for all models are reported in Tables 4 and 6.

Aircraft Windshield data Set:-
The first data set correspond the failure times of 84 for a particular model aircraft windshield. This data are reported in the book "Weibull Models" by Murthy et al.(2004, p.297)[12]. This data consist of 84 failed windshield, the unit for measurement is 1000 h. The data are : 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070,1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281,2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506,2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619,2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663.

We estimated unknown parameters of the distribution by maximum likelihood method as describe in section 5 by using the R code to find the best fit of the data. We use some measures of goodness of fit, including Kolmogorov Smirnov (K-S), For this real data set, we have fitted exponentiated moment exponential binomial(EMEB) distribution, exponentiated moment exponential geometric (EMEG) distribution, exponentiated moment exponential logarithmic (EMEL) distribution, exponentiated moment exponential Poisson (EMEP) distribution.

Table 4:- Criteria for comparison for second data set.

Model				
	$k-s$	*AIC*	*CAIC*	*BIC*
EMEB	0.7311	263.58	264.086	273.303
EMEG	0.742	267.1	267.606	276.823
EMEL	0.7211	265.68	266.186	275.403
EMEP	0.7024	261.75	262.256	271.473

Values of K- S, AIC, AICC, and BIC are listed in Tables 4 and 5. According to the criterion K- S, AIC, AICC and BIC, we found that the *EMEB, EMEG, EMEL and EMEP* distributions are all fit for the data of Aarset and for the aircraft windshield data set. The histogram of two data sets and the estimated PDFs, CDFs and P-P plots for the fitted data model are displayed in Figures (5, 6, 7, 8, 9, 10). It is clear from Tables 4 and 5 and Figures (5, 6, 7, 8, 9, 10) that the EME family of distributions fit to the histogram and therefore could be chosen as the best model for both data set.

Also the plots of the estimated densities and estimated cumulative of the fitted models are achieved in Figures 5 and 6.

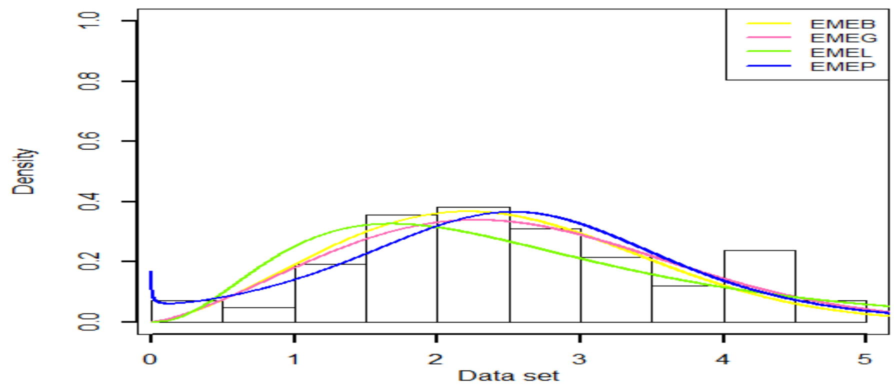

Figure 5:- Estimated densities of models for the first data set.

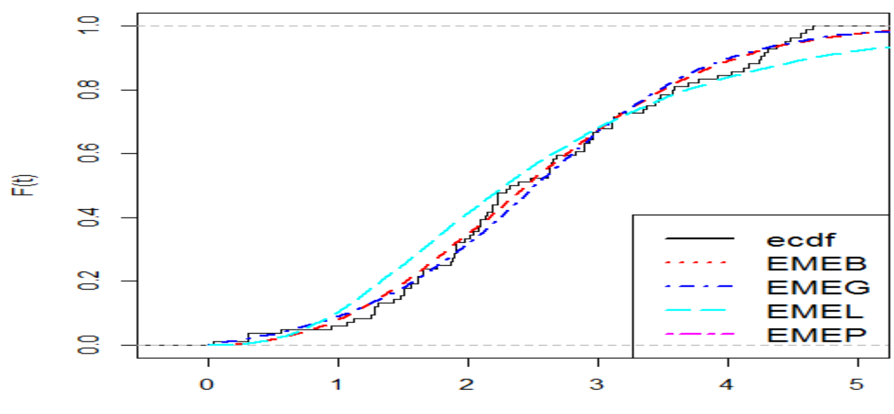

Figure 6:- Estimated cumulative densities of models for the first data set.

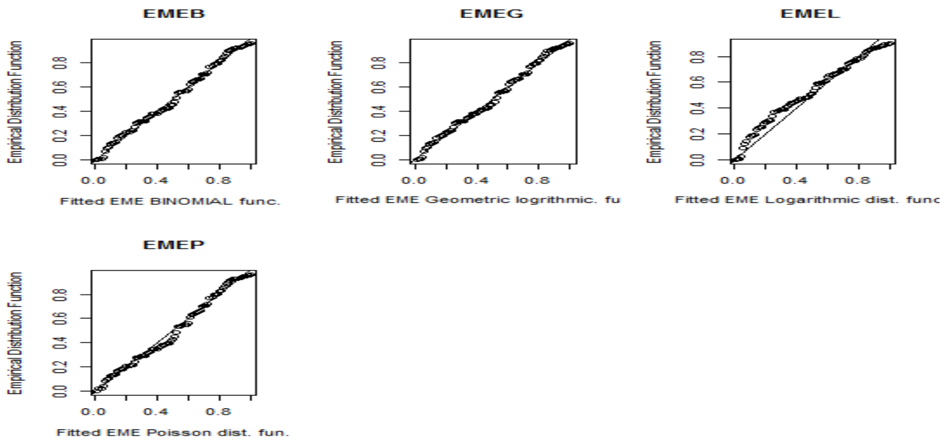

Figure 7:- The probability–probability plots for the aircraft windshield data set

2nd data set:-

The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:
0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 6:- Criteria for comparison for 2nd data set

Model				
	$k-s$	*AIC*	*CAIC*	*BIC*
EMEB	0.823	193.53	194.036	203.253
EMEG	0.832	194.06	194.566	203.783
EMEL	0.853	197.95	198.456	207.673
EMEP	0.844	196.89	197.396	206.613

For the second data set, the values of *k-s, AIC, BIC* and *CAIC* are record in Table 6

The plots of the estimated cumulative and estimated densities of the fitted models are achieved in Figures. 8 and 9 respectively.

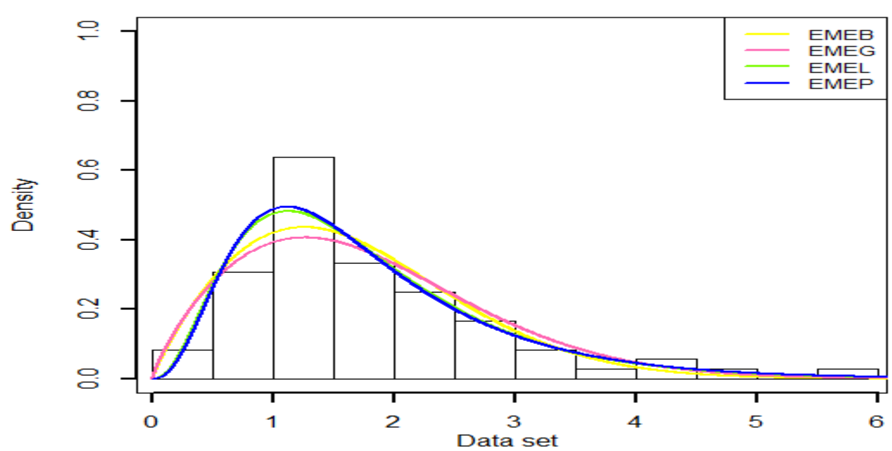

Figure 8:- Estimated densities of models for the Bjerkedal (1960) data set.

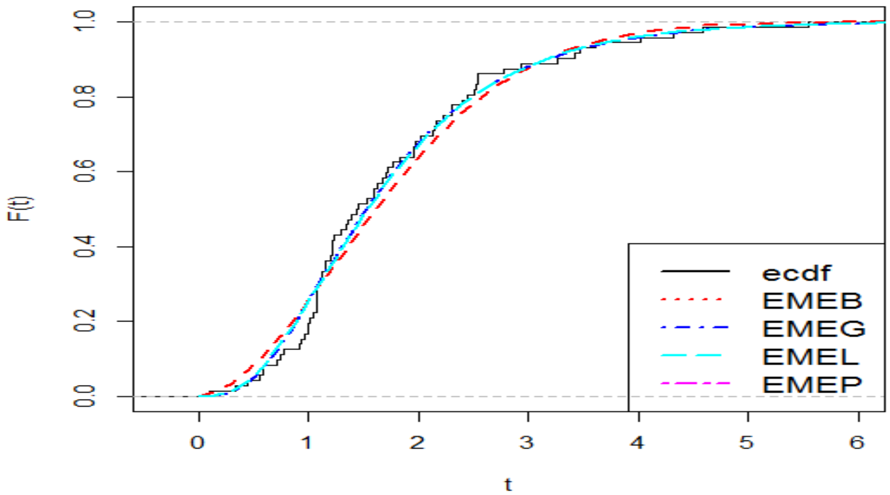

Figure. 9:- Estimated cumulative densities of models for the second data set

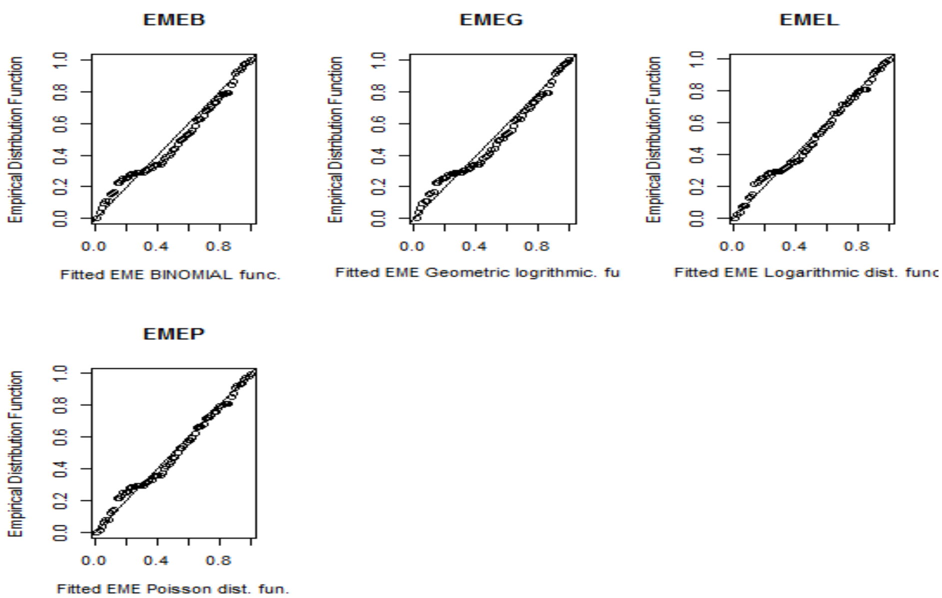

Figure 10:- The probability–probability plots for the Bjerkedal (1960) data set

It is clear from the above two figures that the family of *EMEPS distribution* has the fit on real data set.

Conclusion:-

We introduce a new class of lifetime models called the exponentiated moment exponential power series**.** This new family is obtained by compounding the exponentiated moment exponential distribution and truncated power series distributions. More specifically, the exponentiated moment exponential power series covers several new distributions. Also, mathematical properties of the new family, including expressions for density function, moments, moment generating function, quantile function, order statistics and entropy are provided. The hazard rate function has various shapes such as constant, increasing, decreasing, and bathtub. By simulation procedures it is discovered that the ML estimators are consistent since the bias and MSE approach to zero when the sample size increases. The usefulness of the model associated with this family is illustrated by two real data sets and the new model provides a better fit than the models provided in literature.

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