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RESEARCH ARTICLE

NEARLY QUASI PRIME SUBMODULES

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Abstract

Let R be a commutative ring with identity and M be a unitary of R-module. A proper submodule$N of M$ is called aquasi-prime if whenevera. b x ∈ N ; a, b ∈ R , x ∈ M, implies that either ax ∈ N or bx ∈N. In this paper we say that N is a nearly quasi prime, if whenever abx ∈ N ; a, b ∈ R, x ∈ M, implies that either ax ∈ N + $J(M)$or bx ∈ N + $J(M)$, where $J(M)$is the Jacobson radical of M. Some of the properties of this concept will be investigated. Some characterizations of nearly quasi prime submodules will be given, and we show that under some assumptions quasi prime submodules and nearly quasi prime submodules are coincide.

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Introduction:-

A submodule of an R-module M which Birkenmeier [1] was named prime submodules that they are generalized of prime ideals, which get big importance at last years, many studies and searches are published about prime submodules by many people who care with the subject of commutative algebra and some of them are C.P .Lu, P.F .Smith ,J .Dauns. The definition comes in [1] as following wesay that a proper submodule$N$ of M is called prime if whenever $r \in R$, $m \in M$, $rm \in N$ implies either $m \in N$ or $r \in [N{:}M]$ where $[N{:}M] = \{r \in R : rM \subseteq N\}$. Many generalizations of primesubmodules were studied such a nearly prime, weakly prime , quasi prime , pseudo – prime and on almost prime , see [2 ,3 , 4 ,5].In this article, we give another generalization of prime submodule it is nearly quasi prime submodule if $abm \in N$ for $a, b \in R$and$m \in M$, then either $am \in N + J(M)$ or $bm \in N + J(M)$. We give some characterizations for this concept. Also, we look for the relationships between nearly prime submodules and other well knowssubmodules. .

Basic PropretiesOf Nearly Quasi Prime Submodules:-

In this section we introduce the concept of nearly quasi prime submodule as a generalization of a prime submodule .

Definition(2. 1):-

A proper submodule$N$ of an R – module M is said to be a nearly quasi prime submodule if whenever $r_1 . r_2 x \in N$ for $r_1, r_2 \in R$and $x \in M$, then either r_1x $\in N + J(M)$ or$r_2 x \in N + J(M)$. Equivalently, a proper submodule$N$of an$R$ – module M is said to be a nearly quasi prime if and only if $[N + J(M){:}Rx]$ is a prime ideal of$R$, for each $x \in M$.

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Remarks and Examples (2.2):-

Let M be the Z – module $Z\oplus Z$and $K = 6Z\oplus<0>$. Then K is not nearly quasi prime submodule of M, since $2.3(1,0) \in K$ but $2(1,0) \notin K + J(M)$, $3(1,0) \notin K + J(M)$.

Let N, K be two submodules of an R – module $M$and $N \subseteq K$. If N is a nearly quasi prime submodule of $M$and $J(M) \subseteq J(K)$, then N is nearly quasi prime submodule of K.

Proof: Let$a, b \in R, m \in K$ such that $a.bm \in N$. Since N is nearly quasi prime submodule of$M$, so either $am \in N + J(M)$or $bm \in N + J(M)$. But$J(M) \subseteq J(K)$, so either $am \in N + J(K)$ or $bm \in N + J(K)$. Therefore N is nearly quasi prime submodule of K.

A submodule of a nearly quasi submodule need not be a nearly quasi prime submodule : For instance, $N = <\bar{3}>$ is a Z – submodule of the Z – moduleZ_{12}. It is clear that N is a nearly quasi prime of Z_{12}, but $K = <\bar{6}>$ is not a nearly quasi prime submodule of$M$ since $[K + J(M) : (\bar{1})] = [<\bar{6}> + <\bar{6}> : (\bar{1})] = 6Z$, which is not a prime ideal of Z.

It is clear that every quasi prime submodule is a nearly quasi prime, but the converse is not true in general for examples:

The submodule Z of the Z- module Q is a nearly quasi prime submodule since for each $m \in Q$, $[Z + J(Q) : (m)] = [Z + Q : (m)] = [Q : (m)] = Z$, which is a prime ideal of Z. But it is not quasi prime submodule of M by [3, remark (2.1.2,(4)].

The submodule $Z\oplus<\bar{4}>$ of the Z – module $Z \oplus Z_8$ is a nearly quasi submodule since for each $m \in Z\oplus Z_8$, $[Z\oplus<\bar{4}> + J(Z\oplus Z_8) : (m)] = [Z\oplus<\bar{4}> + J(Z) \oplus J(Z8) : (m)] = [Q : (m)] = Z$, which is a prime ideal of Z. But it is not quasi prime submodule by [3, Rem.2.1.2, (5)].

Consider the Z – module $M = Z\oplus Z_2$ and the Z –submodule$N = (0) \oplus (0)$. Then, for any $x \in M$implies $[N + J(M) : (x)] = [(0) \oplus (0) + J(Z\oplus Z_2) : (x)] = [(0) + Z_2 : (x)] = 2Z$, which is a prime ideal of Z. Therefore, N is a nearly quasi prime submodule of M.

It is clear that every primesubmodule is a nearly quasi prime submodule.

Proof: Let N be a prime submodule of an R – module M. Then N is a quasi prime submodule of M by [3,remark(1)]. Therefore N is a nearly quasi prime submodule of.

But, the converse is not true in general for example $(0) \oplus (0)$ is not prime submodule of $Z\oplus Z_2$, see [3, remark(1.1.3,(8))], but $(0) \oplus (0)$ is a nearly quasi prime submodule of $Z\oplus Z_2$, by (2.2,(5)).

An ideal I is a nearly quasi prime ideal of R if and only if I is a nearly quasi prime R – submodule of R –module .

Consider the Z – module, $= Z\oplus Z$. The submodule $N = 2Z\oplus<0>$ is a nearly quasi prime submodule of M since for any $m \in M$, $[N + J(M) : (m)] = [2Z\oplus<0> + (0)\oplus(0) : (m)] = <0>$, which is a prime ideal of Z .

1. It is clear that every maximal submodule is a nearly quasi prime submodule . But the converse is not true in general for example. Let $M = ZP\oplus Z$as Z – module, $N = ZP\oplus\{0\}$ is a nearly quasi prime submodule(since N is a prime). But N is not maximal by [19,example (2) , chapter].
2. If M is a simple R – module, then the zero submodule is a nearly quasi prime submodule of . Hence $<\bar{0}>$ is the only nearly prime submodule of Z_p(p is prime number).
3. Let$N$ and$W$ be two submodules of an R – module M such that $N \cong W$.If N is a N – prime submodule,then it is not necessary that W is a nearly Prime submodule as the following example explains this :

Consider the Z – module Z, the submodule $2Z$ is a nearly quasi prime submodule of Z and $2Z \cong 30Z$. But 30Z is not a nearly quasi prime submodule of .Since $6.5(1) \in 30Z$,but $5(1) = 5 \notin 30Z + J(Z) = 30Z + 0 = 30Z$ and $6(1) = 6 \notin 30Z + J(Z) = 30Z + 0 = 30Z$.

Consider the Z – module $M = Z\oplus Z$ .The submodule $N = 8Z\oplus 9Z$ is not nearly quasi prime since $2.4(1,9) = (8,72) \in 8Z\oplus 9Z$, $2(1,9) \notin 8Z\oplus 9Z + J(Z\oplus Z) = 8Z\oplus 9Z$ and $4.(1,9) \notin 8Z\oplus 9Z + J(Z\oplus Z) = 8Z\oplus 9Z$.

The following theorem gives some characterizations for nearly quasi prime submodule

Theorem (2.3):-
Let N be a proper submodule of an R – module. Then the following are equivalent:
1. N is a nearly quasi prime submodule of M .
2. $[N + J(M): K]$ is a prime ideal of R, for each submodule K of M .
3. $[N + J(M): (rx)] = [N + J(M): (x)]$, for each $x \in M, r \in R$ and $r \notin [N + J(M): (x)]$.

Proof:
(1)→ (2): Let N be a nearly quasi prime submodule of M .Then $[N + J(M):_R (x)]$ is a prime ideal of R for each $x \in M$ .So $[N + J(M):_R (x)]$ is a prime ideal of R for each x ∈K. And by [3,lemma(1.2.5)], $[N + J(M): K]$ is a prime ideal of R.

(2) → (3) It is clear that $[N + J(M): (x)] \subseteq [N + J(M): (rx)]$. Let $m \in [N + J(M): (rx)]$ for $r \notin [N + J(M): (x)]$ and $x \in M$. Hence, $m(rx) \subseteq N + J(M)$ .It follows that $mr \in [N + J(M): (x)]$, which is a prime ideal by (2).But $r \notin [N + J(M): (x)]$ so $m \in [N + J(M): (x)]$. Thus, $[N + J(M): (rx)] \subseteq [N + J(M): (x)]$. Therefore, $[N + J(M): (rx)] = [N + J(M): (x)]$.

(3) → (1): Let $x \in M$ and $a, b \in R$ such that $ab \in [N + J(M): (x)]$, suppose $b \notin [N + J(M): (x)]$, hence by (3), $[N + J(M): (bx)] = [N + J(M): (x)]$.But $a \in [N + J(M): (bx)]$,so $a \in [N + J(M): (x)]$ and hence N is a nearly quasi prime submodule of M.

The following is an immediate consequence of (.2.3.).
Corollary (2.4):-
Let N be a submodule of an R – module M. If N is a nearly quasi prime submodule of M, then $[N + J(M): M]$ is a prime ideal of R.

The converse of corollary (2.4) is not true in general for example: Let $M = Z\oplus Z$ as a Z – module and $N = 6Z\oplus <0>$, then $[6Z\oplus <0> + J(Z\oplus Z) : Z\oplus Z] = (0)$ is a prime ideal of Z. But N is not a nearly quasi prime submodule of M see (Rem.2.2,(1)).

Corollary (2.5):-
Let N be a submodule of an R – module M .If N is a nearly quasi prime submodule of M, then $[N + J(M): rM] = [N + J(M): M]$, $\forall r \notin [N + J(M): M]$.

Proof:-
Let $a \in [N + J(M): rM]$ so $arM \subseteq N + J(M)$ which means that $ar \in [N + J(M): M]$.But N is a nearly quasi prime submodule of M, so by corollary (2.4), $[N + J(M): M]$ is a prime ideal of R. Hence either $a \in [N + J(M): M]$ or $r \in [N + J(M): M]$, but $r \notin [N + J(M): M]$. So $a \in [N + J(M): M]$, thus $[N + J(M): rM] \subseteq [N + J(M): M]$. But it is clear that $[N + J(M): M] \subseteq [N + J(M): rM]$, so we obtain the result.

The converse of corollary (2.5) is not true for instance: Let $M = Z$ as a Z – module, let $N = 8Z, r = 5, 5 \notin [8Z + J(Z): Z] = [8Z + (0) : Z] = 8Z$ and $[8Z + J(Z) : 5Z] = 8Z$ so $[N + J(M) : rM] = [N + J(M) : M] = 8Z$. But $N = 8Z$ is not nearly quasi prime submodule of Z.

Let R be any ring. A subset S of R is called multiplicatively closed if $1 \in S$ and $ab \in S$ for every $a, b \in S$ .We know that every proper ideal P in R is prime if and only if $R - P$ is a multiplicatively closed subset of R , [7, p.42] . And if N a submodule of an R – module M and S be a multiplicatively closed subset of R, then $N(S) = \{x \in M : \exists t \in S$, such that $tx \in N\}$ be a submodule of M and $N \subseteq N(S)$.

However, the following proposition gives a partial converse of corollary (2.4).

Proposition(2.6):-
Let N be a proper submodule of an R – module M. If $P = [N + J(M): M]$ is a prime ideal of R and$N(S) = N + J(M)$ (where $S = R–P$), then N is a nearly quasi prime submodule of M .

Proof:-
Since P is a prime ideal, so R – P is a multiplicatively closed subset, see[7,p.42]. Let $a, b \in R, m \in M$ such that $abm \in N$ .Suppose that $am \notin N + J(M)$so $a \notin [N + J(M): R(m)]$, hence$a \notin [N + J(M): RM] = P$, then $a \in S$implies $\in N(S) = N + J(M)$. So $bm \in N + J(M)$ which mean that N is a nearly quasi prime submodule of$M$.

Proposition(2.7):-
Let K and N be two submodules of an R – module M. Then N is a nearly quasi prime submodule of M if and only if $abK \subseteq N$ implies either $aK \subseteq N + J(M)$or $bK \subseteq N + J(M)$.

Proof:-
Let $abK \subseteq N$ and suppose $aK \nsubseteq N + J(M), bK \nsubseteq N + J(M)$. So there exists m_1 ,$m_2 \in K$ such that $am_1 \notin N + J(M)$ and $bm_2 \notin N + J(M)$. Since N is nearly quasi prime and $abm_1 \in N$and $am_1 \notin N + J(M)$,so $bm_1 \in N + J(M)$. Also $abm_2 \in N$ and$bm_2 \notin N + J(M)$so $m_2 \in N + J(M)$.

The converse is clear.
The intersection of any collection of nearly quasi prime submodules of an R – module not necessarily nearly quasi prime submodules, $N_1 = <\bar{2}>$ and $N_2 = <\bar{3}>$ are nearly quasi prime submodules of Z_{12} as a Z - module.But $N_1 \cap N_2 = <\bar{6}>$ is not nearly quasi prime submodule of Z_{12} , since $2.3.\bar{1} = \bar{6} \in N_1 \cap N_2 = <\bar{6}>$ but $2.\bar{1} \notin <\bar{6}> + J(Z_{12}) = <\bar{6}>$ and $3.\bar{1} \notin <\bar{6}> + J(Z_{12}) = <\bar{6}>$.

Recall that a ring R is said to be a good ring if $J(M) = J(R)M$ for each R – module M, equivalently, R is a good ring if and only if $J(N) = J(M) \cap N$ for each submodule N of an R – module M, [8, p.236] . And a ring R is called a regular ring if each of its elements is regular, where an element $a \in R$ is said to be regular if $\exists x \in R$ such that axa = a, [8,p.184]. It is know that if $R / J(R)$ is a regular ring, then$R$is a good ring.

Hence ,we have the following Result:-
Proposition(2.8):-
Let R be a good ring. If N is a nearly quasi prime submodule of an R- module M and K be a submodule of M such that $J(M) \subseteq K$ and $K$is not contained in N, then $K \cap N$ is a nearly quasi prime submodule of K.

Proof:-
Since $K \nsubseteq N$, then $K \cap N$is a proper submodule of K . Let $r_1, r_2 \in R$ and $x \in K$ such that $r_1r_2x \in K \cap N$,so $r_1r_2x \in N$. But N is a nearly quasi prime submodule of M so either$r_1x \in N + J(M)$ or $r_2x \in N + J(M)$.Since $x \in K$ implies that either $r_1x \in (N + J(M)) \cap K$ or $r_2x \in (N + J(M)) \cap K$. Since $J(M) \subseteq K$ implies that either $r_1x \in (N \cap K) + (J(M) \cap K)$ or $r_2x \in (N \cap K) + (J(M) \cap K)$ by [8]. But R is a good ring, so either$r_1x \in (N \cap K) + J(K)$ or $r_2x \in (N \cap K) + J(K)$ by [24].Therefore $K \cap N$ is a nearly quasi prime submodule of$K$.

Corollary(2.9):-
Let R be a good ring. If N is a nearly quasi prime submodule of an R- module M and K is a maximal submodule of M is not contained in N, then $N \cap K$ is a nearly quasi prime submodule of$K$.

Proof:-
Since K is a maximal submoduleof$M$, so $J(M) \subseteq K$. Hence the result follows by proposition(2.8).

Corollary(2.10):-
Let $R / J(R)$ is a regular ring. If N is a nearly quasi prime submodule of M and K is a maximal submodule of M is not contained in N, then $N \cap K$is a nearly quasi prime submodule of K.

Proposition (2.11):-
LetN, K be two nearly quasi prime submodules of an R – module M. If $J(M) \subseteq K$or $J(M) \subseteq N$, then$N \cap K$ is a nearly quasi prime submodule of M.

Proof:-
Let $r_1, r_2 \in R$ and $m \in M$ such that $r_1 r_2 m \in N \cap K$, so $r_1 r_2 m \in N$ and $r_1 r_2 m \in K$. But N, K are nearly quasi prime submodulesof$M$, so either $r_1 m \in N + J(M)$ or $r_2 m \in N + J(M)$ and either $r_1 m \in K + J(M)$ or either$r_2 m \in K + J(M)$. Thus either $r_1 m \in (N + J(M)) \cap (K + J(M))$or either$r_2 m \in (N + J(M)) \cap (K + J(M))$. If$(M) \subseteq K$, then either $r_1 m \in (N + J(M)) \cap K = (N \cap K) + J(M)$ by [8].If $J(M) \subseteq N$, then$r_2 m \in (K + J(M)) \cap N = (N \cap K) + J(M)$by [8]. Therefore $K \cap N$ is a nearly quasi prime submodule of M.

Proposition (2.12):-
Let M and M$'$ are R- modules and f: M $\rightarrow$ M$'$ is an epimorphism such that $Kerf \ll M$. IfN is a nearly quasi submodule prime ofM$'$, then $f^{-1}(N)$ is also a nearly quasi prime submodule of M .

Proof:-
Let $r_1, r_2 \in R$ and $m \in M$ such that $r_1 r_2 m \in f^{-1}(N)$so $r_1 r_2 f(m) \in N$. But N is a nearly quasi prime submodule of M so either $r_1 f(m) \in N + J(M')$or $r_2 f(m) \in N + J(M')$.Thus, either $r_1 m \in f^{-1}(N + J(M'))$ or $r_2 m \in f^{-1}(N + J(M'))$ and this implies that either $r_1 m \in f^{-1}(N) + f^{-1}(J(M'))$or $r_2 m \in f^{-1}(N) + f^{-1}(J(M'))$. But $Kerf \ll M$, so either $r_1 m \in f^{-1}(N) + J(M)$ or $r_2 m \in f^{-1}(N) + J(M)]$,[8]. Therefore $f^{-1}(N)$ is a nearly quasi prime submoduleof M.

Proposition (2.13):-
Let ϕ: M $\longrightarrow$ M$'$ be an R- epimorphism. If N is a nearly quasi prime submodule of an R-module M containing $Ker\ \phi$ and$ker\ \phi << M$, then ϕ(N) is a nearly quasi prime submodule of M$'$.

Proof:-
$\phi(N)$is a proper submodule of M' . If not suppose $\phi(N) = M'$, let $m \in M$ such that $\phi(m) \in M' = \phi(N)$, $\exists\, n \in N$ such that $\emptyset(n) = \emptyset(m)$ hence$\emptyset(n - m) = 0$, then $n - m \in ker\emptyset \subseteq N$, then $m \in N$, hence $N = M$ (contradiction), since $N \subsetneq M$. Let $r_1,\ r_2 \in R$, $m' \in M'$ such that $r_1 r_2\ m' \in \emptyset(N)$, we want to show that either $r_1\ m' \in \phi(N) + J(M')$ or $r_2\ m' \in \phi(N) + J(M')$.Since ϕ is an epimorphism and $m' \in M'$, then there exists $n \in N$ such that $\phi(n) = r_1 r_2\ m'$. But $m' = \emptyset(m)$ for some $m \in M$, so $r_1 r_2\ \phi(m) = \phi(n)$ which implies that $r_1 r_2\ m - n \in Ker\ \phi$. But $Ker\ \phi \subseteq$ N so that $r_1 r_2 m - n = n_1$for $n_1 \in N$. Hence, $r_1 r_2 m \in N$. But N is a nearly quasi prime submodulein M, so either $r_1\ m \in N + J(M)$ or $r_2\ m \in N + J(M)$.Thus, either $r_1\ m' \in \phi(N) + J(M')$ or$r_1\ m' \in \phi(N) + J(M')$.Since, $r_2\ \phi(m) \in \phi(N) + \phi(J(M))$.But Ker$\emptyset$ << M and $\emptyset$ is an epimorphism, so either $r_1\ m' \in \phi(N) + J(M')$or $r_2\ m' \in \phi(N) + J(M')$, [8]. Therefore, $\phi\ (N)$ is nearly quasi prime submodule of M'.

Recall that a submodule$K$ of an R – module is called small in M,if every submodule$L$ of M with $K + L = M$ implies $L = M$ notationally $K << M$ [8,p.106].

An R – epimorphism$\emptyset$: M $\rightarrow$ M$'$ is called small epimorphism if Ker $\emptyset \ll M$, [8].
By using these concept ,we have the following:

Let ϕ: M $\longrightarrow$ M$'$ is small epimorphism. If N is a nearly quasi primesubmodule of an R-module Mcontaining$Ker\emptyset$, then ϕ(N) is a nearly quasi prime submodule of M$'$.

Recall that an R-module M is called hollow module if and only i f every submodule in M is small [9].

Corollary (2.15):-
Let M be a hollow R – module and ϕ: M $\longrightarrow$ M$'$ be an epimorphism . If N is a nearly quasi prime submodule of an R-module Mcontaining$Ker\emptyset$, then ϕ(N) is a nearly quasi prime submodule of M$'$.

Recall that an R-module M is called local if M has unique maximal submodule, [10].

Corollary (2.16):-
Let M is a local R – module and ϕ: M $\longrightarrow$ M$'$ be an epimorphism . If N is a nearly quasi primesubmodule of an R-module Mcontaining $Ker\emptyset$, then ϕ(N) is a N - prime submodule of M$'$.

Proof:-
Since M is local R- module, so M is hollow R- module by[11 ,Th.2.6], then the result follows from corollary(2.15).

Corollary (2.17):-
Let N be a submodule of an R- module M and K be a small submodule of M contained in N. Then N / K is a nearly quasi prime submodule of M / Kif N is nearly quasi prime submodule of$M$.

Proof:-
Let $\pi: M \rightarrow M / K$ be the natural epimorphism, then the result follows from proposition (2.13).

The Relation Between Nearly Quasi Prime Submodules And Other Submodules:-
We study in this section the relationships between nearly quasi prime submodules and other submodules such as quasi prime submodules, N –semiprime, prime submodules.

As we have mentioned in section one, that quasi prime submoduleis nearly quasi prime submodule and the converse need not be true in general.

In the following proposition, we give a condition under which the converse is true.

Proposition (3.1):-
IfN is a nearly quasi prime submodule of an R – module M and$J(M) \subseteq N$,then N is quasi prime submodule of M .
Proof:-
It is clear.

Recall that an R – module M is called fully semiprime if for each proper submodule is semiprime.And M is called an almost fully semiprime if each nonzero proper submodule is semiprime,[12] .

Proposition (3.2):-
Let M is an almost fully semiprime R-module which is not fully semiprime.IfN is a nearly quasi prime submodule , then N is a quasi prime submodule of M .

Proof:-
Let M is an almost fully semiprime module and not fully semiprime. Then by [12, lemma (2.10), p.312], $J(M) \subseteq N$. Hence N is a quasi prime submoduleof M by proposition (3.1).

Recall that an R – module M is called co – semisimple if each proper submodule of M is an intersection of maximal submodules,[12].

Proposition (3.3):-
Let M is an almost fully semiprime R- module which is not co-semisimple. If $N$is nearly quasi primesubmodule of M, then N is a quasi prime submodule of M .

Proof:-
Let M is an almost fully semiprime R- module and not co- semisimple. Then by [12, lemma (2.10), p.312], $J(M) \subseteq N$.Hence N is a quasi prime submoduleof M by proposition (3.1).

Proposition (3.4):-
IfN is nearly quasi prime submodule of an R – module M and $(M) = 0$,then N is quasi prime submoduleof M .
Proof:-
It is clear.

Recall that a ring R is called a V – ring if every simple R – module is injective,[13].

Corollary (3.5):-
If N is nearly quasi prime submodule of an R – module M and R is a V – ring,then N is quasi prime submoduleof M.

Proof:-
Since R is a V – ring, so $J(M) = 0$by [13, Theorem.(VILLAMAYOR) ,p.236]. Hence the result follows by proposition (3.4).

Next, an R – module M is said to be F – regular if each submodule of M is pure, [13].
By using this concept, we have the following:

Corollary (3.6):-
If N is a nearly quasi primesubmodule of F - regular R – module M, then N is a quasi prime submodule of M .

Proof:-
Since M is a F – regular R- module, then $J(M) = 0$, [15].Hence the result follows by proposition(3.4) .

Corollary (3.7):-
If N is nearly quasi prime submodule of an R – module M and $R / ann(x)$ is a regular ring for every $0 \neq x \in M$,then N is quasi prime submodule of M.

Proof:-
Since $R / ann(x)$is a regular ring for every$0 \neq x \in M$, then M is a F – regular R- module by [16 ,Theorem. (2.2) . p. 196].Hence the result follows by Corollary (3.6).

Recall that an R – module M is called Z – regular if $\forall\, m \in M, \exists\, f \in M^* = Hom(M,R)$ such that $m = f(m).m$, [15].

Corollary (3.8):-
If N is a nearly quasi prime submodule of Z - regular R – module M,then N is a quasi primesubmodule of M .

Proof:-
Since M is a Z – regular R – module,then$M$ is a F – regular R- module by [17, proposition(2.3) . p. 158].Hence the result follows by Corollary (3.6) .

Now, because of the fact that if M is semi – simple$R$ – module,so $J(M) = 0$ by [8,theorem 9.2.1,p.218],then the following is a consequence of proposition (3.4) ,where an R – module M is called semi – simple if and if only every submodule of M is a direct summand of M,[8].

Corollary (3.9):-
If N is a nearly quasi prime submodule of a semi – simple R – module M,then N is a quasi prime submodule of M .
Now, because of the fact that if R is semisimple then every right and left R – module is semi – simple by [8, corollary (8.2.2), p.196], then the following is a consequence of corollary (3.9).

Corollary (3.10):-
If N is nearly quasi prime submodule of an R – module M and R is a semi – simple ring,then Nisquasi prime submoduleof$M$.

Now, because of the fact that if $M$is a pseudo regular R – module,so $J(M) = 0$ by [17,propostion11,p.4] ,then the following is a consequence of proposition (3.4), where an R – module M is called pseudo regular if and if only every finitely generated submodule of M is a direct submodule,[17].

Corollary (3.11):-
If N is a nearly quasi prime submodule of a pseudo regular R – module M,then $N$is quasi prime submoduleof M.

In the following result, we give another condition for which a nearly quasi Prime submodulebe a quasi-primesubmodule

Corollary (3.12):-
If N is a nearly quasi prime submodule of an R – module M and $J(N) = J(M) \cap N$ for each N submodule of M , then N is quasi prime submodule of M.

Proof:-
Since $J(N) = J(M) \cap N$, so $J(M) = 0$ by [25, proposition(33- 1),p.22]. Hence the result follows by proposition (3.4).

Now, because of the fact R is a good ring if and only if $J(N) = J(M) \cap N$ for each submodule N of an R – module M, then the following is a consequence of proposition (3.4).

As anther consequence of proposition (3.4), we have the following result:

Corollary (3.13):-
If N is a nearly quasi prime submodule of an R – module M and R is a good ring, then N is quasi prime submodule of .

Proof:-
Since R is a good ring,then $J(N) = J(M) \cap N$ for each submodule N of M by [8] .Therefore, $J(M) = 0$ by [25] and hence N is a prime submodule of M .

Now, we can give the following:-
Corollary (3.14):-
If N is a nearly quasi prime submodule of an R – module M and$R / J(R)$ is a regular ring,then N is quasi prime submodule of M.

Proof:-
Since $(M) = J(R) M$, then R is a good ring.Hence the result follows by corollary (3.13).

Recall that an R – module M is called divisible if and only if$rM = M , \forall 0 \neq r \in R$,[8].

By using this concept, we have the following:

Proposition (3.15):-
Let R is PID and M is a divisible R – module such that$J(M) \neq M$. If N is a nearly quasi prime submodule, then N is a quasi prime submodule of M.

Proof:-
Since M is a divisible R – module and$J(M) \neq M$, so $J(M) = 0$ by [25, prop.(1-4) ,p.12]. Hence the result follows immediately from proposition (3.4).

Corollary (3.16):-
Let M be an injective R- module and $J(M) \neq M$ .If N is a nearly quasi primesubmodule,then N is a quasi-primesubmoduleof M.

Proof:-
Since M is an injective R – module,so$M$ is a divisible R – module by [8]. But $J(M) \neq M$,so $J(M) = 0$ by [25 ,prop.(1- 4) ,p.12].Hence the result follows immediately from proposition (3.4).

Now, we can give anther consequences of proposition (3.4). But first we need the following definition: Let M and N be two modules.$M$is said to essentially pseudo – N – injective if for any essentially submodule$A$ of N , any monomorphism $f : A \rightarrow M$ can be extended to some $g \in Hom(N, M)$.$M$is called essentially pseudo – injective if M is essentially pseudo – M – injective,[18].

Corollary (3.17):-
Let M be an essential pseudo – A – injective for any cyclic module A and$J(M) \neq M$.If N is a nearly quasi- prime submodule,then N is a quasi-primesubmoduleof$M$.

Proof:-
Since M be a essential pseudo – A – injective for any cyclic module A, so M is an injective by [18 ,corollary 1, p.4] and so M is a divisible R – module and $J(M) \neq M$,so $J(M) = 0$ by [25 ,prop.(1-4) ,p.12].Hence the result follows immediately from proposition (3.4).

Now, we can give anther consequences of proposition (3.4). But first we need the following definition: Recall that an R – module M is direct injective, if given any direct summand $A$of M, an injection: $i : A \to M$ and every R – monomorphism $f: A \to M$,there is an R – endomorphism $g\ of$ M such that $gof = iA$ [19].

Corollary (3.18):-
Let M be a direct injective R – moduleand $J(M) \neq M$. If N is nearly a quasi-primesubmodule, then N is a quasi-primesubmoduleof M.

Proof:-
Since M be a direct injective R – module, so M is a divisible R -module by [25].Hence the result follows immediately from proposition (3.4).

Recall that an R – module M is called ic – pseudo – injective, if it is ic – pseudo – M – injective. Where an R – module M is said to be ic – (pseudo)- N – injective,if for each ic – submodule$A$ of N , every R – homomorphism (R – monomorphism) from A to M can be extended to an R – homomorphism from N into M.And asubmodule$N$ of M is called ic – submodule,if N is isomorphic to a closed submodule of M,[21] .

Corollary (3.19):-
Let M is an ic–pseudo - injective R – module and $J(M) \neq M$ .If N is nearly quasi prime submodule,then N is quasi primesubmoduleof M.

Proof:-
Since M is anic –pseudo – injective R- module, so M is a divisible$R$ – module by [21, proposition (2.11), p.259]. Hence the result follows immediately from proposition (3.4).

Now, we study the relation between N– prime submodules and N – semi prime submodules .But first we need the following definition:

Recall that a submodule$N$of an R – module M is said to be N – semi prime, if whenever$r^n.x \in N, r \in R, x \in M, n \in Z+$, implies $rx \in N + J(M)$,[22].

Remark (3.20):-
Every nearly quasi prime submodule is N – semi prime submodule .

Proof:-
Suppose N is nearly quasi prime submodule of an R – module M. If $r^2x \in N$ for $r \in R$and $x \in M$, then by definition of nearly quasi prime submodule implies that $r\,x \in N + J(M)$. Hence, N is N – semi prime submodule of M.

The converse is not true for example: Let $M = Z$ as a Z – module and $N = 6Z$. $6Z$is nearly semi prime submodule of Z(since N is a semi prime). But $6Z$ is not nearly quasi prime submodule of Z, since $2.3.1 \in 6Z$. But $2.1 = 2 \notin 6Z + J(Z) = 6Z$and $3.1 = 3 \notin 6Z + J(Z) = 6Z$.

In the following proposition, we give a condition under which the two concepts are equivalent. But first we need the following definition:

Recall that a submodule$N$ of an R – module M is called irreducible if for each submodules L_1 , $L_2$of M such that$L_1 \cap L_2 = N$, then either $L_1 = N$or $L_2 = N$, [23].

Proposition (3.21):-
Let N is an irreducible submodule of an R – module $M$and $(M) \subseteq N$. If N is a N – Semi prime, then N is a nearly quasi prime submodule of M

Proof:-
Since $J\,(M) \subseteq N$ and N is N- semi prime submodule, then N is a semi- prime submodule by [22]. But N is an irreducible submodule of M, so by [26] N is a prime submodule of M and hence N is quasi prime submodule of M Therefore N is nearly quasi prime submoduleof M.

In the following proposition, we give other conditions under which the two concept are equivalent. But first we need the following definitions:

A non – zero R – module M is called secondary module provided that for every element$r \in R$, the endomorphism $m \rightarrow rm$ is either surjective or nilpotent, [4].

Recall that a submodule$N$ of an R – module M is said to be N – prime, if whenever .$x \in N, r \in R, x \in M$, implies that either $x \in N + J(M)$ or $r \in [N + J(M) : M]$,[24].

Proposition (3.22):-
Let N be a submodule of a secondary R – module $M$and $(M) \subseteq N$. If N is a N – semi prime, then N is a nearly quasi prime submodule of .

Proof:-
Let $r \in R$, $m \in M$ such that$rm \in N$. Since M is a secondary module,then either $rM = M$or$r^n M = 0$, then$r^n M \subseteq N$, but N is a N –semi prime submodule of ,hence $rM \subseteq N + J(M)$. If $r\,M = M$, then $m = ry$ for some $\in M$. Thus, $r^2 y = rm \in N$ and hence $y \in N + J(M)$ .This implies that $m \in N + J(M)$ and therefore N is a N – prime submoduleof M. But $J(M) \subseteq N$ so N is a prime submodule of M by [24] and hence N is a nearly quasi prime.

As we have mentioned in section one that prime submodule is nearly quasi prime submodule and the converse need not be true in general.

In the following proposition, we give a condition under which the converse is true.

Proposition (3.23):-
Let N is an irreducible submodule of an R – module M and $J(M) \subseteq N$. If N is a nearly quasi prime submodule of M, then N is a prime submodule of M.

Proof:-
Since $J\,(M) \subseteq N$ and N is a nearly prime submodule, then N is a quasi prime submodule and so N is a semiprimesubmodule of M by [3, remark (2.1.2, 7)]. Hence N is a prime submodule of M by [26, prop.1.10, Ch.2].

References:-

1. Birkenmeier C.F.1976 .On The Cancellation of Quasi –Injective Modules .*Commutative InAlgebra*, 49(2), pp: 101-109.
2. Behboodi, M. Kooly. H.2004.Weakly Prime Modules. Vietnam J Mathematics, 32(3), pp: 303-317.
3. Abdul- Razak H.M., 1999. Quasi-prime Modules and Quasi-prime Submodules. M.D. Thesis, University of Baghdad.
4. KursatHakan Oral.2010.Pseudo Prime and Pseudo IrrducibleSubmodule.World Applied Sciences Journal, 9(12),pp :1350-1352.
5. Hani A. Khashan. 2012 . On Almost Prime Submodule. Acta Mathematica Scientia , 32B (2),pp :645-651.
6. C.P.Lu , .1999 . The ZariskiTopologu on the prime Spectum of aModule, HoustomJ.Math. 25(3) , 417-432.

7. Larsen M.D. and Mccarlthy P.J.1971. Multiplicative Theory of Ideals ,Academic Press, New York and London
8. Kasch, F.1982. Modules and Rings , Academic Press Ins, London .
9. Wisbauer, R 1991.Foundations of Modules and Rings Theory, Gonrdan and Berach Reading.
10. Naoum, A.G. 1995. Regular MultiplictionModules.Periodica Mathematics Hungarica Vol. 31(2),pp.155-162.
11. MajidMohammed ,AbdGhafur Bin Ahmad , .2013. On some properties of hollow and hollow dmension modules. Pure and Applied MathenaticsJournrl. 2(15), 156-161.
12. Behboodi, M. Karamzadeh, O.A.S, and Koohy,H.2004.Modules Whose Certain Submodules Are Prime, Vietnam Journal of Mathematics ,32:2,303-317.
13. Byrd, K.A.,1972. Rings Whose Quasi- Injective Modules Are Injective, Proceedings of the American Mathematical Society ,Volume33,Number 2,June
14. Fieldhouse, D.J., 1969.Pure Theories. Math. Ann. 184,1-18.
15. Zelmanowitz, J.,1973. Regular Modules .Trans.Amer.Math.Soc.,163,341-355.
16. Naoum, A.G. and Yaseen, S.M., 1995. The Regular Submodules of a Module.AnnalesSocietatis Mathematics Polonas.
17. Zhu Zhanmin ,2010. Pseudo FQ- injective Modules. Dep. Mathematics ,Jiaxing University.
18. Al- Ahamadi[1],N.Er[2],S.K.Jain[3], 2003. Modules Which are Invariant under Monomorphisms of their Injective Hulls .Ohio University, Athens,Ohio.
19. W. K. Nicholson , 1975 . Semirgular Modules and rings ,Canad . J. Math. 180 , 1105-1120.
20. C. W. Han , S.J Chio, 2009. Generaliation of the quasi – inlectivemodules ,comm. Korean , 811-813. Math.soc. Vol.10. NO.1
21. Mehdi S.A. and Samir M.S.,2012.IC-Pseudo- Injective Modules. International Journal of Algebra ,Vol.6,no. 6,255-264.
22. Mohammed B.H, 2016. Nearly SemiprimeSubmodules .M.Sc. Thesis ,College of Science ,University of Baghdad.
23. Dauns , J. ,1980. Prime Modules and One- Sided Ideals in "Ring Theory and Algebra III", (Proceeding of the Third Oklahoma Conference),B.R.
24. Quasi-Injective Modules, Comm .Korean Math. Soc. Vol.10.No.1,811-813.
25. Nuhad S. and Adwia J.A.,2015. Nearly Prime Submodules .International J. of Advanced Scientific and Technical Research .Vol.6,2246-9954.