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RESEARCH ARTICLE

SOME INVERSE AND SATURATION RESULTS ON CONVOLUTION OPERATORS

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Abstract

In this paper, firstly we have studied some inverse and saturation results for the family of linear positive convolution operators. We have used Bernstein inequality for proving inverse theorems.Then we have found some linear combinations which are not saturated by construction.

Introduction:-

Consider a family of linear positive convolution operators [1],

$\left(E_{(n,\eta)}f\right)(x) = \left(f * g_{(n)}\right)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g_{(n)}(x-t)dt$, $n \in N$ and $x \in R$ (1)

where $C_{2\pi}$ is the space of 2π-periodic functions with norm,

$\|f\| \coloneqq max|f(x)|$

with kernel η= $\left(g_n(x)\right)_{n>0} \subset L_{2\pi}^1$

depending upon the parameters $n > 0$ and $n \to \infty$.

Here, kernel $\eta = \left\{g_{(n)}\right\}_{n=1}^{\infty}$ be a sequence of even trigonometric polynomials of degree atmost $m(n) = O(n)$, which are normalized by,

$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{(n)}(t)dt = 1$

$g_{(n)}(x) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{(k,n)} \cos kx$ (2)

Here, the operators are uniformly bounded,

$\left\|\left(E_{(n,\eta)}f\right)''\right\| \le A\|f\|$ (3)

and satisfy Bernstein type inequality,

$\left\|\left(E_{(n,\eta)}f\right)''\right\| \le A\varphi(n)^2\left\|\left(E_{(n,\eta)}f\right)\right\|$ (4)

Also, we have,

$\left(E_{(n,\eta)}f\right)'(x) = \left(E_{(n,\eta)}f'(x)\right)$ (5)

For all f with $\varphi(n) > 0$ monotonely increasing to infinity such that,

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$$\sup_{n>0}\left\{\frac{\varphi(n+1)}{\varphi(n)}\right\} = k < \infty$$

For $\delta > 0, 0 < \alpha \le 2$, we have from [2],

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \|f(x+h) - 2f(x) + f(x-h)\|$$

$$= \sup_{0<h\le\delta} \|\Delta_h^2 f(x)\|$$

$$Lip_2\alpha = \{f \in C_{2\pi}; \omega_2(f;\delta) = O(\delta^\alpha), \delta \to 0^+\}$$

By the monotonicity of the modulus of continuity,

$$\omega_2(f;h) \le At^\alpha + Ah^2t^{-2}\omega_2(f;t) \qquad (6)$$

Inverse Results:-

Lemma 2.1. Let Ω be monotonically increasing on $[0, c]$. Then,
$\Omega(t) = O(t^\alpha)$, $t \to 0^+$, if for some , $0 < \alpha < r$ and all $h, t \in [0, c]$
And $m \in N$, such that ,
$h_m \le t \le h_{m-1}$
And
$\Omega(t) \le \Omega(h_{m-1}) \le Bh_{m-1}^\alpha = B(Mh_m)^\alpha \le B(Mt)^\alpha$

Introducing the Steklov means for $\delta > 0$,

$$f_\delta(x) = \frac{1}{\delta^2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2} f(x+s+t)dsdt$$

Also, we have from [3],

$$\|f - f_\delta\| \le \omega_2(f;\delta) \qquad (7)$$

$$\|f_\delta''\| \le \delta^{-2}\omega_2(f;\delta) \qquad (8)$$

Theorem 2.1. [4] If $0 < \alpha < 2$, then we have,
$\|E_{(n,\eta)}f - f\| \le A\varphi(n)^{-\alpha}$
This implies that, $f \in Lip_2\alpha$

Proof. By the assumption and using (3), (4), (5), (7) and (8), we have for $h > 0$,

$$\|\Delta_h^2 f\| \le \|\Delta_h^2(f - E_{(n,\eta)}f)\| + \int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left(E_{(n,\eta)}f\right)''(x+s+t)dsdt$$

$$\le 4\|f - E_{(n,\eta)}f\| + h^2\left\{\left\|\left[E_{(n,\eta)}(f - f_\delta)\right]''\right\| + \|E_{(n,\eta)}f_\delta''\|\right\}$$

$$\le 4A\varphi(n)^{-\alpha} + Ah^2\{\varphi(n)^2\|f - f_\delta\| + \|f_\delta''\|\}$$

$$\le 4A\varphi(n)^{-\alpha} + Ah^2\left\{\varphi(n)^2 + \frac{1}{\delta^2}\right\}\omega_2(f;\delta)$$

$$\le A\delta(n)^\alpha + A\left(\frac{h}{\delta(n)}\right)^2\omega_2(f;\delta(n))$$

for $\delta = \delta(n) = \varphi(n)^{-1}$
If we choose, $\delta(n) \le t \le \delta(n-1) \le k\delta(n)$,
then, $f \in Lip_2\alpha$.

Theorem 2.2. [5] For $0 < \alpha < 2$, we have,
$E_{(n,\eta)}^* f = O\left(\frac{1}{n^\alpha}\right)$, then, $f \in Lip_2\alpha$
where, for the polynomial $e_{(n,\eta)}^* f$ of best approximation, $E_{(n,\eta)}^* f$ is given by,

$$E_{(n,\eta)}^* f = \inf_{e_{(n,\eta)} \in \pi_n} \|f - e_{(n,\eta)}f\| = \|f - e_{(n,\eta)}^* f\|$$

π_n being the set of complex trigonometric polynomials of degree n.

Proof. Let $J_{(n,\eta)}$, for $n \in N$, be a sequence of convolution operators (1) satisfying (2),
$J_{(n,\eta)} f \in \pi_n$,
and $\|J_{(n,\eta)} f - f\| \le A\omega_2\left(f; \frac{1}{n}\right)$ (9)
using Bernstein inequality for trigonmetric polynomials,
$\|e^*_{(n,\eta)} f\| \le n^2 \|e_{(n,\eta)} f\|$
We have,

$$\left\|\left(J_{(n,\eta)} f\right)^{"}\right\| \le \|J_{(n,\eta)}(f - f_{n-1})^{"}\| + \|J_{(n,\eta)} f^{"}_{n-1}\|$$
$$\le n^2 \|J_{(n,\eta)}(f - f_{n-1})\| + A\|f^{"}_{n-1}\|$$
$$\le An^2 \omega_2\left(f; \frac{1}{n}\right) \qquad (10)$$

Using (9) , (10) and theorem 2.1, we have,

$$\|\Delta_h^2 f\| \le \left\|\Delta_h^2\left(f - (e^*_{(n,\eta)} f\,)\right)\right\| + \int\int_{-h/2}^{h/2} \left\|\left(e^*_{(n,\eta)} f\right)^{"}(x + s + t)\right\| dsdt$$

$$\le 4\left(E^*_{(n,\eta)} f\right) + h^2 \left\|\left[e^*_{(n,\eta)}\left(f - J_{(n,\eta)} f\right)\right]^{"}\right\| + h^2 \left\|\left[J_{(n,\eta)} f\right]^{"}\right\|$$

$$\le 4A\left(\frac{1}{n^\alpha}\right) + n^2 h^2 \left\|e^*_{(n,\eta)}\left(f - J_{(n,\eta)} f\right)\right\| + n^2 h^2 A\omega_2\left(f; \frac{1}{n}\right)$$

$$\le A\left[\frac{1}{n^\alpha} + n^2 h^2 A\omega_2\left(f; \frac{1}{n}\right)\right]$$
$$= A\delta_n^\alpha + h\delta_n^{-2}\omega_2(f; \delta_n)$$

with $\delta_n = \left({}^1/_n\right)$
Now, $\frac{\delta_n}{\delta_{n+1}} \le 2$, and using (5), we have, $f \in Lip_2\alpha$.

Some Definitions:-

Definition 3.1.
Fejer type kernels.
Let $q \in L^1(R)$ be normalized by ,

$$\int_{-\infty}^{\infty} q(t)dt = 2\pi \qquad (11)$$

Now, for (2), we will consider even q with its fourier transform $\hat{q}$ with compact support,

$$\hat{q}\,(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(t)e^{-ivt} dt = 0, \ |v| > T, \qquad (12)$$

for some $T > 0$.
Then $q^* = \{q_n^*\}_{n \in N}$ with,

$$q_n^*(x) = \frac{1}{2}\sum_{k=-\infty}^{\infty} nq[nx + 2nk\pi] \qquad (13)$$

is called a kernel of Fejer type.
Closed representation is given by,

$$q_n^*(x) = \frac{1}{2} + \sum_{k=1}^{[nT]} \hat{q}kn^{-1}\, coskx\ , \ x \in R, \qquad (14)$$

Now with the Poisson formula [6] , the singular convolution integral (1) with kernel q may be represented as convolution integral on real line as,

$$\left(E_{(n,q^*)} f\right)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) q_n^*(x - t)dt = \left(E_{(n,\eta)} f\right)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) q_n\left(x - \frac{t}{n}\right) dt \qquad (15)$$

The above condition placed on q guarantee that $\left(E_{(n,q^*)} f\right)(x)$ defines an approximation process on $C_{2\pi}$.
Definition 3.2.
Jackson Kernel. The positive Jackson kernels $j^{(p)} = \left\{J^{(p)}_{(n)}\right\}_{n \in N}$ are given by,

$$J^{(p)}_{(n)} = \frac{1}{\lambda_{(0,n)}(p)}\left\{\frac{\sin(nx/2)}{\sin(x/2)}\right\}^{2p}, \text{ where, } x \in R \qquad (16)$$

This is the closed representation of Jackson kernels.

Here, $\lambda_{(k,n)}(p) = \frac{2}{\pi}\int_0^{\pi}\left\{\frac{\sin(nt/2)}{\sin(t/2)}\right\}^{2p}\cos kt\, dt$, where, $0\leq k \leq (n-1)p$ (17)
Corresponding convergence factors are given by,
$\rho_{(k,n)}\left(j^{(p)}\right) = \frac{\lambda_{(k,n)}(p)}{\lambda_{(0,n)}(p)}$, (18)
Where [1],

$$\rho_{(k,n)} = \begin{cases}(1/\pi)\int_{-\pi}^{\pi} g_{(n)}(t)\cos kt\, dt\,, & 0\leq k \leq m(n)\\ 0\,, & k > m(n)\end{cases} \quad (19)$$

Here, kernel $\{g_{(n)}\}_{n=1}^{\infty} = \eta$ be a sequence of even trigonometric polynomials of degree atmost $m(n) = O(n)$, which are normalized by,
$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{(n)}(t)dt = 1$
The Jackson kernels are not of Fejer type.
Also, $\lambda_{(k,n)}(p)$ can be represented as in [7] [8],

$$\lambda_{(k,n)}(p) = \sum_{j=0}^{2p}(-1)^j\binom{2p}{j}\binom{n(p-j)+p-k-1}{2p-1}\,, \quad 0\leq k \leq (n-1)p \quad (20)$$

Using property of central factorial numbers ,

(i) $t_k^n = T_k^n = 0$, $n < k$
(ii) $T_k^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{k-2j}{2}\right)^n$, $0 \leq k \leq n \in N_0$ (21)

Where, t_k^n is the central factorial numbers of first kind and is uniquely determined coefficients of the polynomials,
$x^{[n]} = \sum_{k=0}^{n} t_k^n x^k$

Similarly, T_k^n is central factorial numbers of second kind and is uniquely determined coefficients of the polynomials,
$x^n = \sum_{k=0}^{n} T_k^n x^{[k]}$
By putting $k = 0$ in (20) and using (21),

$$\lambda_{(0,n)}(p) = \frac{1}{(2p-1)!}\sum_{i=1}^{p} n^{2i-1}t_{2i}^{2p}\sum_{j=0}^{p}(-1)^j\binom{2p}{j}(p-j)^{2i-1} \quad (22)$$

and for $1 \leq k \leq n-p$, we have,

$$\lambda_{(k,n)}(p) = \frac{1}{(2p-1)!}\sum_{i=2}^{p} t_{2i}^{2p}\sum_{m=1}^{i-1}\binom{2i-m}{2m}n^{2i-2m-1}k^{2m}$$
$$\times\sum_{j=0}^{p}(-1)^j\binom{2p}{j}(p-j)^{2i-2m-1}$$
$$+\frac{(-1)^p}{(2p-1)!}\binom{2p}{p}\sum_{i=1}^{p} k^{2i-1}t_{2i}^{2p} + \lambda_{(0,n)}(p) \quad (23)$$

For some, $C_{(i,j)} = C_{(i,j)}\left(j^{(p)}\right)$ and $d_{(k,p)} \neq 0$, polynomial division of (23) by (22) gives,

$$1-\rho_{(k,n)}\left(j^{(p)}\right) = \sum_{j=1}^{p-1}\left(\frac{1}{n^{2j}}\right)\sum_{i=1}^{j} C_{(i,j)}k^{2i} + d_{(k,p)}\,n^{-2p+1} + O(n^{-2p}) \quad (24)$$

We can see from (24),
$j^{(p)} \in S^{(2,p-1)}$ and $j^{(p)} \notin S^{(2,p)}$

For, $p \geq 3$, we have approximation rate higher than $O(n^{-2})$ for the linear combination $\chi = \{\chi_{(n)}\}_{n\in N}$ of even trigonometric polynomials of degree (na_v) , for Jackson kernels,
$\chi_n(x) = \sum_{v=1}^{s}\gamma_v g_{(na_v)}(x)$, $x \in R$, (25)

Definition 3.3.
Central B-splines and Jackson De La Vallee Poussin kernel**.**
We can define central B-splines [9] as the Fourier transforms of the powers of the sinc function,

$$B_m(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(sinc\frac{t}{2}\right)^m cosvtdt \quad (26)$$

$sincx = \frac{sinx}{x}$, $x \in R\backslash\{0\}$, $sinc\ 0 = 1$
Their closed form is given by,

$$B_m(v) = \begin{cases} \frac{1}{(m-1)!}\sum_{j=0}^{[(m/2)-[v]]}(-1)^j \binom{m}{j}\left(\frac{m}{2} - |v| - j\right)^{m-1}, & |v| \le \frac{m}{2} \\ 0 & , |v| > \frac{m}{2} \end{cases} \quad (27)$$

The main properties of B-Splines are,

$B_m(v) \ge 0$, $\quad v \in R,$

$B_m(v) \in C^{m-2}(R)$, $m \ge 2,$

$B_m(v) \in C^{m-1}\left[-\frac{m}{2} + i, -\frac{m}{2} + i + 1\right]$, $\quad 0 \le i \le (m-1)$, $i \in N$ (28)

The function , q is given by,

$$q(x) \equiv q_{(n)}(x) = 1 \Big/ \left\{B_{2p}(0)\left(sinc\frac{x}{2}\right)^{2p}\right\} \quad (29)$$

satisfies (11) and (12).

For $p \in N$, the corresponding Fejer type kernel (13), namely, the kernel of Jackson and de la vallee Poussin $\beta^{(p)} = \left\{P_{(n)}^{(p)}\right\}$, $n \in N$, is given by, according to (2),

$$P_{(n)}^{(p)}(x) = \frac{1}{2} + \sum_{k=1}^{np-1}\frac{B_{2p}(k/n)}{B_{2p}(0)}coskx, \ x \in R, \ (30)$$

For $p = 1$, we have well known Fejer kernel $f = \{F_n\}$, $n \in N$ with,

$$F_n(x) = \frac{1}{2\pi}\left[\frac{\sin(nx/2)}{\sin(x/2)}\right]^2, \quad x \in R, \ (31)$$

and convergence factors,

$$\rho_{(k,n)}(f) = \frac{n-k}{n}, \ 0 \le k \le (n-1), \ (32)$$

For $p = 2$, we have classical kernel of Jackson de La Vallee Poussin ,

$$P_{(n)}^{(2)}(x) = \left(\frac{1}{2n^3} + \frac{cosx}{4n^3}\right)\left[\frac{\sin(nx/2)}{\sin(x/2)}\right]^4, \ x \in R, (33)$$

Convergence factors for above kernel is,

$$\rho_{(k,n)}(\beta^2) = \begin{cases}(4n^3 - 6nk^2 + 3k^3)/4n^3, & 0 \le k \le (n-1) \\ (2n-k)^3/4n^3, & 0 \le k \le (2n-1)\end{cases} \quad (34)$$

Saturation Results:-

Here, we will study saturation theorems that are not saturated.
Consider the best trigonometric approximation for $f \in C_{2\pi}$,

$$\left(I_{(n)}f\right)(x) = I_{(n)}f = \inf_{t_n \in \pi_n}\|f - t_n\| = \|f - t_n^*\| \quad (35)$$

For convergence factors of a kernel η , there holds,

$$\rho_{(k,n)}(\eta) = 1, \ 1 \le k \le n, \quad (36)$$

We have, $E_{(n,\eta)}(t_n, x) = t_n(x)$, $t_n \in \pi_n$, (37)

$$\|E_{(n,\eta)}(f,\cdot) - f(\cdot)\| \le \|E_{(n,\eta)}(f - t_n^*, \cdot)\| + \|f - t_n^*\|$$

$$\|E_{(n,\eta)}(f,\cdot) - f(\cdot)\| \le I_{(n,\eta)}\|f - t_n^*\| + \|f - t_n^*\| \quad (38)$$

For example, the well-known linear combination, $\{\tau_n\}$, $n \in N$, of the Fejer kernel f ,

$$\tau_n(x) = 2F_{2n-1}(x) - F_{n-1}(x) \quad (39)$$

satisfies (36), so that, we have,
$\| E_{(n,\tau)}(f,\cdot) - f(\cdot)\| = O(1)\left(I_{(n)}f\right)$, for, $f \in C_{2\pi}$

We can generalize above method in the following way.
Suppose the convergence factor of η admits an expansion for $\tau = 1$ or 2 , $\mu \in N$,
$$1 - \rho_{(k,n)}(\eta) = \sum_{j=1}^{\mu}(-1)^{j+1}\Psi_j(k)\left(\frac{1}{n^{\tau j}}\right) + h_{(k,\mu)}\left(\frac{1}{n^{\tau\mu+(\tau/2)}}\right) , \quad (40)$$

For $h_{(k,\mu)} \in R$, $1 \le k \le n$, thus in particular, $\eta \in S^{(\tau,\mu)}$.

We can now built a linear combination similar to (25) such that all terms on right hand side of (40) are cancelled.For example, kernels of Jackson and De La Vallee Poussin $\beta^{(p)}$ with its convergence factors .
For $p = 2$, we have,
$$\chi_n(x) = \gamma_1 P_{a_1 n}^2(x) + \gamma_2 P_{a_2 n}^2(x) + \gamma_3 P_{a_3 n}^3(x) \ , \ x \in R \quad (41)$$

Where, $a_i = i, 1 \le i \le 3$, coefficients γ_i can be uniquely determined.
Here, for the linear combination,
$\chi_n(x) = \sum_{\nu=1}^{s}\gamma_\nu g_{(na_\nu)}(x)$ holds (36).

From this, we have the Following:-
Corollary 4.1. The unique linear combination $\chi$of Jackson De La vallee Poussin kernels (41) satisfying (36) is given by,
$$\chi_n(x) = \frac{9}{4}P_{a_1 n}^2(x) - \frac{4}{3}P_{a_2 n}^2(x) + \frac{1}{12}P_{a_3 n}^3(x) , \text{ for, } x \in R \quad (42)$$
Corresponding singular integral for $f \in C_{2\pi}$ is given by,
$\|E_{(n,\chi)}(f,\cdot) - f(\cdot)\| = O(1)E_{(n)}(f)$

The kernels of Jackson $j^{(p)}$ of (16), p≥ 2, admits no expansion of the form (40). Also, here linear combination are not saturated. This will be briefly outlined for $p = 2$. The convergence factors are then given by,
$$\rho_{(k,n)}(j^2) = \frac{1}{4n^3+2n}\begin{cases} 3k^2 - 6nk^2 - 3k + 4n^3 + 2n , & 0 \le k \le n \\ -k^3 + 6nk^2 - 12n^2k + k + 8n^3 - 2n , & n \le k \le 2n \end{cases} \quad (43)$$

Now for j^2 and $a_i = i, 1 \le i \le 3$, leads to another corollary.
Corollary 4.2. [10] The unique linear combination $\tilde{\chi}$ of Jackson De la Vallee poussin kernels (42) satisfying (36), is given by,
$$\tilde{\chi}_{(n)}(x) = -\frac{1}{18}\left(1 + \frac{1}{n^2}\right)J_n^2(x) + \frac{1}{3}\left(4 + \frac{1}{2n^2}\right)J_{2n}^2 - \frac{1}{2}\left(1 + \frac{1}{2n^2}\right)J_{3n}^2(x) + \frac{1}{9}\left(32 + \frac{1}{n^2}\right)J_{4n}^2(x) \quad (44)$$
Corresponding singular integral for $f \in C_{2\pi}$ is given by,
$\|E_{(n,\tilde{\chi})}(f,\cdot) - f(\cdot)\| = O(1)E_{(n)}(f)$

Here, the coefficients depend on n, and are bounded by linear combination $\tilde{\chi} = \left\{\tilde{\chi}_{(n)}(x)\right\}$, $n \in N$, which defines an approximation process and is not saturated by construction.
Conclusion. Jackson de La Vallee Poussin kernels seems more suitable for these linear combinations since the calculation of coefficients is less elaborate.

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