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RESEARCH ARTICLE

Results on the Group Inverse for Block Matrices in Minkowski Space $\mathcal{M}$.

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Abstract

In this paper, existence and the representation of the group inverse for block matrix $M = \begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} (P, Q \in K^{n\times n}, (P^{\sim})^2 = P^{\sim})$ over skew fields in Minkowski space $\mathcal{M}$ is studied. The existence and the representation of the group inverse for other block matrices in Minkowski space $\mathcal{M}$ is also given.

Introduction:-

Suppose K is a skew field. Let $K^{n\times n}$ denote the set of all matrices over K. For $A \in K^{n\times n}$, the matrix $X \in K^{n\times n}$ is said to be the group inverse of A, if it holds that

$$AXA = A, \qquad XAX = X, \qquad AX = XA$$

Denote by $X = A^{\#}$. By Zhuang (1987), if $X = A^{\#}$ exists and it is unique.

Kilicman et al., (2008) extend some results for the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$ in Hilbert Space to the weighted Minkowski inverse $A^{\oplus}_{M,N}$ of an arbitrary rectangle matrix $A \in M_{m,n}$ in Minkowski spaces μ. Hanifa Zekraoui et al., (2013) introduce some new algebraic and topological properties of the Minkowski inverse $A^{\oplus}$ of an arbitrary matrix $A \in M_{m,n}$ in Minkowski space μ. This paper also shows that Minkowski inverse $A^{\oplus}$ in a Minkowski space and Moore-Penrose inverse $A^{\dagger}$ in a Hilbert Space are different in many properties such as the existence, continuity, norm and SVD, and some new conditions like existence, continuity and reverse order law of the Minkowski inverse are also given. Meenakshi (2000) introduced the concept of range symmetric matrix and the existence of the Minkowski inverse of a range symmetric matrix in Minkowski space $\mathcal{M}$. Meenakshi et al., (2006) studied the necessary and sufficient conditions for the product of range symmetric matrices of rank r to be range symmetric in Minkowski space $\mathcal{M}$.

In this paper the notations A^*, and $A^{\sim}$ stands for conjugate transpose and Minkowski adjoint of a matrix A respectively, I_n denote the identity matrix of order $n \times n$. Let C^n be the space of complex n-tuples and we shall index them from 0 to $n-1$, that is $u = (u_0, u_1, u_2, \dots, u_{n-1})$. Let G be the Minkowski metric tensor defined by

$$Gu = (u_0, -u_1, -u_2, \dots, -u_{n-1}).$$

And the Minkowski metric matrix is defined by

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}; \; G^* = G; \; G^2 = I_n.$$

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In [15] the Minkowski inner product on C^n is defined by $(u,v) = [u, Gv]$, where [.,.] denotes the conventional Hilbert (unitary) space inner product. A space with Minkowski inner product is called a Minkowski space (Krishnaswamy et al., 2013). We establish the existence and the representation of the group inverse for block matrix $\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} (P, Q \in K^{n\times n},\ (P^{\sim})^2 = P^{\sim})$ in Minkowski space.

Definition:-

For any $P \in C^{n\times n}$, the Minkowski adjoint of P denoted by $P^{\sim}$ is defined as $P = GP^*G$, where P^* is the usual Hermitian adjoint and G the Minkowski metric matrix of order n.

Some Lemmas:-

Lemma 2.1. *Let $P \in K^{n\times n}$, if $(P^{\sim})^2 = P^{\sim}$, then there is a unitary matrix $A \in K^{n\times n}$ such that $P^{\sim} = (A^*)^{\sim} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} A^{\sim}$, where $r = rank(P^{\sim})$, I_r is the $r \times r$ identity matrix, $I_r \in K^{n\times n}$.*

Proof. Let $P \in K^{n\times n}$ then there are unitary matrices $A_1,\ A_2$ such that

$$P = A_1 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} A_2 \Rightarrow P^* = A_2^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} A_1^* \Rightarrow P^{\sim} = A_2^{\sim} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} A_1^{\sim} = A_2^{\sim} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} A_1^{\sim} A_2^{\sim} (A_2^*)^{\sim}$$

Let $A_1^{\sim} A_2^{\sim} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$ then $P^{\sim} = A_2^{\sim} \begin{pmatrix} P_1 & P_2 \\ 0 & 0 \end{pmatrix} (A_2^*)^{\sim}$

Since $(P^{\sim})^2 = P^{\sim}$, so we have $P_1 (P_1 \quad P_2) = (P_1 \quad P_2)$ because $(P_1 \quad P_2)$ is full row rank, then $P_1 = I_r$.

Let $A^{\sim} = A_2^{\sim} \begin{pmatrix} I_r & -P_2 \\ 0 & I_{n-r} \end{pmatrix}$ then $P^{\sim} = A^{\sim} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} (A^*)^{\sim}$

Lemma 2.2 *Let $P, Q \in K^{n\times n}$, if $(P^{\sim})^2 = P^{\sim}$, $rank(P^{\sim}) = r$, $rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim}Q^{\sim})$ then there is a unitary matrix $A \in K^{n\times n}$, such that $Q^{\sim} = A^{\sim} \begin{pmatrix} Q_1^* & -Q_1^*X \\ -YQ_1^* & YQ_1^*X \end{pmatrix} (A^*)^{\sim}$ and $(Q_1^*)^{\#}$ exists where $Q_1^* \in K^{n\times n}$, $X \in K^{r\times(n-r)}$, $Y \in K^{(n-r)\times r}$.*

Proof. Since $(P^{\sim})^2 = P^{\sim}$, by Lemma 2.1 there is a unitary matrix $A \in K^{n\times n}$, such that

$$P^{\sim} = A^{\sim} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} (A^*)^{\sim}, \qquad Q^{\sim} = A^{\sim} \begin{pmatrix} Q_1^* & -Q_3^* \\ -Q_2^* & Q_4^* \end{pmatrix} (A^*)^{\sim}$$

where $Q_1^* \in K^{r\times r}$, $Q_2^* \in K^{(n-r)\times r}$, $Q_3^* \in K^{r\times(n-r)}$, $Q_4^* \in K^{(n-r)\times(n-r)}$

Then

$$P^{\sim}Q^{\sim} = A^{\sim} \begin{pmatrix} Q_1^* & -Q_3^* \\ 0 & 0 \end{pmatrix} (A^*)^{\sim}, \qquad Q^{\sim}P^{\sim} = A^{\sim} \begin{pmatrix} Q_1^* & 0 \\ -Q_2^* & 0 \end{pmatrix} (A^*)^{\sim}$$

From

$$rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim}Q^{\sim}) \leq rank(Q^{\sim}P^{\sim}) \leq rank(Q^{\sim})$$

we have

$$rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim})$$

From

$$rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim}Q^{\sim}) \leq rank(P^{\sim}Q^{\sim}) \leq rank(Q^{\sim})$$

we get

$$rank(Q^{\sim}) = rank(P^{\sim}Q^{\sim})$$

Thus

$$rank(P^{\sim}) = rank(Q^{\sim}P^{\sim}) = rank(Q^{\sim}) = rank(P^{\sim}Q^{\sim})$$

Since from $rank(Q^{\sim}) = rank(P^{\sim}Q^{\sim})$, we have

$$Q_2^* = YQ_1^*, \quad Q_4^* = YQ_3^*, \qquad Y \in K^{(n-r)\times r}$$

and from $rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim})$, we get

$$Q_3^* = Q_1^*X, \quad Q_4^* = Q_2^*X = YQ_1^*X, \quad X \in K^{r\times(n-r)}$$

So, we get

$$Q^{\sim} = A^{\sim} \begin{pmatrix} Q_1^* & -Q_1^*X \\ -YQ_1^* & YQ_1^*X \end{pmatrix} (A^*)^{\sim}$$

We have $rank(Q_1^*) = rank(Q_1^*)^2 = rank(Q^{\sim})$, that is $(Q_1^*)^{\#}$ exists.

Lemma 2.3 *Let $P \in K^{r\times r}$, $Q \in K^{(n-r)\times r}$, $M = \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} \in K^{n\times n}$. Then the group inverse of M exists in $\mathcal{M}$ if and only if the group inverse of $P^{\sim}$ exists in $\mathcal{M}$ and $rank(P^{\sim}) = rank \begin{pmatrix} P^{\sim} \\ Q^{\sim} \end{pmatrix}$. If the group inverse of M exists in $\mathcal{M}$, then*

$$M^{\#} = \begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix}$$

Proof. Since $M = \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}$, suppose group inverse of $P^{\sim}$ exists in $\mathcal{M}$ and $rank(P^{\sim}) = rank\begin{pmatrix} P^{\sim} \\ Q^{\sim} \end{pmatrix}$. Now $rank(M) = rank\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} = rank\begin{pmatrix} P^{\sim} \\ Q^{\sim} \end{pmatrix} = rank(P^{\sim})$. But $rank(P^{\sim}) = rank(P^{\sim})^2$ as $(P^{\sim})^{\#}$ exists $\Rightarrow rank(M) = rank(M^2)$. Therefore $M^{\#}$ exists in $\mathcal{M}$.

Conversely, suppose the group inverse of M exists in $\mathcal{M}$, then it satisfies the following conditions, (i) $MM^{\#}M = M$, (ii) $M^{\#}MM^{\#} = M^{\#}$ and (iii) $MM^{\#} = M^{\#}M$. Also $rank(M) = rank\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} = rank\begin{pmatrix} P^{\sim} \\ Q^{\sim} \end{pmatrix} \Rightarrow rank(P^{\sim}) = rank\begin{pmatrix} P^{\sim} \\ Q^{\sim} \end{pmatrix}$.

Let $M^{\#} = X = \begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix}$ then,

(i)

$$\begin{aligned} MXM &= \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix}\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} \\ &= \begin{pmatrix} P^{\sim}(P^{\sim})^{\#}P^{\sim} & 0 \\ Q^{\sim}(P^{\sim})^{\#}P^{\sim} & 0 \end{pmatrix} \\ &= \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} \\ MXM &= M \end{aligned}$$

(ii)

$$\begin{aligned} XMX &= \begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix}\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (P^{\sim})^{\#}P^{\sim}(P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2P^{\sim}(P^{\sim})^{\#} & 0 \end{pmatrix} \\ &= \begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix} \\ XMX &= X \end{aligned}$$

(iii)

$$\begin{aligned} MX &= \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P^{\sim}(P^{\sim})^{\#} & 0 \\ Q^{\sim}(P^{\sim})^{\#} & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} XM &= \begin{pmatrix} (P^{\sim})^{\#} & 0 \\ Q^{\sim}((P^{\sim})^{\#})^2 & 0 \end{pmatrix}\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} \\ &= \begin{pmatrix} P^{\sim}(P^{\sim})^{\#} & 0 \\ Q^{\sim}(P^{\sim})^{\#} & 0 \end{pmatrix} \\ XM &= MX. \end{aligned}$$

Lemma 2.4 *Let* $P \in K^{r\times r}$, $Q \in K^{r\times(n-r)}$, *and* $M = \begin{pmatrix} P^{\sim} & Q^{\sim} \\ 0 & 0 \end{pmatrix} \in K^{n\times n}$. *Then the group inverse of* M *exists in* $\mathcal{M}$ *if and only if the group inverse of* $P^{\sim}$ *exists in* $\mathcal{M}$ *and* $rank(P^{\sim}) = rank(P^{\sim} \quad Q^{\sim})$. *If the group inverse of* M *exists in* $\mathcal{M}$, *then,*

$$M^{\#} = \begin{pmatrix} (P^{\sim})^{\#} & ((P^{\sim})^{\#})^2Q^{\sim} \\ 0 & 0 \end{pmatrix}$$

Proof. The proof is same as Lemma 2.3.

Lemma 2.5 *Let* $P, Q \in K^{n\times n}$, *if* $(P^{\sim})^2 = P^{\sim}$, $rank(P^{\sim}) = r$, $rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim}Q^{\sim})$ *then the following conclusions hold:*

(i) $(Q^{\sim}P^{\sim})^{\#}Q^{\sim}P^{\sim}Q^{\sim} = Q^{\sim}$;

(ii) $P^{\sim}(P^{\sim}Q^{\sim})^{\#}=(P^{\sim}Q^{\sim})^{\#},(Q^{\sim}P^{\sim})^{\#}P^{\sim}=(Q^{\sim}P^{\sim})^{\#},(P^{\sim}Q^{\sim})^{\#}P^{\sim}=P^{\sim}(Q^{\sim}P^{\sim})^{\#},(Q^{\sim}P^{\sim})^{\#}Q^{\sim}=Q^{\sim}(P^{\sim}Q^{\sim})^{\#}$;
(iii) $(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(P^{\sim}Q^{\sim})^{\#}=(P^{\sim}Q^{\sim})^{\#},P^{\sim}(Q^{\sim}P^{\sim})^{\#}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}=(P^{\sim}Q^{\sim})^{\#}$;
(iv) $(Q^{\sim}P^{\sim})^{\#}Q^{\sim}P^{\sim}(P^{\sim}Q^{\sim})^{\#}P^{\sim}=(Q^{\sim}P^{\sim})^{\#},Q^{\sim}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}=Q^{\sim}P^{\sim}$.

Proof. Using Lemma 2.2, 2.3 and 2.4 we have

$$P^{\sim}Q^{\sim}=(A^*)^{\sim}\begin{pmatrix}Q_1^* & -Q_1^*X\\ 0 & 0\end{pmatrix}A^{\sim},\qquad Q^{\sim}P^{\sim}=(A^*)^{\sim}\begin{pmatrix}Q_1^* & 0\\ -YQ_1^* & 0\end{pmatrix}A^{\sim}$$

then

$$(P^{\sim}Q^{\sim})^{\#}=(A^*)^{\sim}\begin{pmatrix}(Q_1^*)^{\#} & -(Q_1^*)^{\#}X\\ 0 & 0\end{pmatrix}A^{\sim},\qquad (Q^{\sim}P^{\sim})^{\#}=(A^*)^{\sim}\begin{pmatrix}(Q_1^*)^{\#} & 0\\ -Y(Q_1^*)^{\#} & 0\end{pmatrix}A^{\sim}$$

(i)

$$\begin{aligned}(Q^{\sim}P^{\sim})^{\#}Q^{\sim}P^{\sim}Q^{\sim} &= (A^*)^{\sim}\begin{pmatrix}(Q_1^*)^{\#} & 0\\ -Y(Q_1^*)^{\#} & 0\end{pmatrix}A^{\sim}(A^*)^{\sim}\begin{pmatrix}Q_1^* & 0\\ -YQ_1^* & 0\end{pmatrix}A^{\sim}(A^*)^{\sim}\begin{pmatrix}Q_1^* & -Q_1^*X\\ -YQ_1^* & YQ_1^*X\end{pmatrix}A^{\sim}\\ &= (A^*)^{\sim}\begin{pmatrix}Q_1^* & -Q_1^*X\\ -YQ_1^* & YQ_1^*X\end{pmatrix}A^{\sim}\\ &= Q^{\sim}\end{aligned}$$

Similarly $(ii)-(iv)$ can be proved.

Conclusion:-

Theorem 3.1 *Suppose* $M=\begin{pmatrix}P^{\sim} & P^{\sim}\\ Q^{\sim} & 0\end{pmatrix}$ *where* $P,Q\in K^{n\times n}$, $(P^{\sim})^2=P^{\sim}$, $rank(P^{\sim})=r$, *then*

(i) $M^{\#}$ *exists if and only if* $rank(Q^{\sim})=rank(Q^{\sim}P^{\sim}Q^{\sim})$;

(ii) *If* $M^{\#}$ *exists, then*

$$M^{\#}=\begin{pmatrix}P^{\sim}-(P^{\sim}Q^{\sim})^{\#}+(P^{\sim}Q^{\sim})^{\#}P^{\sim}-(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} & P^{\sim}+(P^{\sim}Q^{\sim})^{\#}P^{\sim}-(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}\\ (Q^{\sim}P^{\sim})^{\#}Q^{\sim}+(Q^{\sim}P^{\sim})^{\#}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}-(Q^{\sim}P^{\sim})^{\#} & -(Q^{\sim}P^{\sim})^{\#}\end{pmatrix}$$

Proof. (i) $rank(M)=rank\begin{pmatrix}P^{\sim} & P^{\sim}\\ Q^{\sim} & 0\end{pmatrix}=rank\begin{pmatrix}0 & P^{\sim}\\ Q^{\sim} & 0\end{pmatrix}=rank(P^{\sim})+rank(Q^{\sim})$.

$$\begin{aligned}rank(M^2)&=rank\begin{pmatrix}P^{\sim}+P^{\sim}Q^{\sim} & P^{\sim}\\ Q^{\sim}P^{\sim} & Q^{\sim}P^{\sim}\end{pmatrix}=rank\begin{pmatrix}P^{\sim}Q^{\sim} & P^{\sim}\\ 0 & Q^{\sim}P^{\sim}\end{pmatrix}=rank\begin{pmatrix}0 & P^{\sim}\\ Q^{\sim}P^{\sim}Q^{\sim} & Q^{\sim}P^{\sim}\end{pmatrix}\\ &=rank\begin{pmatrix}0 & P^{\sim}\\ Q^{\sim}P^{\sim}Q^{\sim} & 0\end{pmatrix}=rank(P^{\sim})+rank(Q^{\sim}P^{\sim}Q^{\sim}).\end{aligned}$$

$M^{\#}$ exists if and only if $rank(M)=rank(M^2)$, that is $rank(Q^{\sim})=rank(Q^{\sim}P^{\sim}Q^{\sim})$.

(ii) Let $X=\begin{pmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{pmatrix}$ where,

$$\begin{aligned}Z_1&=P^{\sim}-(P^{\sim}Q^{\sim})^{\#}+(P^{\sim}Q^{\sim})^{\#}P^{\sim}-(P^{\sim}Q^{\sim})^{\#}-(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim};\\ Z_2&=P^{\sim}+(P^{\sim}Q^{\sim})^{\#}P^{\sim}-(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim};\\ Z_3&=(Q^{\sim}P^{\sim})^{\#}Q^{\sim}+(Q^{\sim}P^{\sim})^{\#}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}-(Q^{\sim}P^{\sim})^{\#};\\ Z_4&=-(Q^{\sim}P^{\sim})^{\#}\end{aligned}$$

Then we will prove that the above matrix satisfies the three conditions of group inverse.

$$\begin{aligned}MXM &= \begin{pmatrix}P^{\sim} & P^{\sim}\\ Q^{\sim} & 0\end{pmatrix}\begin{pmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{pmatrix}\begin{pmatrix}P^{\sim} & P^{\sim}\\ Q^{\sim} & 0\end{pmatrix}\\ &= \begin{pmatrix}P^{\sim}(Z_1+Z_2)P^{\sim}+P^{\sim}(Z_2+Z_4)Q^{\sim} & P^{\sim}(Z_1+Z_3)P^{\sim}\\ Q^{\sim}Z_1P^{\sim}+Q^{\sim}Z_2Q^{\sim} & Q^{\sim}Z_1P^{\sim}\end{pmatrix}\end{aligned}$$

Using Lemma 2.5 we get from (2,1) block of MXM

$$\begin{aligned}Q^{\sim}Z_1P^{\sim}+Q^{\sim}Z_2Q^{\sim} &= Q^{\sim}(P^{\sim}-(P^{\sim}Q^{\sim})^{\#}+(P^{\sim}Q^{\sim})^{\#}P^{\sim}-(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim})P^{\sim}+Q^{\sim}(P^{\sim}+(P^{\sim}Q^{\sim})^{\#}P^{\sim}\\ &\quad -(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim})Q^{\sim}\\ &= Q^{\sim}P^{\sim}-Q^{\sim}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}+Q^{\sim}P^{\sim}Q^{\sim}+Q^{\sim}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}Q^{\sim}\\ &= (Q^{\sim}P^{\sim})^{\#}Q^{\sim}P^{\sim}Q^{\sim}\\ &= Q^{\sim}\end{aligned}$$

Similarly we can get

$$\begin{aligned}&P^{\sim}(Z_1+Z_3)P^{\sim}+P^{\sim}(Z_2+Z_4)Q^{\sim}=P^{\sim};\\ &P^{\sim}(Z_1+Z_3)P^{\sim}=P^{\sim};\\ &Q^{\sim}Z_1P^{\sim}=0\end{aligned}$$

Now

$$XMX = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}$$
$$= \begin{pmatrix} Z_1P^{\sim}Z_1 + Z_2Q^{\sim}Z_1 + Z_1P^{\sim}Z_3 & Z_1P^{\sim}Z_2 + Z_2Q^{\sim}Z_2 + Z_1P^{\sim}Z_4 \\ Z_3P^{\sim}Z_1 + Z_4Q^{\sim}Z_1 + Z_3P^{\sim}Z_3 & Z_3P^{\sim}Z_3 + Z_4Q^{\sim}Z_2 + Z_3P^{\sim}Z_4 \end{pmatrix}$$

Using Lemma 2.6 we get

$$\begin{aligned} Z_1P^{\sim}Z_1 + Z_2Q^{\sim}Z_1 + Z_1P^{\sim}Z_3 &= Z_1 + P^{\sim}(Q^{\sim}P^{\sim})^{\#}Q^{\sim} + P^{\sim}(Q^{\sim}P^{\sim})^{\#}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim} - P^{\sim}(Q^{\sim}P^{\sim})^{\#} \\ &+(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(P^{\sim}Q^{\sim})^{\#} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(P^{\sim}Q^{\sim})^{\#}P^{\sim} \\ &+(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(Q^{\sim}P^{\sim})^{\#}Q^{\sim} \\ &-(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(Q^{\sim}P^{\sim})^{\#}(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim} + (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim}(Q^{\sim}P^{\sim})^{\#} \\ &-(P^{\sim}Q^{\sim})^{\#} + (P^{\sim}Q^{\sim})^{\#}P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} \\ &= Z_1 + (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim} + (P^{\sim}Q^{\sim})^{\#} - (P^{\sim}Q^{\sim})^{\#}P^{\sim} + (P^{\sim}Q^{\sim})^{\#} - (P^{\sim}Q^{\sim})^{\#}P^{\sim} \\ &+(P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim} - (P^{\sim}Q^{\sim})^{\#} + (P^{\sim}Q^{\sim})^{\#}P^{\sim} - (P^{\sim}Q^{\sim})^{\#} \\ &+(P^{\sim}Q^{\sim})^{\#}P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} \\ &= Z_1 \end{aligned}$$

We can easily get

$$Z_1P^{\sim}Z_2 + Z_2Q^{\sim}Z_2 + Z_1P^{\sim}Z_4 = Z_2$$
$$Z_3P^{\sim}Z_1 + Z_4Q^{\sim}Z_1 + Z_3P^{\sim}Z_3 = Z_3$$
$$Z_3P^{\sim}Z_3 + Z_4Q^{\sim}Z_2 + Z_3Q^{\sim}Z_4 = Z_4$$

Finally $MX = XM = \begin{pmatrix} P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} + (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim} & P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} \\ (Q^{\sim}P^{\sim})^{\#}Q^{\sim}P^{\sim} - (Q^{\sim}P^{\sim})^{\#}Q^{\sim} & (Q^{\sim}P^{\sim})^{\#}Q^{\sim}P^{\sim} \end{pmatrix}$

So we have $X = M^{\#}$.

Theorem 3.2 *Suppose* $M = \begin{pmatrix} P^{\sim} & Q^{\sim} \\ P^{\sim} & 0 \end{pmatrix}$ *where* $P, Q \in K^{n\times n}, (P^{\sim})^2 = P^{\sim}, rank(P^{\sim}) = r$*, then*

(*i*) $M^{\#}$ *exists if and only if* $rank(Q^{\sim}) = rank(Q^{\sim}P^{\sim}Q^{\sim})$*;*

(*ii*) *If* $M^{\#}$ *exists, then*

$$M^{\#} = \begin{pmatrix} P^{\sim} - (Q^{\sim}P^{\sim})^{\#} + (P^{\sim}Q^{\sim})^{\#}P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} & (Q^{\sim}P^{\sim})^{\#}Q^{\sim} + (Q^{\sim}P^{\sim})^{\#}(P^{\sim}Q^{\sim})P^{\sim}Q^{\sim} - (P^{\sim}Q^{\sim})^{\#} \\ P^{\sim} + (P^{\sim}Q^{\sim})^{\#}P^{\sim} - (P^{\sim}Q^{\sim})^{\#}P^{\sim}Q^{\sim}P^{\sim} & -(P^{\sim}Q^{\sim})^{\#} \end{pmatrix}$$

Proof. Proof is same as Theorem 3.1.

Theorem 3.3 *If* $\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix}^{\#}$ *exists, where* P*,* $Q \in K^{n\times n}$*,* $(P^{\sim})^2 = P^{\sim}$*,* $rank\ (P^{\sim}) = r$*, then* $P^{\sim}Q^{\sim}$ *and* $Q^{\sim}P^{\sim}$ *are similar.*

Proof. Since $\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix}^{\#}$ exists, using Lemma 2.1 and Lemma 2.2 there is a unitary matrix $A \in K^{n\times n}$, such that

$$P^{\sim} = (A^*)^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}, \quad Q^{\sim} = (A^*)^{\sim}\begin{pmatrix} Q_1^* & -Q_1^*X \\ -YQ_1^* & YQ_1^*X \end{pmatrix}A^{\sim}$$

Hence

$$\begin{aligned} P^{\sim}Q^{\sim} &= (A^*)^{\sim}\begin{pmatrix} Q_1^* & -Q_1^*X \\ 0 & 0 \end{pmatrix}A^{\sim} \\ &= (A^*)^{\sim}\begin{pmatrix} I_r & X \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} Q_1^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & -X \\ 0 & I_{n-r} \end{pmatrix}A^{\sim} \\ Q^{\sim}P^{\sim} &= (A^*)^{\sim}\begin{pmatrix} Q_1^* & 0 \\ -YQ_1^* & 0 \end{pmatrix}A^{\sim} \\ &= (A^*)^{\sim}\begin{pmatrix} I_r & 0 \\ -Y & I_{n-r} \end{pmatrix}\begin{pmatrix} Q_1^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ Y & I_{n-r} \end{pmatrix}A^{\sim} \end{aligned}$$

So $P^{\sim}Q^{\sim}$ and $Q^{\sim}P^{\sim}$ are similar.

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