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RESEARCH ARTICLE

ON δsg* - CLOSED SETS IN TOPOLOGICAL SPACES.

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Abstract

The aim of this paper is to introduce the class of δsg*-closed sets and obtain the characterizations of semi-weakly Hausdorff space. We also introduce the notion of δsg*-closure operator and δsg*-open sets and obtain some properties of them.

Introduction:-

In 1963, Levine introduced and studied the concepts of semi-open sets in topological spaces as a weaker form of open sets [13]. The study of generalized closed sets was initiated by Levine in order to extend the topological properties of closed sets to a larger family of sets in 1970 [12]. Dontchev et.al. [3], Sudha et.al. [17] and Meena et.al. [14] introduced various g-closed sets using δ-closed sets. In 2007, J.H.Park et.al. introduced and studied two concepts namely gδs -closed and δgs -closed sets using δ-semi closure and proved that the class of δgs-closed sets is weaker than the class of gδs -closed sets [7, 8]. In this paper, we introduce and study the class of δsg* -closed sets, which is weaker than the class of δ-semi closed sets and is stronger than the classes of gδs -closed sets and δgs -closed sets. Further, δsg* -closure operator and δsg* -open sets are introduced and their properties are obtained. We use δsg* -closed sets to obtain new characterization of semi-weakly Hausdorff spaces which are the spaces with semi-$T_{1/2}$-semi regularization [7].

Preliminaries:-

We list some definitions in a topological space (X, τ) which are useful in the following sections. The interior (δ-interior) and the closure (δ-closure) of a subset A of (X, τ) are denoted by int(A) (δ-int(A)) and cl(A) (δ-cl(A)) respectively. Throughout the present paper (X, τ) represents non-empty topological space on which no separation axiom is defined, unless otherwise mentioned.

Definition 2.1 A subset A of (X, τ) is called a
1. a semi open set [13] if $A \subseteq cl(int(A))$
2. a δ-open set [18] if it is a union of regular open sets
3. a δ-semi open [10] if $A \subseteq cl(\delta\text{-}int(A))$

The complement of a semi open (resp. δ-open, δ-semi open) set is called a semi closed (resp. δ-closed, δ-semi closed) set. The δ-semi interior of a subset A of (X, τ) is the union of all δ-semi open sets contained in A and is

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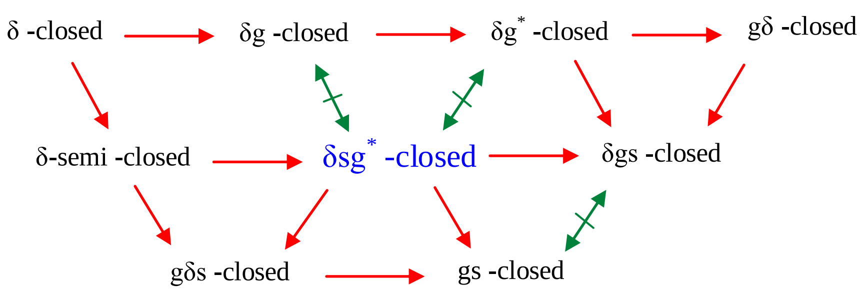
denoted by δ-sint(A) and the δ-semi closure of a subset A of (X, τ) is the intersection of all δ-semi closed sets containing A and is denoted by δ-scl(A).

Definition 2.2 A subset A of a space (X, τ) is said to be
(a) generalized closed (briefly g-closed) [12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(b) generalized semi-closed (briefly gs-closed) [1] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(c) generalized δ-semi-closed (briefly gδs-closed) [7] if $\delta\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(d) δ-generalized closed (briefly δg-closed) [3] if $\delta\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(e) generalized δ-closed (briefly gδ-closed) [4] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ).
(f) δ-generalized star -closed (briefly δg^* -closed) [4] if $\delta\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ).
(g) δ-generalized semi-closed (briefly δgs-closed) [8] if $\delta\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ-open in (X, τ).

Basic Properties of δsg^* - Closed Sets:-

Definition 3.1 [5] A subset A of a topological space (X, τ) is called ***δ semi generalized star -closed*** (briefly δsg^* -closed) ***set*** if $\delta\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in X. The class of all δsg^* -closed sets of (X, τ) is denoted by $\delta SG^*C(X, \tau)$.

Remark 3.2 For a subset of a topological space, from definitions stated above, we have the following diagram of implications:



where none of these implications is reversible as shown by examples in [3,4,9] and the following examples.

Example 3.3 [8] Let X ={a,b,c,d} and τ ={X,ϕ,{a},{c},{a,b},{a,c},{a,b,c},{a,c,d}}. Then
(a) Set A = {a,c,d}. Then A is δgs -closed but neither gs -closed nor gδs -closed in (X, τ).
(b) Set B = {b,c}. Then B is gs -closed but not δgs -closed in (X, τ).
(c) Set C = {c}. Then C is δ-semi closed (hence δgs -closed) but not δg^* -closed in (X, τ).

Example 3.4 Let X ={a,b,c,d} and τ ={X,ϕ,{a},{c},{a,b},{a,c},{a,b,c},{a,c,d}}.
(a) Set A = {b}. Then A is gδs -closed and gδ -closed but neither δg^* -closed nor δ-semi closed in (X, τ).
(b) Set B ={c}. Then B is δ-semi closed (hence δgs) but not gδ -closed in (X, τ).

Example 3.5 Let X ={a,b,c,d} and τ ={X,ϕ,{a},{b},{a,b},{a,b,c}}. Set A = {a,b,c}. Then A is δg^* -closed but not gδs -closed in (X, τ), since $A \in \tau$ but $\delta\text{-}scl(A)=X \not\subset A$.

Example 3.6 Let X = {a,b,c}, τ = {X,ϕ,{a}}. Then
(a) Set A = {b,c}. Then A is δsg^* -closed but not δ-semi closed.
(b) Set B = {a,b}. Then B is gδs -closed but not δsg^* -closed.
(c) Set C = {a,c}. Then C is gs -closed but not δsg^* -closed.
(d) Let X = {a,b,c}, τ = {X,ϕ,{a},{a,b}}. Then the set {b} is gs -closed but not gδs -closed.
(e) Let X = (a,b,c}, τ = {X,ϕ,{a},{b},{a,b}}. Then the set {a} is gδs -closed but not δg^*-closed.

The spaces in which the concepts of g-closed and closed sets are coincide are called *$T_{1/2}$ -spaces* [13]. $T_{1/2}$ -spaces are precisely the spaces in which singleton are open or closed. Now, we observe that in semi-regular $T_{1/2}$-spaces the notions of δsg^* -closed and gs -closed sets coincide. A space (X, τ) is T_d [2] (resp. T_b [2]) if every gs -closed set is g -closed (resp. Closed). A space with semi -$T_{1/2}$ semi -regularization is called *semi weakly Hausdorff* [7].

Theorem 3.7 Let A be a subset of a *$T_{1/2}$ -space* (X, τ) then:
(a) A is δsg^* -closed if and only if A is gδs -closed
(b) If (X, τ) is semi regular then A is δsg^* -closed if and only if A is gs -closed.
(c) If in addition, (X, τ) is T_b (resp. T_d) A is δsg^* -closed if and only if A is closed (resp. g-closed).

Proof:
(a) Let (X, τ) be $T_{1/2}$. Then g-open sets coincide with open sets which leads to (a).
(b) In a semi regular space gδs -closed sets coincide with gs -closed sets [Theorem 2.8 of [7]] Then (a) implies (b).
(c) The proof follows from Theorem 2.8 of [7] and from (a).

Definition 3.8 A ***partition space*** is a space where every open set is closed.

Remark 3.9 In a partition space open sets coincide with δ-open sets and the concepts of δ-closure and δ-semi closure coincide for any set.

Theorem 3.10 For a subset A of a *$T_{1/2}$ partition space* (X, τ) the following are equivalent:
(a) A is δsg^* -closed
(b) A is δg -closed
(c) A is δg^* -closed
(d) A is gδs -closed
(e) A is δgs -closed

Proof: (b) ⇔ (c) ⇔ (d) ⇔ (e) is proved in Theorem 2.6 of [7]
(a) ⇔ (b) In a $T_{1/2}$ -space, g-open sets coincide with open sets and hence by Remark 3.9 the proof follows.

The previous observation leads to the problem of finding the spaces (X, τ) in which the gs-closed sets of (X, τ_s) are δsg^* -closed in (X, τ). While we have not been able to completely resolve this problem, we offer partial solutions. For that reason the spaces with semi-$T_{1/2}$ semi-regularization is called *semi-weakly Hausdorff*. Recall that a space is called *almost weakly Hausdorff* [3] if its semi-regularization is $T_{1/2}$. Clearly almost weakly Hausdorff spaces are semi-weakly Hausdorff, but not conversely.

Example 3.11 [7] Let X = {a,b,c,d} with τ = {X,ϕ,{a},{b},{a,b}}. Then (X, τ) is clearly semi-weakly Hausdorff but not almost weakly Hausdorff.

Theorem 3.12 For a subset A of a *semi weakly Hausdorff space* (X, τ) the following are equivalent:
(a) A is gs -closed in (X, τ_s)
(b) A is δ-semi closed in (X, τ)
(c) A is δsg^* -closed in (X, τ).

Proof: (a) ⇒ (b) Let A ⊆ X be a gs -closed subset of (X, τ_s). Let $x \in$ δ-scl(A). If {x} is δ-semi open, then $x \in A$. If not then X \ {x} is δ-semi open, since X is semi weakly Hausdorff. Assume that $x \notin A$. Since A is gs-closed in (X, τ_s), then δ-scl(A) ⊆ X \ {x}, i.e. $x \notin$ δ-scl(A). By contradiction $x \in A$. Thus δ-scl(A) = A or equivalently A is δ-semi closed.
(b) ⇒ (c) The proof follows from the definition of δsg^* -closed sets.
(c) ⇒ (a) Let A ⊆ U, where U is open in (X, τ_s). Then U is δ -open in (X, τ). Every δ-open set is g-open. Since A is δsg^* -closed in (X, τ), δ-scl(A) ⊆ U. Hence by Lemma 7.3 of [15], scl(A) ⊆ U in (X, τ_s). Thus A is gs -closed in (X, τ_s).

Theorem 3.13 For a space (X, τ) the following are equivalent:
(a) Every g -open set of X is a δ-semi closed set
(b) Every subset of X is a δsg^* -closed set.

Proof: (a) ⇒ (b) Let A ⊆ U, where U is g-open and A is an arbitrary subset of X. By (a), U is δ-semi closed and thus δ-scl(U) ⊆ U. Thus δ-scl(A) ⊆ δ-scl(U) ⊆ U. Hence A is δsg^* -closed.
(b) ⇒ (a) Let U be a g-open set of (X, τ), then by (b) δ-scl(U) ⊆ U or equivalently U is δ-semi closed.

Remark 3.14
(a) Every finite union of δsg^* -closed sets may fail to be a δsg^*-closed set.
(b) Every finite intersection of δsg^* -closed sets may fail to be a δsg^*-closed set.

The following examples support the above remark.

Example 3.15 Let X = {a, b, c}, τ = {X, ϕ, {a}, {b}, {a,b}}. Consider A = {a} and B = {b} then A and B are δsg^* -closed sets but A∪B = {a,b} is not a δsg^* -closed set in (X, τ).

Example 3.16 Let X = {a, b, c, d}, τ = {X, ϕ, {c}, {a,b}, {a,b,c}}. Consider A = {a,b} and B = {a,d} then A and B are δsg^* -closed sets but A∩B = {a} is not a δsg^* -closed set in (X, τ).

Definition 3.17 A topological space (X, τ) is called an ***R_1* -space** if every two different points with distinct closures have disjoint neighborhoods.

Theorem 3.18 For a compact subset A of an *R_1 -topological space* (X, τ) the following conditions are equivalent, when (X, τ) is also $T_{1/2}$.
(a) A is a δsg^* -closed set
(b) A is a gs -closed set

Proof:
(a) ⇒ (b) is clear.
(b) ⇒ (a) Let A ⊆ U, where U is g-open in (X, τ). In a $T_{1/2}$ -space g -open sets coincide with open sets. In R_1-spaces the concepts of closure and δ -closure coincide for compact sets [Theorem 3.6 in [6]]. Thus the rest of the proof follows from the definition of δsg^* -closed sets.

Corollary 3.19 In Hausdorff spaces, a finite set is gs -closed if and only if it is δsg^* -closed.

Theorem 3.20 The intersection of a δsg^* -closed set and a δ -closed set is δsg^* -closed.

Proof: Let A be δsg^* -closed and F be δ -closed. If U is g -open in (X, τ) with A ∩ F ⊆ U, then A ⊆ U ∪ (X \ F) and thus δ-scl(A) ⊆ U ∪ (X \ F). Since every δ -closed set is δ-semi closed, δ-scl(F) ⊆ δ-cl(F) = F and thus, we have δ-scl(A∩F) ⊆ δ-scl(A) ∩ F ⊆ U and hence A∩F is δsg^* -closed.

Definition 3.21 A subset A of (X, τ), a point x∈X is called a ***δ-limit point*** (resp. δ-semi limit point [11]) of A if every δ -open (resp. δ-semi open) set containing x contains a point of A different from x. The set of all δ-limit points (resp. δ-semi limit points) of A is called the δ-derived set (resp. δ-semi derived set [11]) of A and is denoted by δ-D(A) (resp. δ-D_s(A)).

Theorem 3.22 If A and B are δsg^* -closed sets of X such that δ-D(A) ⊆ δ-D_s(A) and δ-D(B) ⊆ δ-D_s(B), then A∪B is δsg^* -closed.

Proof: Since δ-D_s(C) ⊆ δ-D(C) for any subset C of X, by hypothesis δ-D_s(A) = δ-D(A) and δ-D_s(B) = δ-D(B), i.e. δ-cl(A) = δ-scl(A) and δ-cl(B) = δ-scl(B). Let A∪B ⊆ U, where U is g-open in X. Then A ⊆ U and B ⊆ U. Since A and B are δsg^* -closed, δ-scl(A) ⊆ U and δ-scl(B) ⊆ U then δ-cl(A) ⊆ U and δ-cl(B) ⊆ U. Therefore δ-cl(A) ∪ δ-cl(B) ⊆ U, δ-cl(A∪B) ⊆ U. But, δ-scl(A∪B) ⊆ δ-cl(A∪B) and hence δ-scl(A∪B) ⊆ U. Therefore A∪B is δsg^* -closed.

Theorem 3.23 Let A be a subset of (X, τ). Then we have
(a) If A is δsg^* -closed in X, then δ-scl(A) \ A does not contain any non empty closed set.
(b) If A is δsg^* -closed in X and A ⊆ B ⊆ δ-scl(A), then B is also a δsg^* -closed set.

Proof: (a) Let F be a closed set such that F ⊆ δscl(A) \ A. Then A ⊆ X \ F. Since A is a δsg^*-closed set and X \ F is open and hence g-open, δ-scl(A) ⊆ X \ F, i.e. F ⊆ X \ δ-scl(A). Hence F ⊆ δ-scl(A) ∩ (X \ δ-scl(A)) = ϕ. This shows that F = ϕ.
(b) Let U be a g-open set of X such that B ⊆ U then A ⊆ U. Since A is a δsg^* -closed set, δ-scl(A) ⊆ U. Also since B ⊆ δ-scl(A), δ-scl(B) ⊆ δ-scl(δ-scl(A)) = δ-scl(A). Hence δ-scl(B) ⊆ U. Therefore B is also a δsg^*-closed set.

Theorem 3.24 If A is δsg^* -closed in (X, τ) if and only if δ-scl(A) \ A does not contain any non empty g-closed set in (X, τ).

Proof: Necessity - Suppose that A is δsg^* -closed, let F be any g-closed such that F ⊆ δ-scl(A) \ A. Then A ⊆ X \ F and X \ F is g-open in (X, τ). Since A is δsg^*-closed set in (X, τ), δ-scl(A) ⊆ X \ F. Thus, F ⊆ X \ δ-scl(A). Therefore, F ⊆ (δ-scl(A) \ A) ∩ (X \ δ-scl(A)) = ϕ. Hence F = ϕ.

Sufficiency - Suppose that A ⊆ U and U is any g-open set in (X, τ). If A is not a δsg^*-closed set, then δ-scl(A) ⊆ U and hence δ-scl(A) ∩ (X \ U) ≠ ϕ. We have a non empty g-closed set δ-scl(A) ∩ (X \ U) such that δ-scl(A) ∩ (X \ U) ⊆ δ-scl(A) ∩ (X \ A) = δ-scl(A) \ A which contradicts the hypothesis.

Corollary 3.25 If A is a δsg^*-closed subset of (X, τ), then A is δ-semi closed if and only if δ-scl(A) \ A is g-closed.

Theorem 3.26 If A is g-open and δsg^* -closed in (X, τ), then A is a δ-semi closed set of X.

Proof: If A is g-open and δsg^* -closed. Let A ⊆ A, where A is g-open and δ-scl(A) ⊆ A which implies δ-scl(A) = A. Hence A is δ-semi closed.

Theorem 3.27 Let A ⊆ Y ⊆ X. Then
(a) If Y is open in (X, τ) and A is δsg^* -closed in X, then A is δsg^* -closed relative to Y.
(b) If Y is δsg^* -closed and g-open in (X, τ) and A is δsg^* -closed relative to Y, then A is δsg^* -closed in X.

Proof: (a) Let A ⊆ Y∩G, where G is g-open. Since A is δsg^* -closed in (X, τ), δ-scl(A) ⊆ Y∩G ⊆ G, which implies Y∩δ-scl(A) ⊆ Y∩G which is g-open. Therefore Y∩δ-scl(A) ⊆ G. Then A is δsg^* -closed relative to Y, as Y∩δ-scl(A) is the δ-scl(A) relative to Y. That is Y∩δ-scl(A) = $\delta\text{-scl}_Y(A)$.
(b) Let G be a g-open subset of (X, τ) such that A ⊆ G. Then A ⊆ G∩Y. Since A is δsg^* -closed relative to Y, then δ-scl(A) ⊆ G∩Y, i.e. δ-scl(A)∩Y ⊆ G∩Y from Theorem 4.2.25 of [16] and Theorem 3.26, δ-scl(A) = δ-scl(A∩Y) = δ-scl(A) ∩ δ-scl(Y) = δ-scl(A) ∩ Y. Therefore δ-scl(A) ⊆ G∩Y ⊆ G. Hence A is δsg^* -closed in X.

δsg^* - Open Sets:-

Definition 4.1 A subset A of (X, τ) is called ***δsg^** -open** if its complement X \ A is δsg^* -closed.

Lemma 4.2 For a subset A of (X, τ), δ-scl(X \ A) = X \ δ-sint(A).

Theorem 4.3 A subset A of (X, τ) is δsg^* -open if and only if G ⊆ δ-sint(A) whenever A ⊇ G and G is g -closed.

Proof: Assume that A is δsg^* -open. Then X \ A is δsg^* -closed. Let G be a g -closed set in (X, τ) contained in A. Then X \ G is a g -open set in (X, τ) containing X \ A. Since X \ A is δsg^* -closed, δ-scl(X \ A) ⊆ X \ G, equivalently G ⊆ δ-sint(A).

Conversely assume that G ⊆ δ-sint(A), whenever G ⊆ A and G is g -closed in (X, τ). Let X \ A ⊆ F, where F is g -open. Then X \ F ⊆ A. By criteria, X \ F ⊆ δ-sint(A). This implies δ-scl(X \ A) ⊆ F. Thus X \ A is δsg^* -closed. Hence A is δsg^* -open.

Theorem 4.4 If δ-sint(A) ⊆ B ⊆ A and A is δsg^* -open in (X, τ), then B is δsg^* -open in (X, τ).

Proof: Follows from Lemma 4.2 and Theorem 3.23(b).

Theorem 4.5 If A is δsg^* -open in (X, τ) and F is g -open such that δ-sint(A) ∪ (X \ A) ⊆ F then F = X.

Proof: Let F be a g -open set such that δ-sint(A) ∪ (X \ A) ⊆ F. Then (X \ F) ⊆ (X \ δ-sint(A)) ∩ A, i.e. (X \ F) ⊆ δ-scl(X \ A) \ (X \ A). Since (X \ A) is δsg^* -closed, by Theorem 3.24, X \ F = ϕ and hence F = X.

Definition 4.6 A point x∈X is called the δsg^* -cluster point of A if A∩U ≠ ϕ for every δsg^* -open set U of X containing x. The set of all δsg^* -cluster points of A is called the ***δsg* -closure of A***, denoted by δsg^*Cl(A) and the δsg^* -interior of A, denoted by δsg^*Int(A), is defined as the union of all δsg^* -open sets contained in A.

Corollary 4.7 Let A be a subset of a topological space (X, τ). Then the following properties hold.
(a) $\delta sg^*Cl(A) = \cap \{F \in \delta SG^*C(X, \tau) : A \subset F\}$
(b) $\delta sg^*Cl(\delta sg^*Cl(A)) = \delta sg^*Cl(A)$

Remark 4.8 For a subset A of (X, τ), A ⊆ gscl(A) ⊆ δsg^*cl(A) ⊆ δ-scl(A) ⊆ δ-cl(A).

Theorem 4.9 Let A be any subset of (X, τ). If A is δsg^* -closed in (X, τ) then δsg^*cl(A) = A.

Proof: Let A be δsg^* -closed in (X, τ). By definition, δsg^*cl(A) = ∩ {F ⊆ X : A ⊆ F and F is δsg^*-closed in (X, τ)}. Since A is δsg^* -closed, F in the above intersection is A and hence δsg^*cl(A) = A.

Remark 4.10 δsg^* -closure of a set need not be a δsg^* -closed set. Let X = {a,b,c,d} and τ = {X,ϕ,{c},{a,b},{a,b,c}}

δsg^*cl{a} = ∩ {all δsg^* -closed sets containing {a}}
= {a,b}∩{a,d}∩{a,b,d}∩{a,c,d}∩X
= {a} ≠ δsg^* -closed set

Theorem 4.11 For any two subsets A and B of (X, τ). Then the following statements are true:
(a) $\delta sg^*cl(\phi) = \phi$ and $\delta sg^*cl(X) = X$
(b) If A ⊆ B, then $\delta sg^*cl(A) \subseteq \delta sg^*cl(B)$
(c) $\delta sg^*cl(A) \cup \delta sg^*cl(A) \subseteq \delta sg^*cl(A \cup B)$
(d) $\delta sg^*cl(A \cap B) = \delta sg^*cl(A) \cap \delta sg^*cl(B)$

Proof:
(a), (b) follow from Definition 4.6.
(c) Since A ⊆ A∪B and B ⊆ A∪B, by (b), $\delta sg^*cl(A) \subseteq \delta sg^*cl(A\cup B)$ and $\delta sg^*cl(B) \subseteq \delta sg^*cl(A\cup B)$. Hence $\delta sg^*cl(A) \cup \delta sg^*cl(B) \subseteq \delta sg^*cl(A\cup B)$.
(d) Since A∩B ⊆ A and A∩B ⊆ B, by (b), $\delta sg^*cl(A\cap B) \subseteq \delta sg^*cl(A)$ and $\delta sg^*cl(A\cap B) \subseteq \delta sg^*cl(B)$. Hence $\delta sg^*cl(A\cap B) \subseteq \delta sg^*cl(A) \cap \delta sg^*cl(B)$. Conversely, $\delta sg^*cl(A) \cap \delta sg^*cl(B) = [\cap\{F \subseteq X \setminus A \subseteq F \text{ and } F \in \delta sg^*C(X, \tau)\}] \cap [\cap\{F \subseteq X \setminus B \subseteq F \text{ and } F \in \delta sg^*C(X, \tau)\}] \subseteq \cap\{F \subseteq X \setminus A\cap B \subseteq F \text{ and } F \in \delta sg^*C(X, \tau)\} = \delta sg^*cl(A\cap B)$.

The converse part of inequality in (c) is disproved in the following example.

Example 4.12 Let X = {a,b,c,d}, τ = {X, ϕ ,{c},{a,b},{a,b,c}}. Let A = {a,b}, B = {c} and A∪B = {a,b,c} then δsg^*cl{a,b} = {a,b}, δsg^*cl{c} = {c} and δsg^*cl{a,b,c} = X. Hence δsg^*cl(A∪B) = X but not contained in δsg^*cl(A) ∪ δsg^*cl(B).

Remark 4.13 As intersection of δsg^* -closed sets is not a δsg^* -closed set, δsg^*cl(A) need not be the smallest δsg^* -closed set containing A as seen in the following example.

Example 4.14 Let X = {a,b,c,d} and τ = {X, ϕ ,{c},{a,b},{a,b,c}}

δsg^*cl{b} = ∩ {all δsg^* -closed sets containing {b}}
= {a,b}∩{b,d}∩{a,b,d}∩{b,c,d}∩X
= {b} ≠ smallest δsg^* -closed set containing {b}.

Remark 4.15 The collection of δ-semi open sets is denoted as $\tau_{\delta s}$.

Definition 4.16 Let U be any subset of (X, τ). Using δsg^* -closure operator, a new class of sets denoted by $\delta sg^*\tau^\#$ is defined as follows.

$$\delta sg^*\tau^\# = \{U : \delta sg^*cl(X \setminus U) = X \setminus U\}$$

Proposition 4.17 For any topology τ, we have $\tau_\delta \subseteq \tau_{\delta s} \subseteq \delta sg^*\tau^\#$.

Proof: Obvious from Remark 4.8 and Definition 4.16.

Lemma 4.18 For any $A \subseteq X$, $A \subseteq \delta sg^*cl(A) \subseteq \delta\text{-}scl(A) \subseteq \delta\text{-}cl(A)$.

Proof: It follows from Lemma 3.4 and Proposition 3.2 of [5].

Remark 4.19 The relations in Lemma 4.18 may be proper as seen from the following example.

Example 4.20 Let X = {a, b, c} and τ = {X, ϕ, {a}, {b}, {a,b}}. Let A = {a} then $\delta sg^*cl(A)$ = {a}, δ-scl(A) = {a}, δ-cl(A) = {a,c} and so $A \subseteq \delta sg^*cl(A) \subseteq \delta\text{-}scl(A) \subseteq \delta\text{-}cl(A)$.

Definition 4.21 The intersection of all g -open subsets of X containing A is called the ***g-kernel of A*** and is denoted by g-ker(A).

i.e., $g\text{-}ker(A) = \cap \{U / U \text{ is g -open in } (X, \tau) \text{ and } A \subseteq U\}$

Theorem 4.22 Every singleton is either g -closed or δsg^* -open in (X, τ).

Proof: Let $a \in X$. Suppose that {a} is not g -closed in X, then X \ {a} is not g -open and the only g -open set containing X \ {a} is the space X itself. That is $X \setminus \{a\} \subseteq X$. Therefore $\delta\text{-}scl(X \setminus \{a\}) \subseteq X$ and X \ {a} is δsg^* -closed and hence {a} is δsg^* -open.

Note 4.23 Theorem 4.22 gives a decomposition for (X, τ) as $X = X_1 \cup X_2$ where $X_1 = \{x \in X / \{x\} \text{ is g -closed}\}$ and $X_2 = \{x \in X / \{x\} \text{ is } \delta sg^* \text{ -open}\}$

Theorem 4.24 For a subset A of (X, τ), the following properties are equivalent:

(a) A is δsg^* -closed

(b) $\delta\text{-}scl(A) \subseteq g\text{-}ker(A)$

(c) (i) $\delta\text{-}scl(A) \cap X_1 \subseteq A$

(ii) $\delta\text{-}scl(A) \cap X_2 \subseteq g\text{-}ker(A)$

Proof:

(a) ⇒ (b) Let $x \notin g\text{-}ker(A)$. Then there exist a set $U \in GO(X, \tau)$ such that $A \subseteq U$ and $x \notin U$. Since A is δsg^* -closed, $\delta\text{-}scl(A) \subseteq U$ and $x \notin \delta\text{-}scl(A)$.

(b) ⇒ (a) Let U be a g-open set containing A, then by Definition 4.21, $g\text{-}ker(A) \subseteq U$. By (b), $\delta\text{-}scl(A) \subseteq g\text{-}ker(A)$. Therefore $\delta\text{-}scl(A) \subseteq U$. Hence A is δsg^* -closed.

(b) ⇒ (c) (i) Let $x \in \delta\text{-}scl(A) \cap X_1$ then $x \in \delta\text{-}scl(A)$ and $x \in X_1$. Consider $x \in \delta\text{-}scl(A)$ then by (b), $x \in g\text{-}ker(A)$. Now consider $x \in X_1$ then {x} is g -closed. Suppose if $x \notin A$ and say U = X \ {x} then U is g-open and $A \subseteq U$. i.e. U is a g-open set containing A. By Definition 4.21, $g\text{-}ker(A) \subseteq U$. Already $x \in g\text{-}ker(A)$ which implies $x \in U$. But this is a contradiction to U = X \ {x}. ∴ $x \in A$.

(ii) Always $\delta\text{-}scl(A) \cap X_2 \subseteq \delta\text{-}scl(A)$. By (b), $\delta\text{-}scl(A) \subseteq g\text{-}ker(A)$. Hence $\delta scl(A) \cap X_2 \subseteq g\text{-}ker(A)$.

(c) ⇒ (b) $\delta\text{-}scl(A) = \delta\text{-}scl(A) \cap X = \delta\text{-}scl(A) \cap [X_1 \cup X_2] = [\delta\text{-}scl(A) \cap X_1] \cup [\delta\text{-}scl(A) \cap X_2] \subseteq A \cup g\text{-}ker(A)$ by (c). Therefore $\delta\text{-}scl(A) \subseteq g\text{-}ker(A)$.

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