Journal Homepage: - www.journalijar.com
INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)
Article DOI: 10.21474/IJAR01/3475
DOI URL: http://dx.doi.org/10.21474/IJAR01/3475
ISSN NO. 2320-5407

RESEARCH ARTICLE

Homogeneous Finsler Square Metrics of Douglas Type.

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Manuscript Info

Manuscript History

Received: 31 January 2017
Final Accepted: 05 March 2017
Published: March 2017

Key Words: Homogeneous Finsler space, Square Metrics, Randers type and Douglas Metrics.

Abstract:

In this paper, we study homogenous Finsler square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ of Douglas type, and we investigate the necessary and sufficient conditions for the homogenous Finsler square metric to be Douglas metric, then if has following properties:

(1) it is a Berwald metric or Randers type, and
(2) it is a Riemannian metric.

1. **Introduction**

A Finsler metric F on a manifold M is a homogeneous continuous function $F : TM \rightarrow [0; +\infty)$ where F is smooth on the slit tangent bundle TM_o satisfying nonnegativity ($F(y) > 0$ for any $y \neq 0$) and strong convexity (the fundamental tensor $g_{ij} := [\frac{1}{2}F^2]_{y^iy^j}$ is positive defnite on TM_o). Here $(x^i; y^i)$ denote the natural system of coordinates of TM. [1]

The notion of (α, β)-metric in Finsler spaces was introduced by M. Matsumoto [4] as a generalization of Randers metric $L = \alpha + \beta$, where α is a regular Riemannian metric $\alpha = a_{ij}(x)y^iy^j$, i.e., $det(a^{ij}) \neq 0$ and β is a one-form $\beta = b_i(x)y^i$ and studied by many authors ([5], [6], [8], and [9]). A Finsler metric $L(\alpha, \beta)$ on a differentiable manifold M^n is called an (α, β)-metric, if L is a positively homogeneous function of degree one in α and β. There are several important (α, β)-metrics, namely Randers metric $L = \alpha + \beta$ Kropina metric $L = \frac{\alpha^2}{\beta}$, Matsumoto metric $L = \frac{\alpha^2}{(\alpha-\beta)}$, generalized Kropina metric $L = \frac{\alpha^{n+1}}{\alpha^n}$ and Z. Shen's square metric $L = \frac{(\alpha+\beta)^2}{\alpha}$.

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In 1929, L. Berwald constructed the following famous Finsler metric[10]

$$F = \frac{(\sqrt{(1-|x|^2)|y|^2 + <x,y>^2} + <x,y>)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + <x,y>^2}}.$$

This metric, defined on the unit ball $B^n(1)$ with all the straight line segments as its geodesics, has constant flag curvature $K = 0$. In a modern point of view, Berwald's metric belongs to a special kind of Finsler metrics called Berwald type or square metrics given as the form

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{1.1}$$

where α is a Riemannian metric and β is a 1-form[1]. It is known that above equation is a regular Finsler metric if and only if the length of β with respect to α, denoted by b, satisfies $b < 1$.

(α, β)-metrics form an important class of Finsler metrics that can be expressed in the form

$$F = \alpha\phi(\frac{\beta}{\alpha}),$$

where $= \sqrt{a_{ij}(x)y^i y^j}$ is a Riemann metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha < b_0$ on a manifold. It is well known that $F = \alpha\phi(\frac{\beta}{\alpha})$, is a positive definite Finsler metric if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition[3]

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0. \tag{1.2}$$

In 1927, J. Douglas introduced the Douglas curvature for Finsler metrics[11]. Douglas curvature is an important projectively invariant in Finsler geometry. It it also a non-Riemannian quantity, since all the Riemannian metrics have vanishing Douglas curvature inherently. Finsler metrics with vanishing Douglas curvature are called Douglas metrics. Roughly speaking, a Douglas metric is a Finsler metric which is locally projectively equivalent to a Riemannian metric[12].

Douglas metrics form a rich class of Finsler metrics including locally projectively at Finsler metrics and Berwald metrics, the later are those metrics whose Berwald curvature vanishes[7].

In this present article, we study homogenous Finsler square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ of Douglas type, and we investigate the necessary and sufficient conditions for the homogenous Finsler square metric to be Douglas metric, then if has following properties:

(1) it is a Berwald metric or Randers type, and
(2) it is a Riemannian metric.

2. **Preliminaries**

Definition 2.1. *A locally projectively flat (α, β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$ is said to be trivial, if α is locally projectively flat and β is parallel with respect to α.*

A result of Z. Shen and G. Civi Yildirim [8] says that a Berwald's metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ is locally projectively flat if and only if the spray coefficients of α are given in an adapted coordinate system by

$$G^i_\alpha = \xi y^i - 2\tau\alpha^2 b^i, \tag{2.1}$$

for some 1-form $\xi = \xi_i(x)y^i$ and some scalar function $\tau = \tau(x)$, and at the same time the covariant derivative of β is given by

$$b_{i|j} = 2\tau\{(1+2b^2)a_{ij} - 3b_i b_j\}. \tag{2.2}$$

Later on, B. Li, Z. Shen and Y. Shen, found a sufficient and necessary condition for (α, β)-metrics to be locally projectively flat in dimension $n \ge 3$ [12]. It says that for a projectively flat (α, β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$ on an open subset $U \subseteq R^n$ with $n \ge 3$, if we add

Theorem 2.1. *Let $s = \frac{\beta}{\alpha}$ and let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric on an open subset $U \subseteq R^n (n \geq 3)$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i(x) y^i \neq 0$. Let $b := \|\beta\|_\alpha$. Suppose that the following conditions hold:*

- *F is not of Randers type, i.e.,$F \neq \sqrt{c_1\alpha^2 + c_2\beta^2} + c_3\beta$ for any constants c_1, c_2 and c_3,*
- *β is not parallel with respect to α,*
- *$db \equiv 0$ or $db \neq 0$ everywhere or b is constant on U.*

Then F is a Douglas metric if and only if $\phi(s)$ satisfies the following ODE,

$$\{1 + (k_1 + k_3)s^2 + k_2 s^4\}\phi''(s) = (k_1 + k_2 s^2)\{\phi(s) - s\phi'(s)\}, \tag{2.3}$$

where k_1, k_2, k_3 are constants with $k_2 \neq k_1 k_3$ and the covariant derivative $\nabla\beta = b_{i|j} y^i dx^j$ of β with respect to α satisfies equation (2.2).

We can see that the function $\phi(s) = (1 + s)^2$ satisfies equation (2.3) with $(k_1, k_2, k_3) = (2, 0, -3)$ or $(k_1, k_2, k_3) = (-3, 0, 2)$.

We used the following results which proved by G. Yang in [18].

Theorem 2.2. *Let $F = \alpha\phi(s), s = \beta/\alpha$ be a regular (α, β)-metric on an open subset $U \subset R^2$, where $\phi(0) = 1$. Suppose that β is not parallel with respect to α and F is not of Randers type. Let F be a Douglas metric. Then one has one of the following two cases.*

- *$\phi(s)$ satisfies (2.3).*
- *F can be written as*

$$F = \widetilde{\alpha} \pm \frac{\widetilde{\beta^2}}{\widetilde{\alpha}}, \quad (\widetilde{\alpha} := \sqrt{\alpha^2 - k\beta^2}, \quad \widetilde{\beta} := c\beta), \tag{2.4}$$

where k, c are constants with $c \neq 0$

Corollary 2.1. *Let $F = \alpha \pm \beta^2/\alpha$ be a two-dimensional (α, β)- metric. Then F is a Douglas metric if and only if β satisfies*

$$r_{ij} = 2\tau\{(1 \pm 2b^2)a_{ij} \mp 3b_i b_j\} + \frac{3}{\pm 1 - b^2}(b_i s_j + b_j s_i), \tag{2.5}$$

where $\tau = \tau(x)$ is a scalar function. Note that

- *$F = \alpha + \frac{\beta^2}{\alpha}$ is positive if and only if $b^2 < 1$,*
- *$F = \alpha - \frac{\beta^2}{\alpha}$ is positive if and only if $b^2 < \frac{1}{2}$.*

Every Finsler metric F on a manifold M induced a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ which determines the geodesics. By definition, a Finsler metric F is a Berwald metric if the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_x M$ at every point x, i.e., $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$. Riemannian metrics are special Berwald metrics. In fact, Berwald metrics are almost Riemannian in the sense that every Berwald metric is affinely equivalent to a Riemannian metric, i.e., the geodesics of any Berwald metric are the geodesics of some Riemannian metric [2]. The Douglas metrics are more generalized ones than Berwald metrics. A Finsler metric is called a Douglas metric if the spray coefficients $G^i = G^i(x, y)$ are in the following form:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i. \tag{2.6}$$

Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. The study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities.

Definition 2.2. *Two (α, β)-metrics $F_1 = \alpha_1\phi_1(\frac{\beta_1}{\alpha_1})$ and $4F_2 = \alpha_2\phi_2(\frac{\beta_2}{\alpha_2})$ are said to be of same type if there is an element $\Pi \in G$ such that $\Pi(\phi_1) = \phi_2$. In this case, the functions $\phi_1(s)$ and $\phi_2(s)$ are said to be equivalent. G is called the representation group of (α, β)-metrics.*

For example, all the functions equivalent to $(1 + s)$ will provide Randers type metrics. Conversely, if $F = \alpha\phi(\frac{\beta}{\alpha})$ is of Randers type, then $\phi(s)$ must be equivalent to $(1 + s)$. Actually, the functions for Randers type metrics, which are given by $\phi(s) = \sqrt{1 + us^2} + vs$, can be expressed as $\phi(s) = g_u \circ h_v(1 + s)$. Notice that all the functions are always asked to satisfy $\phi(0) = 1$. Suppose that a given locally projectively flat (α, β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$ is neither locally Minkowskian nor of Randers type, then $\phi(s)$ must be a solution of (2.3) according to Z. Shen's result. Due to the non-uniqueness, if we rewrite the metric as $F = \widetilde{\alpha}\psi(\frac{\widetilde{\beta}}{\widetilde{\alpha}})$, then the new function $\psi(s)$, which is equivalent to $\phi(s)$, must be also a solution of (2.3) with some different parameters.

Theorem 2.3. *A Finsler metric F on a manifold $M(dimM \geq 3)$ is locally projectively flat if and only if F is a Douglas metric with scalar flag curvature.*

In [15] shows an (α, β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$ is a Berwald metric if and only if β is parallel with respect to α, i.e., $b_{i|j} = 0$, regardless of the choice of a particular ϕ, and [12] the authors obtained a characterization of (α, β)-metrics of Douglas type.

3. Homogeneous Finsler Square Metrics Of Douglas Type

Recall that the group $I(M; F)$ of isometries of a Finsler manifold $(M; F)$is a Lie transformation group of M [14]. If $I(M; F)$ acts transitively on M, then $(M; F)$ is called homogeneous. Thus, the homogeneous Finsler manifold M can be written as the form $M = G/H$, where G is a Lie group acting isometrically and transitively on M, and H is the isotropy subgroup at a point in M. Moreover, if the Lie algebra of G, g, has a decomposition

$$g \quad = \quad \eta + m \quad (direct \quad sum \quad of \quad subspaces),$$

where η is the Lie algebra of H and m is a subspace of g satisfying

$$Ad(h)(m) \quad \subset \quad m \quad for \quad all \quad h \in H.$$

Then the homogeneous Finsler manifold $(G/H; F)$ is called reductive. In this case, the tangent space $T_o(G/H)$, where $o = eH$ is the origin, can be canonically identified with m. Note that the isotropy subgroup $I_x(M, F)$ of $I(M; F)$ at a point $x \in M$ is compact [14], and M can be written as

$$M \quad = \quad I(M; F)/I_x(M, F).$$

Then $M = I(M; F)/I_x(M, F)$ is a reductive homogeneous manifold.

Let $(G/H; F)$ be a reductive homogeneous (α, β)-space of the form $F = \alpha\phi(s)$ where $s = \frac{\beta}{\alpha}$ with a Riemannian metric α and a 1-from β on G/H . In this section, we assume that α and β are both G-invariant. Consider the underlying homogeneous Riemannian manifold $(G/H; \alpha)$. Let $< ., . >$ denote the corresponding inner product on m. According to [16], the form β corresponds to a vector u in the subspace

$$V \quad = \quad \{u \in M | Ad(h)u = u \quad for \quad all \quad h \in H\}.$$

We also assume that $\beta \neq 0$, or equivalently, $u \neq 0$.

We use the local coordinate system developed in [17], which is given as follow.

Let $u_1, u_2, ..., u_n$ be an orthonormal basis of m with respect to the inner product $< .,. >$ and $u_n = \frac{u}{|u|}$. Then there exists a neighborhood U of $o = H$ in G/H such that the map

$$(exp(x^1 u_1) exp(x^2 u_2) ... exp(x^n u_n)) \longmapsto (x^1, x^2, ..., x^n),$$

defines a local coordinate system on U.

Since α and β are both G-invariant, $b := \|\beta\|_\alpha = |u|$ is a constant. By [17], at the origin $o = H$, we have

$$\begin{aligned} a_{ij} &= \delta_{ij}, \quad b_i = b\delta_{ni}, \\ b_{i|j} &= \frac{b}{2}(\langle [u_i, u_j], u_n\rangle - \langle [u_n, u_i], u_j\rangle - \langle [u_n, u_j], u_i\rangle), \\ s_{ij} &= \frac{b}{2}\langle [u_i, u_j], u_n\rangle, \quad r_{ij} = -\frac{b}{2}(\langle [u_n, u_i], u_j\rangle + \langle [u_n, u_j], u_i\rangle). \end{aligned} \tag{3.1}$$

Note that $s_n = b^i s_{in} = b s_{nn} = 0$.

Theorem 3.4. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a homogeneous Finsler square metric on G/H. Then F is a Douglas metric if and only if F is a Berwald metric or F is a Douglas metric of Randers type.*

Proof. Suppose that $F = \alpha\phi(\frac{\beta}{\alpha})$ is a homogeneous (α, β)-metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ on G/H, where the Riemannian metric α and the 1-from β are both G-invariant. We only need to compute at the origin $o = H$. By (3.1), it is obvious that $b_{n|n} = 0$ at the origin. We now prove the theorem in the following two cases.

Case 1: $dim(G/H) \geq 3$. Suppose that F is a Douglas metric, and F is neither a Berwald metric nor of Randers type. Since b is a constant, it follows from Theorem 2.3 and (2.2) that

$$b_{n|n} = 2\tau\{1 - b^2\} = 0. \tag{3.2}$$

Since α and β are both G-invariant, the scalar function $\tau = \tau(0)$ is a constant.

By the assumption that F is not a Berwald metric, it follows that $\tau \neq 0$. So we have

$$(1 - b^2) = 0. \tag{3.3}$$

By (2.3), we have

$$\phi''(s) = \{\phi(s) - s\phi'(s)\}\frac{k_1 + k_2 s^2}{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2}. \tag{3.4}$$

Plugging (3.4) into (1.2), we get

$$\begin{aligned} \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) &= \{\phi(s) - s\phi'(s)\}\{1 + \frac{b^2 - s^2) + (k_1 + k_2 s^2)}{1 + (k_1 + k_3)s^2 + k_2 s^4}\} \\ &= \{\phi(s) - s\phi'(s)\}\{1 + \frac{1 + k_1 b^2 + (k_2 b^2 + k_3)s^2}{1 + (k_1 + k_3)s^2 + k_2 s^4}\}. \end{aligned} \tag{3.5}$$

By taking $b = s$ in (1.2) we can see that $\phi(s) - b\phi'(s) > 0$, is always positive as long as F is a Finsler metric. So the (1.2) implies the following

$$\frac{1 + k_1 b^2 + s^2(k_3 + k_2 b^2)}{1 + (k_1 + k_3)s^2 + k_2 s^4} > 0, \forall |s| \leq b < b_0. \tag{3.6}$$

Taking $s = 0$ we have $1 + k_1 b^2 > 0$ than $1 + k_1 s^2 \geq min\{1, 1 + k_1 b^2\}$.

On other hand, we have $(k_1, k_2, k_3) = (2, 0, -3)$ and by taking $s = b$ in (2.3), then

$$\begin{aligned} \phi''(s) &= \{\phi(s) - s\phi'(s)\}\frac{2 + 0s^2}{1 + (2 + 0s^2)s^2 - 3s^2}, \\ &= \{\phi(s) - s\phi'(s)\}\frac{2}{1 - s^2}. \end{aligned} \tag{3.7}$$

And when $(k_1, k_2, k_3) = (-3, 0, 2)$ we have

$$\begin{aligned}\phi''(s) &= \{\phi(s) - s\phi'(s)\}\frac{-3+0s^2}{1+(-3+0s^2)s^2+s^2}, \\ &= \{\phi(s) - s\phi'(s)\}\frac{-3}{1+s^2}. \end{aligned} \tag{3.8}$$

It is clear that the solution of (3.7) and (3.8) is given by

$$\phi(s) = \frac{c}{1 \mp s^2}, \tag{3.9}$$

for some constant c. This implies that is of Randers type, which is a contradiction. This completes our proof in this case.

Case 2: $dim(G/H) = 2$. By Theorems 2.1 and 2.2, we only need to prove the theorem under the assumption that F is given by $F = \alpha \pm \frac{\beta^2}{\alpha}$, where the Riemannian metric α and the 1-from β are both G-invariant. We will also use the above local coordinate system and setting $n = 2$. Note that $b_2 = b$ and $s_2 = 0$. By Theorem 2.2,

- if $F = \alpha + \frac{\beta^2}{\alpha}$ is a Douglas metric, then at the origin $o = H$, we have
$$r_{22} = b_{22} = 2\tau(o)(1 - b^2).$$
Since $b^2 < 1$ when $F = \alpha + \frac{\beta^2}{\alpha}$ is positive definite, we conclude that $\tau(o) = 0$.
- If $F = \alpha - \frac{\beta^2}{\alpha}$ is a Douglas metric, then at the origin $o = H$, we have
$$r_{22} = b_{2|2} = 2\tau(o)(1 + b^2) = 0.$$
Thus we also have $\tau(o) = 0$. Since α and β are both G-invariant, the scalar function $\tau(x)$ is a constant. Therefore $\tau = \tau(o) = 0$, which implies that F is a Berwald metric.

This completes the proof of Theorem under the assumption that both α, β are G-invariant.

Let u be a vector corresponding to β in the subspace V given in the above. Then the condition for F to be a Berwald metric is equivalent to the following:

$$\langle [v, w]_m, u\rangle = 0 \quad for \quad all \quad v, w \in m, \tag{3.10}$$
$$\langle [u, v_1]_m, v_2\rangle + \langle [u, v_2]_m, v_1\rangle = 0 \quad for \quad all \quad v_1, v_2 \in m. \tag{3.11}$$

By Theorem 3.4 and a direct observation, we have

Theorem 3.5. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a homogeneous Finsler square metric on G/H, where the Riemannian metric α and the 1-from β are both G-invariant. Suppose the Lie algebra g of G is perfect, i.e., $g = [g, g]$, then F is a Douglas metric if and only if F is a Riemannian metric.*

Proof. By Theorem 3.4, if $F = \alpha\phi(\frac{\beta}{\alpha})$ is a Douglas metric, then it is a Berwald metric or a Douglas metric of Randers type. Let $g = \eta + m$ denote a reductive decomposition of g. If F is a Douglas metric of Randers type, then F can be expressed in the form $F = \widetilde{\alpha} + \widetilde{\beta}$, where α and β are both G-invariant, and β is a closed 1-form. In this case, there exists a vector $u \in m$ satisfying (3.10), where the inner product $\langle ., .\rangle$ is corresponding to $\widetilde{\alpha}$, and the vector u is corresponding to $\widetilde{\alpha}$ with respect to the inner product. Now the condition that $Ad(h)(u) = u$ for all $h \in H$ is equivalent to

$$[v, u] = 0 \quad for \quad all \quad v \in \eta. \tag{3.12}$$

Since the inner product $\langle ., .\rangle$ is G-invariant, we have

$$\langle [v, w_1]_m, w_2\rangle + \langle [v, w_2]_m, w_1\rangle = = 0 \quad for \quad all \quad v \in \eta, w_1, w_2 \in m. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\langle [v, w]_m, u\rangle = 0 \quad for \quad all \quad v \in \eta, w \in m. \tag{3.14}$$

Now the assumption $g = [g, g]$ implies that there exists two vectors $w, v \in g$ such that $[w, v] = u$. Let $w = w_1 + w_2$ and $v = v_1 + v_2$, where $w_1, v_1 \in m$ and $w_2, v_2 \in \eta$. Then we have

$$[w, v] = [w_1, v_1] + [w_1, v_2] + [w_2, v_1] + [w_2, v_2]. \tag{3.15}$$

Therefore by (3.10) and (3.14), we have

$$\begin{aligned} \langle u, u\rangle &= \langle u, [w, v]_m\rangle \\ &= \langle u, [w_1, v_1]_m\rangle + \langle u, [w_1, v_2]_m\rangle + \langle u, [w_2, v_1]_m\rangle \\ &= 0. \end{aligned}$$

Thus $u = 0$, which implies that F is a Riemannian metric. If $F = \alpha\phi(\frac{\beta}{\alpha})$ is a Berwald metric with α and β both G-invariant, then there exists a vector $u \in m$ satisfying (3.10). Then a similar argument shows that F is also a Riemannian metric.

4. **Conclusion**

The important example of Finsler space are different type of (α, β)-metric are Randers metric, Kropina metric and other special (α, β)-metric. In [13] Ramesha M and S. K. Narasimhamurthy are devoted the necessary and sufficient conditions for a Finsler space with a special (α, β)-Metric $F = C_1\alpha + C_2\beta + \frac{\beta^2}{\alpha}$: $C_2 \neq 0$, to be a Douglas space and also to be Berwald space. H. Liu and S. Deng in [19] have shown the necessary and sufficient conditions for Randers homogeneous of Douglas type to be Berwaled and Riemannian metric.

In this present article we consider homogenous Finsler square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ of Douglas type, and we investigate the necessary and sufficient conditions for the homogenous Finsler square metric to be Douglas metric, then if has following properties:

(1) it is a Berwald metric or Randers type, and
(2) it is a Riemannian metric.

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