Journal Homepage: -www.journalijar.com
INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)
ISSN NO. 2320-5407
Article DOI:10.21474/IJAR01/3974
DOI URL: http://dx.doi.org/10.21474/IJAR01/3974

RESEARCH ARTICLE

STRONGLY HOLLOW-δ-LIFTING MODULES.

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Manuscript Info

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Manuscript History

Received: 16 February 2017
Final Accepted: 08 March 2017
Published: April 2017

Abstract

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In this paper, we introduce and study the concept of strongly hollow-δ-lifting modules as a proper generalization of strongly hollow-δ-lifting modules. We call an R-module M is strongly hollow-δ-lifting if for every submodule N of M with $\frac{M}{N}$ hollow, then there is a fully invariant direct summand K of M such that K⊆N and $\frac{N}{K} \ll_{\delta} \frac{M}{K}$. Several characterizations and properties of strongly hollow-δ-lifting modules are obtained. Modules related to strongly hollow-δ-lifting modules are given.

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Introduction:-

N. Orhan, D. keskin and R. Tribak introduced the concept hollow-lifting modules as a generalization of lifting modules. An R-module M is called Hollow –lifting if every submodule N of M such that M/N is hollow has coessential submodule that is a direct summand of M. Also, following[10], an R-module M is strongly hollow-lifting if every submodule N of with $\frac{M}{N}$ hollow, then there exists a stable(fully invariant) direct summand K of M such that K⊆N and $\frac{N}{K} \ll \frac{M}{K}$. On other direction, an R-module M is called hollow-δ-lifting if for everysubmodule$A$ of M with $\frac{M}{A}$ hollow, then there exists a direct summand$B$ of M such that $B \subseteq$ A and$\frac{A}{B} \ll_{\delta} \frac{M}{B}$ [12].

In this paper we introduce and study the concept of strongly hollow-δ-lifting modules as generalization of strongly hollow-lifting and as a strong concept of hollow-δ-lifting modules. We call an R-module M is strongly hollow-δ-lifting if for every submodule N of M with $\frac{M}{N}$ hollow, there exists a fully invariant direct summand K of such that K⊆N and $\frac{N}{K} \ll_{\delta} \frac{M}{K}$. Throughout this paper R will denote arbitrary associative ring with identity and all R-modules are unitrary left R-module, N⊆M will mean N is a submodule of an R-module M. Let M be an R-module and N be a submodule of M. N is called δ-small submodule of M (denote $N \ll_{\delta} M$) if N+K ≠M,for any proper submodule K of M such that $\frac{M}{K}$ singular [11]. A submodule N of M is called δ-coclosed in M if whenever K be a submodule of M with $\frac{M}{K}$ singular and $\frac{N}{K} \ll_{\delta} \frac{M}{K}$ implies N=K [4]. If N and K are submodules of M, then N is δ-supplement submodule of K in M if M=N+K and $N \cap K \ll_{\delta} K$, if every submodule has a δ-supplement submodule in M, then M is called δ-supplemented module [3]. A submodule A of an R-module M is called stable if f(A)⊆ A for each homomorphism f:A→M . An R-module M is stable if every submodule of M is stable [9] Recall that a submodule K of M is fully invariant if g(K)⊆ K for all f∈End(M) , an R-module M is duo if every submodule of M is fully invariant [6].

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Strongly hollow-δ-lifting Modules:-
As a proper generalization of strongly hollow-lifting modules and as a strong concept of hollow-δ-lifting modules, we introduce the following concept:

Definition (2.1):- An R-module M is called strongly hollow-δ-lifting if for every submodule A of M with $\frac{M}{A}$ hollow, there is a fully invariant direct summand B of M such that $B \subseteq N$ and $\frac{A}{B} \ll_{\delta} \frac{M}{B}$.

Examples and Remarks (2.2):-
Evrey strongly hollow-δ-lifting is hollow-δ-lifting .But the converse is not true in general. In fact, M=$Z_4 \oplus Z_8$ is hollow-δ-lifting Z-module. But M is not strongly hollow-δ-lifting, since A=$Z_4 \oplus \frac{4Z}{8Z}$ is a submodule of M with $\frac{M}{A}$ ≅4Z hollow, while A is not δ-small submodule and not contain any a fully invariant direct summand B of M such that B⊆A and $\frac{A}{B} \ll_{\delta} \frac{M}{B}$. Therefore M is not strongly hollow-δ-lifting.
2.Evrey strongly hollow-lifting is strongly hollow-δ-lifting. But the converse is not true in general.
3.Let M be an indecomposable R-module. Then M is strongly hollow-δ-lifting if and only if M is hollow-δ-lifting module.

Proof:- Let A be a submodule of M with $\frac{M}{A}$ hollow and since M is hollow-δ-lifting, then there is a direct summand B of M such that B⊆A and $\frac{A}{B} \ll_{\delta} \frac{M}{B}$. Since M is an indecomposable ,therefore B=(0) ,so $\frac{A}{(0)} \ll_{\delta} \frac{M}{(0)}$ such that B=(0) is a fully invariant direct summand of M. Thus M is strongly hollow-δ-lifting.
(2)⟹(1) It is clear.

Proposition(2.3):- If M is hollow-δ-lifting module, then every δ-coclosed submodule B of M with $\frac{M}{B}$ hollow is direct summand.

Proof:-Let M is strongly hollow-δ-lifting module and let A be δ-coclosed submodule such that $\frac{M}{A}$ hollow. Thus there is a fully invariant direct summand B of M such that B⊆A and $\frac{A}{B} \ll_{\delta} \frac{M}{B}$. But A is δ-coclosed, so A=B and hence A is a fully invariant direct summand of M.■

Recall that an R-module M is hollow-SS-module if every direct summand D of M with $\frac{M}{D}$ hollow is stable. Equivalently, M is hollow-SS-module if and only if every direct summand D of M with $\frac{M}{D}$ is fully invariant [10].

Now, the following proposition gives characterization of strongly hollow-δ-lifting by using hollow-SS-module and in the same time we give another a condition under which a hollow-δ-lifting module is strongly hollow-δ-lifting module.

Proposition(2.4):- If M is strongly hollow-δ-lifting , then M is hollow SS-module and hollow-δ-lifting.

Proof:- Let M is strongly hollow-δ-lifting, then M is hollow-δ-lifting. We want to show that M is hollow SS-module ,let B be a direct summand of M with $\frac{M}{B}$ hollow and hence B is δ-coclosed by proposition(2.3),B is fully invariant and hence M is hollow SS-module.■

Recall that an R-module M is satisfies the condition(δ^*), if for every direct summands A and B of M and A∩B$\ll_{\delta}$B, then A∩B=0 [2].

Now, the following proposition, we prove that this proposition valid on strongly hollow-δ-lifting module without extra condition(δ^*).

Proposition(2.5):- Let M be a strongly hollow-δ-lifting module. If A and B are direct summands of M such that $\frac{M}{A \cap B}$ hollow, then A∩B is a direct summand of M.

Proof:-Suppose that A and B are direct summands of M. Since $\frac{M}{A\cap B}$ hollow, therefore $\frac{M}{A}$ and $\frac{M}{B}$ are hollow[1,proposition(2.1.2)].Since M is strongly hollow-δ-lifting by proposition(2.4) , then M is hollow SS-module and hence we have A and B are fully invariant submodule. By[7], if A and B are fully invariant direct summands of M, then A∩B is a direct summand of M.■

Proposition(2.6):-An R-module M is strongly hollow-δ-lifting if and only if for every submodule A of M with $\frac{M}{A}$ hollow, there is a fully invariant direct summand B of M with B⊆A such that M=B⊕L, and $A\cap L\ll_{\delta}L$.

Proof:- Let M is strongly hollow-δ-lifting module and let A be a submodule of M with $\frac{M}{A}$ hollow, then there is a fully invariant direct summand B of M such that B⊆A and $\frac{A}{B}\ll_{\delta}\frac{M}{B}$ .Let M=B⊕ L, where L be a submodule of M . Let f: $\frac{M}{B}\rightarrow L$ be an mapping defined by f(m+B)=l, where m=b+l such that b∈B and l∈L. But $\frac{A}{B}\ll_{\delta}\frac{M}{B}$, so by[2,lemma(1.2.7)] and since $f(\frac{A}{B})=\{f(a+k+B)\ |a\in A \text{ and } k\in(A\cap L)\}=\{k|\ k\in(A\cap L)\}=A\cap L\ll_{\delta}L$.

Conversely, let A be a submodule of M with $\frac{M}{A}$ hollow. By assumption there is a fully invariant direct summand B of M with B⊆A such that M=B⊕L and $A\cap L\ll_{\delta}L$. Let $\frac{A}{B}\ll_{\delta}\frac{M}{B}$. Let $\frac{A}{B}+\frac{Y}{B}=\frac{M}{B}$ such that $\frac{M}{Y}$ singular . Hence, M=A+Y .By modular ,A=A∩M=A∩(B⊕L)=B⊕(A∩L),so M=B⊕((A∩L)+Y=(A∩L)+Y. But $(A\cap L)\ll_{\delta}L$, therefore $(A\cap L)\ll_{\delta}M$. Also, since $\frac{M}{Y}$ singular . Thus M=Y and so $\frac{A}{B}\ll_{\delta}\frac{M}{B}$. Therefore , M is strongly hollow-δ-lifting.■

Proposition(2.7):-Let M be an R-module. Then the following statements are equivalent:
1. M is strongly hollow-δ-lifting module.
2. Evrey submodule A of M with $\frac{M}{A}$ hollow, can be written as A=B⊕S with B is a fully invariant direct summand of M and $S\ll_{\delta}M$.
3. Every submodule A of M with $\frac{M}{A}$ hollow, can be written as A=B+S with B is a fully invariant direct summand of M and $S\ll_{\delta}M$.

Proof:- (1⟹2). Let A be a submodule of M with $\frac{M}{A}$ hollow. Since M is strongly hollow-δ-lifting, then there is a fully invariant direct summand B of M such that B⊆A and $\frac{A}{B}\ll_{\delta}\frac{M}{B}$. Let M=B⊕L, where L be a submodule of M. By modular law, A=A∩M=A∩(B⊕L)=B⊕(A∩L). By the same argument of proposition (2.6), we have $A\cap L\ll_{\delta}L$. Let S=A∩L, so A=B⊕S, where B is a fully invariant direct summand of M and $S\ll_{\delta}M$.
(2⟹3) It is clear.
(3⟹1).Let A be a submodule of M with $\frac{M}{A}$ hollow. From(3),A=B+S, where B is a fully invariant direct summand of M and $S\ll_{\delta}M$. Let M=B⊕L,where L be a submodule of M. Let $\frac{A}{B}+\frac{Y}{B}=\frac{M}{B}$ such that $\frac{M}{Y}$ singular ,then M=A+Y and so M=B+S+Y=S+Y. Since $\frac{M}{Y}$ singular and since $S\ll_{\delta}M$,therefore M=Y. Hence $\frac{A}{B}\ll_{\delta}\frac{M}{B}$ and so M is strongly hollow-δ-lifting.■

Proposition(2.8):- Let M=N⊕K be a duo module such that N and K are strongly hollow-δ-lifting modules. Then M is strongly hollow-δ-lifting.

Proof:-Let M=N⊕K be a duo module and let B be a submodule of M with $\frac{M}{B}$ hollow. Now, $\frac{M}{B}=\frac{N\oplus K}{B}=\frac{N\oplus B}{B}\oplus\frac{K\oplus B}{B}$ by[5,lemma(2.2.12)]. Since $\frac{M}{B}$ hollow we can assume that $\frac{M}{B}=\frac{N\oplus B}{B}$,so $\frac{N\oplus B}{B}\cong\frac{N}{N\cap B}$ and hence $\frac{N}{N\cap B}$ hollow. Since B is fully invariant submodule of M, then B=(B∩N)⊕ (B∩K) by [8]. Since N and K are strongly hollow-δ-lifting. Thus $B\cap N=H_1\oplus L_1$, where H_1 is a fully invariant direct summand of N and $L_1\ll_{\delta}N$. In the same way ,we have $B\cap K=H_2\oplus L_2$,where H_2 is a fully invariant direct summand of K and $L_2\ll_{\delta}K$. Then B=H⊕L,where $H=H_1\oplus H_2$ is a fully invariant direct summand of M and $L=L_1\oplus L_2\ll_{\delta}M$ and hence M=N⊕K is strongly hollow-δ-lifting. By proposition(2.7).■

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