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RESEARCH ARTICLE

Bayesian Estimation of the Scale Parameter for Inverse Gamma Distribution under Linex Loss Function

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Abstract

The object of the present paper is finding the best estimator for the scale parameter of Inverse Gamma distribution using Linex loss function and Squared error loss function with non-informative prior, and compared with Bayes estimators under quadratic loss function with the same prior. The comparison was made on the performance of these estimators with respect to the mean square error (MSE) and the mean percentage error (MPE). The results showed that the Bayes' estimator under Linex loss function is the best.

INTRODUCTION

In Bayesian estimation, we consider two types of loss functions. The first is squared error loss function which classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second is the Linexloss function (exponential where the name LINEX is justified by the fact that is, this loss function rises approximately linearly on one side of zero and approximately exponentially on the other side) loss function which is asymmetric.

LINEX Loss Function and its Properties

Thompson and Basu (1996) identified a family of loss functions $L(\Delta)$, where Δ is either the estimation error $(\hat{\theta} - \theta)$ or the relative estimation error $(\hat{\theta} - \theta) / \theta$, such that

- $L(0) = 0$
- $L(\Delta) >(<) L(-\Delta) > 0$ for all $\Delta > 0$
- $L(\cdot)$ is twice differentiable with $L'(0) = 0$ and $L''(\Delta) > 0$ for all $\Delta \neq 0$.
- $0 < L'(\Delta) >(<) - L'(-\Delta) > 0$ for all $\Delta > 0$.

Such loss function is useful whenever the actual losses are nonnegative, increases with estimation error, overestimation is more (less) serious than under estimation of the same magnitude and losses increase at a faster (slower) rate with overestimation error.

Considering the loss function:

$$L_*(\Delta) \propto b \exp(a\Delta) + c\Delta + d$$

and with the restriction $L_*(0)=0$, $(L_*)'(0)=0$, we get $d=-b$ and $c=-ab$, see Thompson and Basu (1996). The resulting loss function is:

$$L_*(\Delta) \propto b[\exp(a\Delta) - a\Delta - 1]$$

Which is considered as a function of θ and $\hat{\theta}$, is called the Linex loss function, a and b are constants with b >0 so that, the loss function is nonnegative. [2]

In Bayesian analysis the unknown parameter is regarded as being the value of a random variable from a given probability distribution, with the knowledge of some information about the value of parameter prior to observing the data $x_1, x_2 \ldots x_n$.

The object of the present paper is to obtain Bayesian estimates of the scale parameter for Inverse Gamma distribution using Linexloss function with non-informative prior. The comparison was based on a Monte Carlo study. The efficiency for the estimators was compared with respect to the mean square error (MSE) and the mean percentage error (MPE).

Bayes' Estimators

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n, the n items have an independent and identically Inverse Gamma distribution, with probability density

$$f(x,\lambda,\theta) = \frac{\theta^{\lambda}}{\Gamma_{\lambda}} x^{-\lambda-1}\ e^{-\theta/x}, x \geq 0 \qquad (1)$$

Where $\boldsymbol{\lambda}$ is the location parameter and $\boldsymbol{\theta}$ is the scale parameter.

Bayes' estimators for the scale parameter $\boldsymbol{\theta}$ was considered with Linex loss function and quadratic loss function with non-Informative prior, which represented by Jeffreys prior distribution of (θ) where:

$$p(\theta) \propto \sqrt{I(\theta)} \qquad (2)$$

$p(\theta) = k\sqrt{I(\theta)}$, I(θ) is a Fisher information [5]

$$p(\theta) = k\sqrt{-nE(\frac{\partial^2 lnf}{\partial\theta^2})}\frac{\partial^2}{\partial\theta^2}$$

$$lnf(x,\lambda,\theta) = \frac{-\lambda}{\theta^2}$$

$$lnf(x,\lambda,\theta) = \lambda ln\theta - ln\Gamma_{\lambda} - (\lambda+1)lnx - \frac{\theta}{x}\frac{\partial}{\partial\theta}lnf(x,\lambda,\theta) = \frac{\lambda}{\theta} - \frac{1}{x}$$

Hence:

$$p(\theta) = \frac{k}{\theta}$$

The posterior distribution for θ given random sample $X=(x_1,x_2,\ldots,x_n)$ is:

$$h(\lambda,\theta|x_1,x_2,\ldots,x_n) = \frac{\frac{\theta^{n\lambda}}{(\Gamma_{\lambda})^n}\prod_{i=1}^{n} x^{-\lambda-1}e^{-\theta\sum_{i=1}^{n}1/x_i}}{\int_0^{\infty}\frac{\theta^{n\lambda}}{(\Gamma_{\lambda})^n}\prod_{i=1}^{n} x^{-\lambda-1}e^{-\theta\sum_{i=1}^{n}1/x_i}d\theta} \qquad (3)$$

$$h(\lambda,\theta|x_1,x_2,\ldots,x_n) = \frac{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^{n\lambda}\theta^{n\lambda-1}\quad e^{-\theta\sum_{i=1}^{n}1/x_i}}{\Gamma_{(n\lambda)}} \qquad (4)$$

We notice that:

$$\theta \sim \Gamma_{\left(n\lambda,\ \frac{1}{\sum_{i=1}^{n}1/x_i}\right)}$$

i-Bayes Estimation of θ under Squared Error Loss Function

The squared error loss function is:

$$l_1(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2 \qquad (5)$$

We drive the corresponding Bayes estimator for θ using risk function where

$$\hat{R}_1(\hat{\theta},\theta) = \int_0^{\infty}(\hat{\theta}-\theta)^2\ h(\lambda,\theta|x)\ d\theta \qquad (6)$$

$$= \hat{\theta}^2 - 2\hat{\theta}E(\theta|x) + E(\theta^2|x)$$

Let:

$$\frac{\partial}{\partial\theta}\hat{R}_1(\hat{\theta},\theta) = 0 \Rightarrow \hat{\theta}_1 = E(\theta|x)$$

Hence:

$$\hat{\theta}_1 = \frac{n\lambda}{\sum_{i=1}^{n} 1/x_i} \qquad (7)$$

ii- Bayes Estimation of θ under Linex Function

Zellner (1968) proposed the Linex loss function[6]

$$l_2(\hat{\theta},\theta) = b\left[e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1\right]$$

Where, b>0 , a≠0. Then, the risk function is

$$\hat{R}_2(\theta,\theta) = \int_0^\infty b\left[e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1\right] h(\lambda,\theta|x)d\theta \qquad (8)$$

$$= be^{a\hat{\theta}}E[e^{-a\theta}] - ab\hat{\theta} + abE[\theta] - b$$

$$\frac{\partial\hat{R}_2}{\partial\hat{\theta}} = ab\, e^{a\hat{\theta}}E[e^{-a\theta}] - ab$$

Let: $\frac{\partial\hat{R}_2}{\partial\hat{\theta}} = 0$

On simplification, we get:

$$\hat{\theta} = \frac{-1}{a}\ \mathrm{Ln}\left[E(e^{-a\theta})\right] (9)$$

$$E[e^{-a\theta}] = \int_0^\infty e^{-a\theta}\, h(\lambda,\theta|x)d\theta$$

$$= \int_0^\infty e^{-a\theta} \frac{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^{n\lambda} \theta^{n\lambda-1}\ e^{-\theta\sum_{i=1}^{n} 1/x_i}}{\Gamma(n\lambda)} d\theta$$

$$= \left[\frac{\sum_{i=1}^{n} 1/x_i}{a + \sum_{i=1}^{n} 1/x_i}\right]^{n\lambda}$$

After substitution on (9), we find that:

$$\hat{\theta}_2 = \frac{-1}{a}\ \mathrm{Ln}\left[\frac{\sum_{i=1}^{n} 1/x_i}{a+\sum_{i=1}^{n} 1/x_i}\right]^{n\lambda} \qquad (10)$$

Simulation and Results

In this section, Monte–Carlo simulation study is performed to compare the methods of estimation by using mean square Errors (MSE's) and the mean percentage errors (MPE's), as follows:

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{R}(\hat{\theta}_i - \theta)^2}{R} \text{and} MPE(\hat{\theta}) = \frac{\frac{\sum_{i=1}^{R}|\hat{\theta}_i - \theta|}{\theta}}{R}$$

Where R is the number of replications.

We generated R= 2000 samples of size n = 20, 50 and 100, to represent small, moderate and large sample sizes from Inverse Gamma distribution with θ = 1.5, 3
We chose the values of a; (a = -2, 0.5, 1, 3). The results were summarized and tabulated in the following tables for each estimator and for all sample sizes.

From tables (1) and (2), it appears that, $\hat{\theta}_2$ which represented Bayes estimator under Linex loss function, is better than $\hat{\theta}_1$which represented Bayes estimator under Squared error loss function, when (a > 0).
We can see clearly that, when (θ= 1.5) $\hat{\theta}_2$became the best when (a=3) with all sample sizes, While with (θ= 3), $\hat{\theta}_2$ became the best when (a=1) with all sample sizes.

In general, we can say that, the Bayes' estimator under Linex loss function when (a > 0) is the better than Bayes estimator under Squared error loss function with all sample sizes and we reach to the estimator become better when a = 1 with a large values of θ while it was better with large value of of a (a=3) with small values of θ.

Table 1: MSE's and MPE's of estimated the scale parameter ofInverse Gamma distribution with θ=1.5.

n	Criteria	$\hat{\theta}_1$	$\hat{\theta}_2$			
			a = -2	a=0.5	a = 1	a = 3
20	EXP.	1.5260	1.5665	1.5158	1.5060	1.4690
	MSE	0.0419	0.0504	0.0405	0.0392	0.0364
	MPE	0.1059	0.1146	0.1045	0.1034	0.1014
50	EXP.	1.5103	1.5258	1.5065	1.5027	1.4878
	MSE	0.0157	0.0169	0.0155	0.0153	0.0148
	MPE	0.0663	0.0687	0.0659	0.0655	0.0647
100	EXP.	1.5060	1.5136	1.5041	1.5022	1.4947
	MSE	0.0075	0.0078	0.0075	0.0074	0.0073
	MPE	0.0457	0.0465	0.0456	0.0454	0.0450

Table 2: MSE's and MPE's of estimated the scale parameter ofInverse Gamma distribution with θ = 3.

n	*Criteria*	$\hat{\theta}_1$	$\hat{\theta}_2$			
			a = -2	a= 0.5	a = 1	a = 3
20	EXP.	3.0512	3.2212	3.0124	2.9749	2.8367
	MSE	0.1677	0.2554	0.1569	0.1496	0.1499
	MPE	0.1059	0.1283	0.1034	0.1019	0.1051
50	EXP.	3.0206	3.0836	3.0054	2.9904	2.9323
	MSE	0.0629	0.0748	0.0612	0.0601	0.0600
	MPE	0.0663	0.0721	0.0655	0.0650	0.0655
100	EXP.	3.0120	3.0427	3.0044	2.9969	2.9673
	MSE	0.0301	0.0330	0.0296	0.0293	0.0292
	MPE	0.0457	0.0477	0.0454	0.0452	0.0454

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