RESEARCH ARTICLE

QUASI-INJECTIVE GAMMA MODULES.

Mehdi S. Abbas, Saad Abdulkadhim Al-Saadi and Emad Allawi Shallal
Department of Mathematics, College of Science, Al-Mustansiriyah University, Iraq.

Abstract

In this paper we introduce the concept of quasi-injective gamma modules as a proper generalization of injective gamma modules and quasi-injective modules. An $R_{\Gamma}$-module $M$ is called quasi-injective if for any $R_{\Gamma}$-submodule $A$ of $M$ and $R_{\Gamma}$-homomorphism $f$ from $A$ to $M$ there is an $R_{\Gamma}$-endomorphism of $M$ which extends $f$. We are extending some results from module theory to gamma module theory, we established that every $R_{\Gamma}$ - module has quasi-injective hull which is unique up to isomorphism. Moreover, if $M$ is quasi-injective, then $\text{End}_{R_{\Gamma}}(M)/\langle \text{End}_{R_{\Gamma}}(M) \rangle$ is regular $\Gamma$ - ring.

1. Introduction

The notion of $\Gamma$-ring was first introduced by N. Nobusawa [9] and then Barnes [3] generalized the definition of Nobusawa’s gamma rings. R. Ameri and R. Sadeghi [2] studied gamma module, gamma submodule, homomorphism of gamma modules. They obtained some basic results of gamma modules. The authors in [1] introduced and studied the concept of injective gamma module, divisible gamma module and essential gamma submodule. They proved that every gamma module can be embedded in injective gamma module.

In this paper, we introduce and study the concept of quasi-injective gamma modules as a proper generalization of injective gamma modules and quasi-injective modules.

2. Preliminaries

Let $R$ and $\Gamma$ be two additive abelian groups, $R$ is called a $\Gamma$ - ring (in the sense of Barnes), if there exists a mapping $: R \times \Gamma \times R \rightarrow R$, written $\cdot (r, \gamma, s) \leftrightarrow r\gamma s$ such that $(a + b)\alpha c = a\alpha c + b\alpha c$ and $a(\alpha + \beta)c = a\alpha c + a\beta c$, $a(\alpha + \beta)c = a\alpha c + a\beta c$ and $(a\alpha b)c = a(\alpha bc)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$ [3]. A subset $A$ of $\Gamma$-ring $R$ is said to be a left(right) ideal of $R$ if $A$ is an additive subgroup of $R$ and $R\alpha A \subseteq A$ ($\alpha R \subseteq A$), where $R\alpha A = \{ra : r \in R, \alpha \in \Gamma, a \in A\}$. If $A$ is both right and left ideal, we say that $A$ is an ideal of $R$ [3]. An element $1$ in $\Gamma$-ring $R$ is unity if there exists element, say $1$ in $R$ and $\gamma \in \Gamma$ such that $r = 1\gamma r \rightarrow r\gamma 1$ for every $r \in R$, unities in $\Gamma$ - rings differ from unities in rings, it is possible for a $\Gamma$ - ring have more than one unity [7].

Let $R$ be a $\Gamma$-ring and $M$ be an additive abelian group. Then $M$ together with a mapping $: R \times \Gamma \times M \rightarrow M$, written $\cdot (r, \gamma, m) \leftrightarrow r\gamma m$ such that

1. $r(a_1 + a_2)m = r_1 a_1 + r_2 a_2 m$
2. $r(a + b)m = r_1 a_1 + r_2 a_2 m$
3. $r(a + b)m = r_1 a_1 + r_2 a_2 m$
4. $r(a + b)m = r_1 a_1 + r_2 a_2 m$

Corresponding Author: Mehdi S. Abbas
Address: Department of Mathematics, College of Science, Al-Mustansiriyah University, Iraq.
for each \( r, r_1, r_2 \in R \), \( \alpha, \beta \in \Gamma \) and \( m, m_1, m_2 \in M \), is called a left \( R \) \(-\)module, similarly one can defined right \( R \) \(-\)module [2]. A left \( R \) \(-\)module \( M \) is unitary if there exist elements, say \( 1 \) in \( R \) and \( \gamma_{e} \in \Gamma \) such that \( 1 \gamma_{e} = m = m \) for each \( m \in M \).

Let \( M \) be an \( R_{\Gamma} \) \(-\)module. An empty subset \( N \) of \( M \) is said to be an \( R_{\Gamma} \) \(-\)submodule of \( M \) (denoted by \( N \subseteq M \)) if \( N \) is a subgroup of \( M \) and \( R \Gamma N \subseteq N \), where \( R \Gamma N = \{ r \alpha : r \in R, \alpha \in \Gamma, n \in N \} \) [2]. An \( R_{\Gamma} \) \(-\)module \( M \) is called simple if \( R \Gamma M \neq 0 \) and the only \( R_{\Gamma} \) \(-\)submodules of \( M \) are \( M \) and \( 0 \) [4]. If \( X \) is a nonempty subset of \( M \), then the \( R_{\Gamma} \) \(-\)submodule of \( X \) generated by \( X \) denoted by \( \langle X \rangle \) [2]. A left \( R_{\Gamma} \) \(-\)module \( M \) is called simple if \( M \) and the only \( R_{\Gamma} \) \(-\)submodules of \( M \) are \( M \) and \( 0 \) [4].

Let \( M \) and \( N \) be two \( R_{\Gamma} \) \(-\)modules. A mapping \( f : M \to N \) is called homomorphism if \( R_{\Gamma} \) \(-\)modules (simply \( R_{\Gamma} \) \(-\)homomorphism) if \( f(x + y) = f(x) + f(y) \) and \( f(r \alpha) = r f(\alpha) \) for each \( x, y \in M \), \( r \in R \) and \( \gamma \in \Gamma \). An \( R_{\Gamma} \) \(-\)homomorphism is \( R_{\Gamma} \) \(-\)monomorphism if it is one-to-one and \( R_{\Gamma} \) \(-\)epimorphism if it is onto, the set of all \( R_{\Gamma} \) \(-\)homomorphisms from \( M \) into \( N \) denoted by \( \text{Hom}_{R_{\Gamma}}(M, N) \). In particular, if \( M = N \), then \( \text{Hom}_{R_{\Gamma}}(M, N) \) denote by \( \text{End}_{R_{\Gamma}}(M) \). If \( M \) is \( R_{\Gamma} \) \(-\)module, \( \text{End}_{R_{\Gamma}}(M) \) is a \( \Gamma \) \(-\)ring with the mapping \( \cdot : \text{End}_{R_{\Gamma}}(M) \times \Gamma \to \text{End}_{R_{\Gamma}}(M) \) \(-\)ring with the mapping \( \cdot : \text{End}_{R_{\Gamma}}(M) \times \Gamma \to \text{End}_{R_{\Gamma}}(M) \).

If \( M \) and \( N \) are two \( R_{\Gamma} \) \(-\)modules. Then \( M \) is called \( N \) \(-\)injective if for any \( R_{\Gamma} \) \(-\)submodule \( A \) of \( N \) and \( R_{\Gamma} \) \(-\)homomorphism \( f : A \to M \) there exists an \( R_{\Gamma} \) \(-\)homomorphism \( g : N \to M \) such that \( g f = f \) where \( i \) is the inclusion mapping. An \( R_{\Gamma} \) \(-\)module \( M \) is injective if it is \( N \) \(-\)injective for any \( R_{\Gamma} \) \(-\)module \( N \) [1]. An \( R_{\Gamma} \) \(-\)submodule \( N \) of \( R_{\Gamma} \) \(-\)module \( M \) is essential (denote by \( N \subseteq_{e} M \)) if every nonzero \( R_{\Gamma} \) \(-\)submodule of \( M \) has nonzero intersection with \( N \), in this case we say that \( M \) is an essential extension of \( N \) [1]. It is proved in [1], that every gamma module can be embedded in injective gamma module. The minimal injective extension of \( M \) is called injective hull (denote by \( \text{E}(M) \)) which is unique up to isomorphism.

3. Quasi-injective gamma module

In this section we introduce the concept of quasi-injective gamma modules as a generalization of injective gamma modules.

Definition 3.1 An \( R_{\Gamma} \) \(-\)submodule \( N \) of \( R_{\Gamma} \) \(-\)module \( M \) is called direct summand if there exists an \( R_{\Gamma} \) \(-\)submodule \( K \) of \( M \) such that \( M = N + K \) and \( N \cap K = 0 \), in this case \( M \) is written as \( M = N \oplus K \).

The \( R_{\Gamma} \) \(-\)submodules \( 0 \) and \( M \) are always direct summand of \( M \).

Definition 3.2 Let \( M \) be an \( R_{\Gamma} \) \(-\)module of an \( R_{\Gamma} \) \(-\)module \( M \). A complement of \( N \) in \( M \) is any \( R_{\Gamma} \) \(-\)submodule denoted by \( N \) \(-\)complement of \( N \) which is maximal with respect to the property \( N \subseteq_{e} M \). By Zorn's lemma, one can be show that every submodule of gamma module has a complement submodule which is not unique in general.

Definition 3.3 An \( R_{\Gamma} \) \(-\)submodule \( A \) of an \( R_{\Gamma} \) \(-\)module \( M \) is called closed in \( M \) if it has no proper essential extension in \( M \), that is, the only solution of the relation \( A \subseteq_{e} K \subseteq M \) is \( A = K \). It is easy to see that every direct summand of \( M \) is closed.

In this lemma, we see that every \( R_{\Gamma} \) \(-\)submodule of \( A \) \(-\)direct summand of an essential \( R_{\Gamma} \) \(-\)submodule.

Lemma 3.4 Let \( N \) be an \( R_{\Gamma} \) \(-\)submodule of a \( R_{\Gamma} \) \(-\)module \( M \). Then \( N \oplus N^{e} \subseteq_{e} M \).

Proof. For each \( R_{\Gamma} \) \(-\)submodule \( K \) of \( M \) such that \( K \cap (N \oplus N^{e}) = 0 \), if \( a \in N \cap (N \oplus K) \), then \( a = b + k \) where \( b \in N^{e} \) and \( k \in K \), so \( k = a - b \in K \cap (N \oplus N^{e}) = 0 \), hence \( a = b \in N \cap N^{e} = 0 \), so \( N \cap (N^{e} \oplus K) = 0 \), by maximality of \( N^{e} \) we have \( N^{e} = N^{e} \oplus K \), so \( K = 0 \), hence \( N + N^{e} \subseteq_{e} M \).

Lemma 3.5 Let \( N \) be an \( R_{\Gamma} \) \(-\)submodule of an \( R_{\Gamma} \) \(-\)module \( M \) and \( K \) a complement of \( N \) in \( M \). Then:

1. There exists a complement \( L \) of \( K \) in \( M \) such that \( N \leq L \).
2. L is a maximal essential extension of N.
3. If N is closed, then N = L.

**Proof.**
1. By Zorn's lemma there exists a complement L of K which contains N.
2. For any A ≤ L, since L ∩ K = 0, then A ∩ K = 0. Let 0 ≠ x = a + k ∈ (A + K) ∩ N where a ∈ A and k ∈ K.
   Then k = x - a ∈ K ∩ N = 0, so k = 0 and x = a ∈ N ∩ A, hence N ∩ A ≠ 0, so N ≤ L. If P is an Rf - submodule of M contains L properly, then P ∩ K ≠ 0 and (P ∩ K) ∩ N = P ∩ (K ∩ N) = P ∩ 0 = 0, thus P is not essential extension of N.
3. Follows from (2).

**Definition 3.6** An Rf - module M is called quasi-injective if for any Rf - submodule A of Q and for any Rf - homomorphism f:A → M there exists an Rf - endomorphism g of M such that gi = f where i is the inclusion mapping of A into M.

In fact, M is a quasi-injective if and only if M is M-injective [1].

The proof of the following propositions follow from proposition(3.13) in [1].

**Proposition 3.7** An Rf - module M is quasi-injective if f(M) ⊆ M for every f ∈ End(E(M)).

**Corollary 3.8** Let M be a quasi-injective Rf - module and \( \{A_\lambda : \lambda \in \Lambda \} \) be a family of an independent set of Rf - submodules of M, then M ∩ (⊕_\lambda A_\lambda) = ⊕_\lambda(M ∩ A_\lambda).

**Corollary 3.9** Let M be a quasi-injective Rf - module, then:
1. Every Rf - submodule of M is essential in a direct summand of M.
2. If an Rf - submodule N of M isomorphic to a summand of M, then N is a summand of M.

**Proof.**
1. Assume N ≤ M and E(M) = E_1 ⊕ E_2 where E_1 = E(N), by proposition(3.7) M = (M ∩ E_1) ⊕ (M ∩ E_2), by lemma(3.3) in [1], N ≤ M ∩ E_1.
2. Assume N ≤ K and K is a direct summand of M, then there exists Rf - submodule K_1 of M such that M = K ⊕ K_1 and Rf - isomorphism α:N → K, from [1] K is M-injective, so α can be extended to an Rf - homomorphism β:M → K such that α = βi where i is the inclusion mapping, so M = Im(i) ⊕ Ker(β), hence N is a summand of M.

**Proposition 3.10** Let N be a closed Rf - submodule of an Rf - module M. If M is quasi-injective, then N is M-injective.

**Proof.** Let K is Rf - submodule of M and f:K → N is an Rf - homomorphism, define Ω = {(K',f'):K ≤ K' ≤ M,f(extended of f to K')} by Zorn's lemma Ω has a maximal element (K_,f_), since M is quasi-injective, then f_ can extended to an Rf - homomorphism g:M → M. If g(M) ⊆ N, let L be a complement of N in M, since N closed, then N is complement of L, since N ⊆ N + g(M) so [N + g(M)] ∩ L = 0. Let 0 ≠ x = a + b where a ∈ N and b ∈ g(M), if b ∈ N, then x = a + b ∈ N ∩ L = 0 contradiction, so b ∉ N and b = x - a ∈ L_∩ N. Define S = {m ∈ M: g(m) ∈ L_∩ N}, S is an Rf - submodule contains K, take t ∈ M such that g(t) = b, then t ∈ S but t ∉ K, if π:L_∩ N → N is the projection Rf - homomorphism, then πg:M → N and (πg)(k) = π(g(k)) = π(f(k)) = f(k) for each k ∈ K, thus πg extending of f which is contradiction, therefore g(M) ⊆ N.

**Corollary 3.11** Every closed Rf - submodule N of an quasi-injective Rf - module M is a direct summand of M, moreover, N is quasi-injective.

**Proof.** Let N: identity map of N. Then by proposition(3.10) there exists f:M → N such that f = I_N where i is inclusion mapping, so Im(i) ⊕ Ker(f) = M, hence N ⊕ Ker(f) = M. By lemma(1.5) in [1], we have N is quasi-injective.

**Corollary 3.12** Let M be Rf - module. Then M is quasi-injective if and only if M ⊕ M is quasi-injective.

**Proof.** If M is quasi-injective Rf - module, then by [1, proposition 1.4] M is M ⊕ M - injective, by [1, lemma 1.5] M ⊕ M is quasi-injective.

The proof of the following propositions follow from proposition(1.3) in [1].
Proposition 3.13  An $R_f$- module $M$ is quasi-injective if and only if for each $R_f$- submodule $B$ of a cyclic $R_f$- submodule $A$, each $R_f$- homomorphism $\alpha: B \to M$ can be extending to an $R_f$ - homomorphism $\beta: A \to M$.

Examples and Remarks 3.14
1. Every simple $R_f$- module is quasi-injective.
2. Every injective $R_f$- module is quasi-injective, the converse is not true, for example, let $R = \Gamma = Z$ and $M = Z_2$, then $M$ is quasi-injective from (1) but not injective since $1 \neq 2. m. x$ for any $x \in M$, so it is not divisible [1].
3. Let $F$ be a field, $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y, z \in F \right\}$, and $\Gamma = \left\{ \begin{pmatrix} \gamma & \beta \\ \lambda & \mu \end{pmatrix} : \gamma, \beta, \lambda, \mu \in F \right\}$. $F$ is a $\Gamma$- ring with usual multiplication of matrices, consider $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in F \right\}$, $B = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F \right\}$ and $C = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in F \right\}$, then $M = A \otimes F$ and $B \cong A$, from corollary (3.9), $R$ is not quasi-injective $R_f$- module.

4. Direct sum of two quasi-injective $R_f$- modules need not be quasi-injective, for example, let $R = \Gamma = Z$, $M_1 = Z_2$ and $M_2 = Q$ from example (2) and example (2.3) in [1] $M_1$ and $M_2$ are quasi-injective. But $M = M_1 \oplus M_2$ is not quasi-injective, since if the $R_f$- homomorphism $f: 0 \oplus \mathbb{Z} \to M$ by $f(0, n) = (n, 0)$ extended to $R_f -$ endomorphism $g$ of $M$, then $g(0, 1) = g(2.1 \cdot (0, 1)) = 2.1. g(0, 1) = 2.1. (x, y) = (2x, 2y)$. Hence $g(0, 1) = (2x, 2y) = (0, 2y)$ contradiction.

5. If $R_f$- module $M$ contains a copy of $R$ as $R_f$- module, then $M$ is quasi-injective if and only if it is injective.

6. An $R_f$- module $M$ is quasi-injective if and only if for each $R_f$- submodule $N$ of $M$, each $R_f$- homomorphism $f: N \to M$ can be extended to $R_f$ - endomorphism of $M$. For each $R_f$- submodule $N$ of $M$ and each $R_f$- homomorphism $f: N \to M$, define $g: N \oplus N^c \to M$, by $g(n + n') = f(n) + n'$ where $n \in N$ and $n' \in N^c$ since $N \oplus N^c$ is essential by lemma (3.4), then $g$ can be extended to $R_f -$ endomorphism $h$ of $M$, clear that $h$ is extending of $f$.

Lemma 3.15 Let $M$ be a quasi-injective $R_f$- module. Then $M$ is injective if there exist an $R_f$- epimorphism from $M$ to $E(M)$.

Proof. Let $f: M \to E(M)$ be an $R_f$- epimorphism. Then there exists an $R_f$- endomorphism $h: E(M) \to E(M)$ such that $f = hi$, since $M$ is quasi-injective, then $h(M) \subseteq M$, hence $E(M) = f(M) = hi(M) = h(M) \subseteq M$, so $M = E(M)$, thus $M$ is injective $R_f$- module.

The annihilator of a left $R_f$- module $M$ define by $Ann_R(M) = \{ r \in R : r \Gamma M = 0 \}$ and the annihilator of $m \in M$ define by $Ann_R(m) = \{ r \in R : r \Gamma m = 0 \}$ [4].

We denote $\ell_{R_f}(M)$ and $\ell_{R_f}(m)$ instead of $Ann_R(M)$ and $Ann_R(m)$.

Definition 3.16 Let $M$ be an $R_f$- module, $a \in M$ and $\gamma \in \Gamma$. The left Annihilator of $m$ in $R$ with respect to $\gamma$ define by $\ell_{R_f}(m) = \{ r \in R : r \Gamma m = 0 \}$.

It's clear that $\ell_{R_f}(M) \subseteq \ell_{R_f}(m) \subseteq \ell_{R_f}(m)$, in fact $\ell_{R_f}(M) = \cap_{m \in M} \ell_{R_f}(m) = \cap_{m \in M} \ell_{R_f}(m)$.

The following proposition gives a characterization of quasi-injective gamma modules.

Proposition 3.17 An $R_f$- module $M$ is quasi-injective if and only if for each left ideal $L$ of $R$, each $R_f$- homomorphism $\psi: L \to M$ with $\ell_{R_f}^\psi(a) \subseteq Ker(\psi)$ for some $a \in M$, $\psi$ can be extending to an $R_f$- homomorphism from $R$ to $M$.

Proof. Assume $M$ is quasi-injective $R_f$- module, $L$ a left ideal of $R$ and $\psi$ an $R_f$- homomorphism from $L$ to $M$ with $\ell_{R_f}^\psi(a) \subseteq Ker(\psi)$ for some $a \in M$. Then $L \gamma \alpha$ is $R_f$- submodule of $(\alpha)$ and $\alpha(\ell_{R_f}^\psi(a)) = f(r)$ for any $r \gamma \alpha \in L \gamma \alpha$, if $r \gamma \alpha = 0$, then $\ell_{R_f}^\psi(a) \subseteq Ker(\psi)$, so $f(r) = 0$, hence $\alpha$ is well defined and easily to show $\alpha$ is $R_f$- homomorphism, also $\ell_{R_f}^\psi(a) = \ell_{R_f}^\psi(a)$, hence $\alpha$ can extends to an $R_f$- homomorphism $\beta: a \to M$, define $g: R \to M$ by $g(r) = \beta(r \gamma \alpha)$ for each $r \in R$. Since $g(r) = \alpha(\ell_{R_f}^\psi(a)) = \beta(r \gamma \alpha) = f(r)$ for each $r \in L$, then $g$ extended to $f$. Conversely, Assume $B$ is an $R_f$- submodule of $M$ and $\alpha B \to M$ is $R_f$- homomorphism. By Zorn's lemma there exists a maximal element $(B_\alpha, \alpha)$ such that $B \leq B$, and $\alpha$ extends $x \to B_\alpha$, if $B = M$ the proof complete, if not there exists $a \in M$ and $a \in B_\alpha$, take $L = \{ r \in R : r \gamma \alpha \in B_\alpha \}$, then $L$ is left ideal of $R$, define $\psi: L \to M$ by $\psi(r) = \alpha(\ell_{R_f}^\psi(a))$ for each $r \in L$, if $r = 0$, then $\psi(r \gamma \alpha) = 0$, hence $\alpha = \alpha(\ell_{R_f}^\psi(a)) = \psi(r)$, therefore $\psi$ is well if and it is $R_f$- homomorphism and for each $r \in \ell_{R_f}^\psi(a)$, then $\psi(r \gamma \alpha) = 0$, so $\psi(\ell_{R_f}^\psi(a)) = \psi(r)$, so $r \in Ker(\psi)$, thus $\ell_{R_f}^\psi(a) \subseteq Ker(\psi)$, by hypothesis $\psi$ extended to $R_f$- homomorphism $\lambda: R \to M$. Define $R_f$- submodule $C$ by
C = B + Rγ a and β : C → M by β(b + rγ a) = α(b) + λ(r) for each b ∈ B and r ∈ R, if b + rγ a = 0, then b = −rγ a ∈ Bγ , so r ∈ L and hence λ(r) = ϕ(r) = αr(γ a) = −αγ (b) . thus β(b + rγ a) = 0, so β is well defined and it is Rγ - homomorphism, for each b ∈ B , a(b) = αγ (b) = αγ (b) + λ(0) = β(b), a contradiction with maximality of (Bγ , αγ ) , so Bγ = M, thus M is quasi-injective.

**Proposition 3.18** If M is an Rγ-module, E = EndRγ (E(M)) and Q = MΓE, where MΓE = {xyf : x ∈ M, α ∈ Γ and f ∈ E}, then:
1. Q is an quasi-injective Rγ-submodule of E(M) containing M.
2. Q is the intersection of quasi-injective Rγ-submodule of E(M) containing M.
3. M = Q if and only if M is quasi-injective.
4. Q is the smallest quasi-injective Rγ-submodule of E(M) that contains M, furthermore, Q is essential extension of M.

**Proof.**
1. Since M = 1E(M)(M) = 1E(M)(1γ M) = Mγ 1E(M) ⊆ Q, this shows that Q contains M. For all x ∈ M, α ∈ Γ and f ∈ E, then xyf = f(1γ x), M ⊆ E(M), clearly that Q is an Rγ-submodule of E(M). If N is an Rγ-submodule of Q and f N → Q is an Rγ-homomorphism, by injectivity of E(M), there exists φq : E(M) → E(M) which extends f. Since φqf(xyγ ) = φq(1γ x) = φq(1γ yx) = φq(1γ yq(1γ x)) = xyq(φq(1γ x)) ∈ Q for x ∈ M, α ∈ Γ and f ∈ E, therefore φq(Q) ⊆ Q, so if define ϕ = φq1Q, then ϕ_{1Q} = f and thus Q is quasi-injective Rγ-module.
2. Let Q′ be a quasi-injective Rγ-submodule of E(M) containing M. By proposition(3.7) and part(1) f(Q′) ⊆ Q′ for f ∈ E, since M ⊆ Q′, then Q = MΓE ⊆ Q′ΓE ⊆ E(1Q′) ⊆ E(Q′) ⊆ Q′ and this shows that Q is the smallest one. Now for any family of a quasi-injective Rγ-submodules {Qα}α∈A of E(M) each of which contains M, then Q ⊆ α∈A Qα , but α∈A Qα ⊆ Q, since Q ⊆ {Qα}α∈A. Thus Q = α∈A Qα.
3. Follows from(1) and from(2).
4. It’s clear from(2) Q is the smallest quasi-injective Rγ-submodule of E(M) contains M, since M essential in E(M), hence Q is essential in E(M) [1].

**Definition 3.19** Let M be an Rγ-module. A quasi-injective hull of M denoted by Q(M) is a quasi-injective Rγ-module containing M such that for any Rγ-monomorphism f from M into a quasi-injective Rγ-module N, extends to an Rγ-monomorphism from Q(M) into N. In fact, Q(M) = MΓEndRγ (E(M)).

**Lemma 3.20** Every Rγ-module has a quasi-injective hull which is unique up to isomorphism.

**Proof.** Let M be an Rγ-module, for each quasi-injective extension N of M and Rγ-monomorphism f from M into N, let E = EndRγ (E(M)), E′ = EndRγ (E(N)) and Q = MΓE. By proposition(3.7), we have MΓE ⊆ N. Since E(N) is injective Rγ-module, there exists Rγ-homomorphism g : Q → E(N) such that g_{1Q} = iQf, where i : M → E(N) is the inclusion mapping of M into Q(E(N)) , if x ∈ Ker(g) ∩ M, then f(x) = g(x) = 0, so f(x) = 0, hence x ∈ Ker(f) = 0, thus g is Rγ-monomorphism, hence g(Q) is quasi-injective and so E′(g(Q)) ⊆ g(Q), take X = N ∩ g(Q), then XE′ ⊆ X, so by proposition(3.7), X is quasi-injective, hence g^{-1}(X) is quasi-injective Rγ-submodule of E(M) contains M, by proposition(3.18) Q = g^{-1}(X), hence g(Q) = X ⊆ N. If there exists another quasi-injective hull T of M, then there exists an Rγ-monomorphism g : T → Q such that g i = i where i inclusion mapping from M to Q, for each xy ∈ Q, g g(xy) = g g(xy) = g i_{iQ} f(xy) = h(1γ x) = xy, so g g = iQ, hence g is Rγ-isomorphism.

**Definition 3.21** Let M be an Rγ-module and I a left ideal of R. M is called I-bounded if for each left ideal J of R, there exists an element m in M with \ell_{Rγ}(m) ≤ J if and only if 1 ≤ J.

Every Rγ-module M is R-bounded, since 0 ∈ M and \ell_{Rγ}(0) = R and M is 0-bounded if there exists an element m in M with \ell_{Rγ}(m) ≤ J for each ideal J of R.

**Remarks 3.22** Let I be a left ideal of a ring R and M is an I-bounded Rγ-module. Then.
1. I is the minimal ideal of R with the property I = \ell_{Rγ}(m) for some m ∈ M. Since I ≤ 1, so by definition(3.21), there exists m ∈ M such that \ell_{Rγ}(m) ≤ 1. On the other hand, since \ell_{Rγ}(m) ≤ \ell_{Rγ}(m) again by definition(3.21)
we have \( l \leq \ell_{R_f}(m) \), so \( l = \ell_{R_f}(m) \). For the minimality, if there exists a left ideal \( I_1 \) of \( R \) such that \( I_1 = \ell_{R_f}(x) \) for some \( x \in M \), since \( \ell_{R_f}(x) \leq I_1 \), by definition(3.21) we have \( l \leq I_1 \).

2. \( I \) is two-sided ideal of \( R \). Since \( l = \ell_{R_f}(m) \), then \( I \Gamma M \leq I \Gamma m = 0 \). So \( (I \Gamma R)(\Gamma M) \leq I \Gamma M = 0 \), hence \( I \Gamma R \leq \ell_{R_f}(M) \leq \ell_{R_f}(m) = I \), therefore \( I \) is two-sided ideal.

3. Suppose there exists \( m \in M \) such that \( l \leq \ell_{R_f}(m) \), then there is no element \( x \in M \) such that \( \ell_{R_f}(x) \leq \ell_{R_f}(m) \) which is a contradiction. Thus \( l \leq \bigcap_{m \in M} \ell_{R_f}(m) \) for each \( m \in M \).

4. Since \( l \leq \ell_{R_f}(m) \) for each \( m \in M \), then \( l \leq \bigcap_{m \in M} \ell_{R_f}(m) = \ell_{R_f}(M) \), so \( M \) is \((R/I)_f \) – module by the rule \( (r + l, y, m) \mapsto rym \) for each \( r \in R, y \in \Gamma \) and \( m \in M \).

The following proposition gives a characterization of quasi-injective gamma modules.

**Theorem 3.23** Let \( M \) be an \( I \) – bounded \( R_f \) – module. Then \( M \) is quasi-injective if and only if it is injective as an \((R/I)_f \) – module.

**Proof.** Assume \( M \) is injective – module. Let \( K/I \) is an \( R_f \) – submodule of \( R/I \) and \( f: K/I \to M \) an \( R_f \) – homomorphism. Define \( \alpha: K \to M \) by \( \alpha(r) = f(r + I) \) for each \( r \in K \), if \( r = 0 \), then \( f(r + I) = 0 \), hence \( \alpha \) is well-defined and it’s easily to show that \( \alpha \) is \( R_f \) – homomorphism. Since \( I \leq \ker(f) \), then \( I = \ell_{R_f}(M) \leq \ker(\alpha) \). So \( \ell_{R_f}^{-1}(a) \subseteq \ker(\alpha) \), hence by proposition(3.17) \( \alpha \) can extends to an \( R_f \) – homomorphism \( \beta: R \to M \). Define \( g: R/I \to Q \) by \( g(r + I) = \beta(r) \) for each \( r \in R \). If \( r + 1 = I \), then \( r \in I \), so \( \beta(r) = \alpha(r) = 0 \), thus \( g(r + I) = 0 \), therefore \( g \) is well-defined and for each \( r \in K \), \( g(r + I) = \beta(r) = \alpha(r) = f(r + I) \), then \( M \) is injective as \((R/I)_f \) – module.

**Corollary 3.24** Let \( M \) be \( 0 \) – bounded \( R_f \) – module. Then \( M \) is quasi-injective if and only if it is injective.

**Examples 3.25**

1. If \( R \) is simple, then \( M \) is bounded – module. Since \( R \Gamma \neq 0 \), then there exists a non-zero element \( r \in R \), since \( r = 1 \), so \( 1 \notin \ell_{R_f}(r) \), thus \( \ell_{R_f}^{-1}(r) \neq R_f \), hence \( \ell_{R_f}(r) = 0 \). In particular, \( Z_2 \) as \((Z_2)_Z \) – module is bounded – module.

2. The \( Z_2 \) – module \( Z_2 \) is not \( 0 \) – bounded. Take the ideal \( J = 3Z \), since \( Z_2 = \{0, 1\} \), \( \ell_{R_f}(0) = Z \) and \( \ell_{R_f}(1) = \{n \in Z: n \text{ is even}\} \), so there is not \( m \in Z_2 \) such that \( \ell_{R_f}(m) \leq J \) but \( 0 \leq J \). By example(3.14)(2) \( Z_2 \) is quasi-injective but not injective, this example show that the condition of \( 0 \) – bounded in corollary(3.24) cannot be dropped.

3. The \( Z_2 \) – module \( Z_2 \) is not \( 0 \) – bounded for each ideal \( J \) of \( Z \). Since \( \ell_{R_f}(n) = 0 \leq J \) for any nonzero \( n \) in \( Z \).

4. Let \( R = Z_{12} \), \( \Gamma = Z \) and \( M = Z_{12} \). Take \( I_1 = \{0\}, I_2 = \{0, 6\}, I_3 = \{0, 4, 8\}, I_4 = \{0, 3, 6, 9\}, I_5 = \{0, 7, 4, 6, 8, 10\} \), then \( \ell_{R_f}(0) = Z_{12} \), \( \ell_{R_f}(I_1) = \ell_{R_f}(5) = \ell_{R_f}(7) = \ell_{R_f}(11) = I_1 \), \( \ell_{R_f}(2) = I_2 \), \( \ell_{R_f}(3) = \ell_{R_f}(9) = I_3 \), \( \ell_{R_f}(4) = \ell_{R_f}(8) = I_4 \). Relations \( I_2 \leq \ell_{R_f}(6) = I_5 \), hence \( Z_{12} \) is \( I_2 \) bounded, \( I_2 \) bounded, \( I_3 \) bounded, \( I_4 \) bounded, \( I_5 \) bounded and \( Z_{12} \) bounded. So \( Z_{12} \) is quasi-injective as \((Z_{12}/I_2)_Z \) – module (j=1,2,...,6) by theorem(3.23).

**Lemma 3.26** If direct sum of every pair of quasi-injective \( R_f \) – modules is quasi-injective, then every quasi-injective is injective.

**Proof.** For any ideal \( I \) of \( R \) and \( R_f \) – homomorphism \( f: I \to M \), since \( M \oplus E(R) \) is quasi-injective, then there exists an \( R_f \) – endomorphism \( g \) of \( M \) such that \( i_M f = g_iR_f i \), where \( i_M(\ell_R I) \) is the inclusion mapping of \( M(R, I) \) into \( M \oplus E(R) \). Define \( \overline{g} : R \to M \) by \( \overline{g} = \pi_1 g \pi_2 \) where \( \pi_1 \) is the projection of \( M \oplus E(R) \) into \( M \), then \( \overline{g}(n) = \pi_1 g \pi_2(\pi_1(n)) = \pi_1 i_M f(n) = f(n) \) for each \( n \in I \), so by proposition(1.7) in [1] \( M \) is injective.

Let \( R \) be a \( \Gamma \) – ring, the radical \( J(R) \) of \( R \) is the set of all elements of \( R \) which annihilates all simple \( R_f \) – modules [6]. An element \( a \) in \( \Gamma \) – ring \( R \) is called left quasi-regular if there exists \( a \) in \( R \) such that \( a + a' + a' \gamma a = 0 \) for each \( \gamma \in \Gamma \), an ideal \( I \) of \( R \) is left quasi-regular if each its elements is left quasi-regular [10].

**Theorem 3.27** [10] Let \( R \) be \( \Gamma \) – ring. Then the radical \( J(R) \) of \( R \) is left quasi-regular ideal of \( R \) contains every left quasi-regular ideal of \( R \).
An element $x$ of a $\Gamma$-ring $R$ is called regular if there exists $s \in R$ such that $x = xasyx$ for some $\gamma, \alpha \in \Gamma$ and $R$ is regular if each element of $R$ is regular [8].

**Theorem 3.28** Let $M$ be a quasi-injective $R_t$-module and $E = End_{R_t}(M)$, then $J(E) = \{f : E : \text{Ker}(f) \text{ is essential} \}$.

**Proof.** Let $K = \{f \in E : \text{Ker}(f) \text{ essential} \}$, for each $f, g \in K$, since $\text{Ker}(f) \cap \text{Ker}(g) \subseteq \text{Ker}(f - g)$, then $\text{Ker}(f - g)$ is essential $R_t$-submodule of $M$, so $f - g \in K$ and for each $f \in K, \gamma \in \Gamma$ and $h \in E$, if $N$ is non-zero $R_t$-submodule of $M$, since $\text{Ker}(f) \leq M$, then $h^{-1}(\text{Ker}(f)) \leq N$ by [1, lemma(3.3)], so there exists $n' \neq 0 \in N \cap h^{-1}(\text{Ker}(f))$, hence $h(n) \in \text{Ker}(f)$, and $h(n)(\gamma)h(n) \in \text{Ker}(f)$, so $(\gamma h)(n) = f(h(n)) = 0$, hence $n \in N \cap \text{Ker}(\gamma h) = 0$, so $\text{Ker}(\gamma h) \leq M$, hence $\gamma h \in K$, thus $\text{ETG} \subseteq K$, this show that $K$ is an ideal of $E$. Now for each $f \in K$, define an $R_t$-homomorphism $h : M \to M$ by $h(x) = 1\gamma f(x)$ for each $x \in M$, since $\text{Ker}(f) \leq \text{Ker}(h)$, then $\text{Ker}(h) \leq M$ but $\text{Ker}(h) \cap \text{Ker}(I - h) = 0$ where $I = id(M)$, hence $\text{Ker}(I - h) = 0$ and $I - h : M \to \text{Im}(I - h)$ is an $R_t$-isomorphism, so there exists $g : M \to M$ such that $g (I - h)(M) = (I - h)^{-1}$, hence $g(I - h) = I$. Define an $R_t$-homomorphism $\tau : M \to M$ by $\tau(x) = g(x) - f(x) - I(x)$ for each $x \in M$, then $g(x) = \tau(x) + f(x) + I(x)$, so $g = g - gh = t + f + I - gyf$, hence $t + f - gyf = 0$ for each $\gamma \in \Gamma$, therefore $\gamma$ is quasi-regular by theorem(3.27) $f \in J(E)$, thus $K \subseteq J(E)$. For each $f \in J(E)$, let $K \subseteq M$ with $K \cap \text{Ker}(f) = 0$, then $\gamma f \in J(E)$, since $\gamma f$ is quasi-regular by theorem(3.11) in [10], thus there exists $h \in E$ such that $\gamma f + h + hgyf, f = 0$, that is, $\gamma f + h = 1$, hence $\text{Ker}(I + gyf) = 0$ but $K \subseteq \text{Ker}(I + gyf)$, so $K = 0$, therefore $\text{Ker}(f) \leq M$, thus $J(E) \subseteq K$, so $K = J(E)$. For each $\tilde{f} = f + K \in E/K$, take $B = (\text{Ker}(f))^\gamma$ in $M$, since $\text{Ker}(f) = 0$, then $fB$ is an $R_t$-isomorphism and $\tilde{f}_{|B}^{-1} : \tilde{f}(B) \to B$ is $R_t$-isomorphism, so $\tilde{f}_{|B}^{-1}$ can be extended to $R_t$-homomorphism $g : M \to M$ such that $g(f|B) = f_{|B}^{-1}$, so $(g, f)(b) = g((f|B)(b)) = g((f|B)(b)) = b$ for each $b \in B$, hence $gyf, f = id(B)$, since $(f_{|B}, gyf, f - \tilde{f})B = (f_{|B}, gyf - f)(B) - (f-B) = 0$. Then $(f_{|B}, gyf, f - \tilde{f})(B + \text{Ker}(f)) = 0$, so $B \oplus \text{Ker}(f) \leq \text{Ker}(f_{|B}, gyf, f - \tilde{f})$ but $B \oplus \text{Ker}(f) \leq M$, then $\text{Ker}(f_{|B}, gyf, f - \tilde{f}) \leq M$ and $f_{|B}, gyf, f - f \in \text{Ker}(f)$, so $f_{|B}, gyf, f + K = f + K$, take $\tilde{f} = \tilde{f}_{|B}, gyf, f$, hence $E/K$ is regular, thus $E/J(E)$ is regular.

**References:**