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RESEARCH ARTICLE

THEOREM ON ALTERNATING SERIES INVOLVNG POSITIVE NUMBERS AND THE FACTORIAL.

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Abstract

The major objective of this paper is to present generalized solution on alternating series of some consecutive natural or whole numbers in addition to Pascal's identity or binomial coefficients with the notion of factorial n.

Introduction:-

Factorial function is not a new concept for any mathematical reader. Factorials in addition to binomial coefficients are normally defined in a rather narrow structure. So-called Pascal's Triangle with the study of factorials has always been a focus of need so referring the reader to [1] for more information. Furthermore, a sequence is a set of terms that occur in a specific order and the sum of finite or infinite terms of a sequence is known as a series. Some special types of series are Harmonic series, Arithmetic series, Geometric series, Alternating series etc. This paper develops a new generalized expression, comprised of alternating series and binomial coefficients, that leads to simple factorial of a number.

Here are some basic concepts related to the topic.

Preliminary Results:-

Definition 1: [3]

Let X be a nonempty set, then a function f:N→ X whose domain is a set of natural numbers, is called an infinite sequence in X. If the domain of f is the finite set of numbers {1, 2,3,…,n} then it is called a finite sequence.

Definition 2:-

Let $\{a_n\}$ be a sequence. An expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Containing finite terms of the sequence $\{a_n\}$ is called a finite series. Symbolically, it is represented as $\sum_{1}^{n} a_i$ .Otherwise it is an infinite Series.

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Definition 3:-

A series with both negative and positive terms is called mixed series. An important type of mixed series is **alternating series**, which is any series, for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n a_n \qquad a_n \geq 0$$

$$a_n = (-1)^{n+1} a_n \qquad a_n \geq 0$$

i.e. the terms of the series are alternately positive and negative [3]. In other words, if $a_n \geq 0$ for all n, then the series

$$a_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n \cdots$$

and the series

$$a_n = -a_1 + a_2 - a_3 + \cdots + (-1)^n a_n \cdots$$

are called alternating series

Definition 4:- [2]

Pascal's Identity or Pascal's Rule is a useful approach in combinatorics theory for dealing with combinations and to simplify complex expressions. Here is the Pascal's triangle with power 0 in the first row, power 1 in the second row and so on.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

Lemma 1:-

$$T_x^j = \sum_{i=0}^{n} (-1)^i \, {}^nC_i \, (x-i)^j = 0 \qquad j = 0,1,2,3,\ldots,n-1 \quad (1.1)$$

Lemma 2:-

$$T_n^0 = \sum_{i=0}^{n} (-1)^i \, {}^nC_i = 0 \qquad \forall n > 0$$

Theorem:-

An alternating series consisting of (n+1) terms yields n factorial, if the terms are the product of Pascal's coefficients of any power n with the corresponding (n+1) consecutive positive decreasing numbers exponentiated with n.

$$T_x^n = \sum_{i=0}^{n} (-1)^i \; {}^{n}C_i \left(x-i\right)^n = n! \qquad \forall x \ge n,\ n \in N,\ x \in N$$

Proof:- Using mathematical induction

For $x = n$

$$T_n^n = \sum_{i=0}^{n} (-1)^i \; {}^{n}C_i \left(n-i\right)^n \tag{1.2}$$

$$T_n^n = n\sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \left(n-1-i+1\right)^{n-1}$$

$$T_n^n = n\sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \sum_{l=0}^{n-1} {}^{n-1}C_l \left(n-1-i\right)^{n-1-l}$$

$$T_n^n = n\begin{bmatrix} {}^{n-1}C_0 \sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \left(n-1-i\right)^{n-1} + \\ {}^{n-1}C_1\left(0\right) + {}^{n-1}C_2\left(0\right) + \cdots + {}^{n-1}C_{n-1}\left(0\right) \end{bmatrix} \qquad \text{by Lemma 1}$$

$$T_n^n = n \sum_{i=0}^{n-1} (-1)^i \; {}^{n-1}C_i \left(n-1-i\right)^{n-1}$$

$$T_n^n = n\left(n-1\right)\sum_{i=0}^{n-2} (-1)^i \; {}^{n-2}C_i \left(n-2-i\right)^{n-2}$$

and so on generalizing

$$T_n^n = n\left(n-1\right)\left(n-2\right)\left(n-3\right)\cdots 2\sum_{i=0}^{1} (-1)^i \; {}^{1}C_i \left(1-i\right)^1$$

$$T_n^n = n!$$

For $x = n+1$

$$T_{n+1}^n = \sum_{i=0}^{n} (-1)^i \; \frac{n!}{i!\left(n-i\right)!}\frac{1}{\left(n+1-i\right)}\left(n+1-i\right)^{n+1}$$

$$T_{n+1}^n = \frac{1}{n+1}\sum_{i=0}^{n} (-1)^i \; {}^{n+1}C_i \left(n+1-i\right)^{n+1}$$

$$T_{n+1}^n = \frac{1}{n+1}\sum_{i=0}^{n+1} (-1)^i \; {}^{n+1}C_i \left(n+1-i\right)^{n+1}$$

$$T_{n+1}^n = \frac{1}{n+1}\left(n+1\right)!$$

$$T_{n+1}^n = n!$$

For $x = n+2$

$$T_{n+2}^{n} = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^{i}\ {}^{n+1}C_i\left(n+2-i\right)^{n}\left(n+1-i\right)$$

$$T_{n+2}^{n} = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^{i}\ {}^{n+1}C_i\left(n+2-i\right)^{n}\left(n+1-i\right)$$

$$T_{n+2}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\left(n+2-i\right)^{n}\left(n+2-i-1\right)$$

$$T_{n+2}^{n} = \frac{1}{n+1}\left\{\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\left(n+2-i\right)^{n+1}-0\right\}$$

Put $n+1=m$ within summation in the expression

$$T_{n+2}^{n} = \frac{1}{n+1}\left\{\sum_{i=0}^{m}(-1)^{i}\ {}^{m}C_i\left(m+1-i\right)^{m}\right\}$$

$$T_{n+2}^{n} = \frac{1}{n+1}m!$$

$$T_{n+2}^{n} = \frac{1}{n+1}(n+1)!$$

$$T_{n+2}^{n} = n!$$

Suppose equation holds for $x=n+k$.

Now for $x=n+k+1$

$$T_{n+k+1}^{n} = \sum_{i=0}^{n}(-1)^{i}\ {}^{n}C_i\left(n+k+1-i\right)^{n}$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\left(n+k+1-i\right)^{n}\left(n+1-i\right)$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\left(n+k+1-i\right)^{n}\left(n+k+1-i-k\right)$$

$$T_{n+k+1}^{n} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^{i}\ {}^{n+1}C_i\left(n+k+1-i\right)^{n+1}$$

Put $n+1=m$ within summation in the expression

$$T_{n+k+1}^{n} = \frac{1}{n+1}T_{m+k}^{m}$$

$$T_{n+k+1}^{n} = \frac{1}{n+1} m! \qquad \text{as it holds for } k$$

$$T_{n+k+1}^{n} = \frac{1}{n+1} (n+1)! \qquad \text{back substitution}$$

$$T_{n+k+1}^{n} = n!$$

Hence, completes the induction.

Conclusion:-

It is applicable on several problems of number theory. A lot more can be explored in future by extending this work. .

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