ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

INCLUSION THEOREM ON TWO ABSOLUTE SUMMABILITY METHODS: A GENERALIZED FORM

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Manuscript Info

Manuscript History:

Received: 12 June 2014
Final Accepted: 26 July 2014
Published Online: August 2014

Key words:
:$(\bar{N},p_n)$ means, Cesàro means, Absolute summability.

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Abstract

Bor [1] (see also [2]) has investigated inclusion theorems establishing $|\bar{N},p_n|_k \subset |C,1|_k, |C,1|_k \subset |\bar{N},p_n|_k,\ k \geq 1.$ In the present paper, we extend the result of Bor [2] with different set of conditions and established that $|\bar{N},p_n,\theta_n|_k$ is equivalent to $|C,1|_k,\ k \geq 1.$

Introduction

Suppose $\sum a_n$ is an infinite series with sequence of partial sums $\{s_n\}$ where s_n is defined by $s_n = a_0 + a_1 + \cdots + a_n$. Also, let $u_n = na_n$. Let σ_n denotes the n^{th} Cesàro means of order 1 of the sequence $\{s_n\}$ and t_n denotes the n^{th} Cesàro means of order 1 of the sequence $\{u_n\}$.

In Order to appreciate the work already done in this field, we require the following definitions:

Definition 1.1 The series $\sum a_n$ is said to be absolutely summable $(C,1)$ of order k or simply summable $|C,1|_k$, if (cf, [3])

$$\sum_{n=1}^{\infty} n^{k-1}\,|\sigma_n - \sigma_{n-1}|^k < \infty \tag{1.1}$$

Since $t_n = n(\sigma_n - \sigma_{n-1})$(cf. [5]) , condition (1.1) can also be written as ,

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty \tag{1.2}$$

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \text{as} n \to \infty\ (P_{-1} = p_{-1} = 0\ ) \tag{1.3}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu \tag{1.4}$$

Defines the sequence (T_n) of the Riesz means or simply $(\bar{N},p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n).

Definition 1.2 The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k\ \ k \geq 1$,if (cf. [4])

$$\sum_{n=1}^{\infty} \left[\frac{P_n}{p_n}\right]^k |T_n - T_{n-1}|^k < \infty \tag{1.5}$$

In the special case $p_n = 1$, for all values of n, $|\bar{N},p_n|_k$ summability is same as $|C,1|_k$ summability. Let (θ_n) be any sequence of positive constants.

Definition 1.3 The series $\sum a_n$ is said to be summable $|\bar{N},p_n,\theta_n|_k,\ k \geq 1$, if (cf. [6])

$$\sum_{n=1}^{\infty} \theta_n^{k-1}|T_n - T_{n-1}|^k < \infty \tag{1.6}$$

2. Results already proved.

Refer the results concerning the relationship between $|\bar{N},p_n|_k$ summability and $|C,1|_k$ summability.

Theorem 2.1 (cf. [1]) Let (p_n) be a sequence of positive real constants such that, as $n \to \infty$,

$$\text{(i)} \quad np_n = O(P_n) \qquad \text{(ii)} \quad P_n = O(np_n) \tag{2.1}$$

If $\sum a_n$ is summable $|C,1|_k$, then it is also summable $|\bar{N},p_n|_k,\ k \geq 1$.
Theorem 2.2 (cf. [2]) Let(p_n) be a sequence of positive real constants such that it satisfies the condition (2.1). If $\sum a_n$ is summable $|\bar{N},p_n|_k$, then it is also summable $|C,1|_k,\ k \geq 1$.
Theorem 2.3 (cf. [2]) Suppose(p_n) is a sequence of non-negative real constants such that $P_n = \sum_{\nu=0}^{n} p_\nu \neq 0,\ P_n \to \infty$, as $n \to \infty$, and that (2.1) holds. Then summability$|C,1|_k$ is equivalent to summability $|\bar{N},p_n|_k,\ k \geq 1$.

3. Main result

We now state our main theorem which is an extension of Theorem 2.3 with simple set of conditions.
Theorem 3.1 Let (p_n) be a sequence of positive real numbers such that$P_n = \sum_{\nu=0}^{n} p_\nu \neq 0,\ P_n \to \infty$, as $n \to \infty$, and that it satisfies condition (2.1). Let (θ_n) be any sequence of positive real numbers satisfying following conditions

$$n = O(\theta_n) \tag{3.1}$$

$$\frac{1}{n} = O(\frac{1}{\theta_n}) \tag{3.2}$$

$$\frac{P_n}{p_n} = O(\theta_n) \tag{3.3}$$

$$\theta_n = O\left(\frac{P_n}{p_n}\right) \tag{3.4}$$

Then summability$|C,1|_k$, is equivalent to summability $|\bar{N},p_n,\theta_n|_k,\ k \geq 1$.

Proof of Theorem 3.1

We first establish that $|\bar{N},p_n,\theta_n|_k \subset |C,1|_k (k \geq 1)$.

We have

$$\begin{aligned} T_n &= \frac{1}{P_n}\sum_{i=0}^{n} p_i s_i \\ &= \frac{1}{P_n}\sum_{i=0}^{n} p_i \sum_{\nu=0}^{i} a_\nu \\ &= \frac{1}{P_n}\sum_{\nu=0}^{n} a_\nu \sum_{i=\nu}^{n} p_i \\ &= \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu \qquad (P_{-1} = 0, \text{by convention}) \end{aligned}$$

Consider,

$$\begin{aligned} \Delta T_{n-1} &= T_n - T_{n-1} \\ &= \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}(P_{n-1} - P_{\nu-1})a_\nu \\ &= [\frac{1}{P_{n-1}} - \frac{1}{P_n}]\sum_{\nu=1}^{n} P_{\nu-1}a_\nu \\ &= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu \end{aligned}$$

$$\frac{P_n P_{n-1}}{p_n}\Delta T_{n-1} = \sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

$$\frac{P_{n-1}P_{n-2}}{p_{n-1}}\Delta T_{n-2} = \sum_{\nu=1}^{n-1} P_{\nu-1}a_\nu$$

Hence,

$$a_n = \frac{P_n}{p_n}\Delta T_{n-1} - \frac{P_{n-2}}{p_{n-1}}\Delta T_{n-2},\quad n \geq 1. \tag{3.5}$$

It may be easily seen that (3.5) also holds for n=1, since in this case $P_{-1} = 0$.
By t_n, we denote the n^{th} (C,1) mean of the sequence (na_n) i.e. $t_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu$

Since, by (3.5)

$$a_\nu = \frac{P_\nu}{p_\nu}\Delta T_{\nu-1} - \frac{P_{\nu-2}}{p_{\nu-1}}\Delta T_{\nu-2},$$

So

$$\begin{aligned} t_n &= \frac{1}{n+1}\sum_{\nu=1}^{n} \nu(\frac{P_\nu}{p_\nu}\Delta T_{\nu-1} - \frac{P_{\nu-2}}{p_{\nu-1}}\Delta T_{\nu-2}) \\ &= \frac{1}{n+1}\sum_{\nu=1}^{n-1} \nu\frac{P_\nu}{p_\nu}\Delta T_{\nu-1} + \frac{nP_n}{(n+1)p_n}\Delta T_{n-1} - \frac{1}{n+1}\sum_{\nu=1}^{n} \nu\frac{P_{\nu-2}}{p_{\nu-1}}\Delta T_{\nu-2} \\ &= \frac{1}{n+1}\left\{\sum_{\nu=1}^{n-1} \nu\frac{P_\nu}{p_\nu}\Delta T_{\nu-1} - \sum_{\nu=1}^{n-1}(\nu+1)\frac{P_{\nu-1}}{p_\nu}\Delta T_{\nu-1}\right\} + \frac{nP_n}{(n+1)p_n}\Delta T_{n-1} \\ &= \frac{1}{n+1}\{\sum_{\nu=1}^{n-1}\frac{1}{p_\nu}\Delta T_{\nu-1}[\nu P_\nu - (\nu+1)P_{\nu-1}]\} + \frac{nP_n}{(n+1)p_n}\Delta T_{n-1} \end{aligned}$$

Since $\nu P_\nu - (\nu+1)P_{\nu-1} = \nu P_\nu - (\nu+1)(P_\nu - p_\nu) = (\nu+1)p_\nu - P_\nu$

We have,

$$\begin{aligned} t_n &= \frac{1}{n+1}\{\sum_{\nu=1}^{n-1}\frac{1}{p_\nu}\Delta T_{\nu-1}[(\nu+1)p_\nu - P_\nu]\} + \frac{nP_n}{(n+1)p_n}\Delta T_{n-1} \\ &= \frac{1}{n+1}\sum_{\nu=1}^{n-1}(\nu+1)\Delta T_{\nu-1} - \frac{1}{n+1}\sum_{\nu=1}^{n-1}\frac{P_\nu}{p_\nu}\Delta T_{\nu-1} + \frac{nP_n}{(n+1)p_n}\Delta T_{n-1} \\ &= t_{n1} + t_{n2} + t_{n3} \qquad \text{(say)} \end{aligned}$$

To prove the theorem, by Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}|t_{nr}|^k < \infty \text{ for } r = 1,2,3.$$

Now by applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}|t_{n1}|^k &= \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\Delta T_{v-1}\right|^k \\
&= \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}\frac{v+1}{v}\ v\Delta T_{v-1}\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{k+1}}\{\sum_{v=1}^{n-1}v|\Delta T_{v-1}|\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}v^k|\Delta T_{v-1}|^k\times\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}v^k|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m}v^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{1}{n^2} \\
&= O(1)\sum_{v=1}^{m}v^k|\Delta T_{v-1}|^k\frac{1}{v} \\
&= O(1)\sum_{v=1}^{m}v^{k-1}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}|\Delta T_{v-1}|^k \quad \text{(By (3.1))} \\
&= O(1)\text{as } m\to\infty.
\end{aligned}$$

Again, by applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}|t_{n2}|^k &= \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}\frac{P_v}{p_v}\Delta T_{v-1}\right|^k \\
&\le \sum_{n=2}^{m+1}\frac{1}{n^{k+1}}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}|\Delta T_{v-1}|\right\}^k \\
&\le \sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}\left[\frac{P_v}{p_v}\right]^k|\Delta T_{v-1}|^k\times\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}\left[\frac{P_v}{p_v}\right]^k|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m}\left[\frac{P_v}{p_v}\right]^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{1}{n^2} \\
&= O(1)\sum_{v=1}^{m}\left[\frac{P_v}{p_v}\right]^k\frac{1}{v}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m}\theta_v^k\frac{1}{\theta_v}|\Delta T_{v-1}|^k \quad \text{(By (3.2) and (3.3))} \\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}|\Delta T_{v-1}|^k \\
&= O(1) \quad \text{as } m\to\infty
\end{aligned}$$

Finally, by applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}|t_{n3}|^k &= \sum_{n=1}^{m}\frac{1}{n}\left|\frac{nP_n}{(n+1)p_n}\Delta T_{n-1}\right|^k \\
&= O(1)\sum_{n=1}^{m}\frac{1}{n}\left[\frac{P_n}{p_n}\right]^k|\Delta T_{n-1}|^k \\
&= O(1)\sum_{n=1}^{m}\theta_n^{k-1}|\Delta T_{n-1}|^k \quad \text{(By (3.2) and (3.3))} \\
&= O(1) \text{ as } m\to\infty
\end{aligned}$$

Therefore from above estimates, we finally arrive at the conclusion that

$$\sum_{n=1}^{m}\frac{1}{n}|t_{nr}|^k = O(1)\text{as } m\to\infty, \text{ for } r = 1,2,3.$$

So ,$\sum_{n=1}^{\infty}\frac{1}{n}|t_n|^k<\infty$

Hence $\sum a_n$ is summable $|C,1|_k$

Conversely, we shall now establish the following inclusion:$|C,1|_k\subset|\bar{N},p_n,\theta_n|_k, k\ge 1$

We have $$T_n = \frac{1}{P_n}\sum_{v=0}^{n}p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v. \tag{3.6}$$

Therefore $$T_n-T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}v^{-1}va_v. \tag{3.7}$$

By applying Abel's partial summation summation formula to the sum $\sum_{v=1}^{n}P_{v-1}v^{-1}va_v$, we get

$$\begin{aligned}
\sum_{v=1}^{n}P_{v-1}v^{-1}va_v &= \sum_{v=1}^{n-1}\{P_{v-1}v^{-1}-P_v(v+1)^{-1}\}\sum_{i=1}^{v}ia_i + P_{n-1}n^{-1}\sum_{i=1}^{n}ia_i \\
&= \sum_{v=1}^{n-1}\{P_{v-1}v^{-1}-P_vv^{-1}+P_vv^{-1}-P_v(v+1)^{-1}\}(v+1)t_v+\frac{n+1}{n}P_{n-1}t_n \\
&= \sum_{v=1}^{n-1}\{-p_v\,v^{-1}+P_vv^{-1}(v+1)^{-1}\}(v+1)t_v+\frac{n+1}{n}P_{n-1}t_n \\
&= \sum_{v=1}^{n-1}-p_v\,v^{-1}(v+1)t_v+\sum_{v=1}^{n-1}P_v\,v^{-1}t_v+\frac{n+1}{n}P_{n-1}t_n
\end{aligned}$$

Referring (3.7), we get

$$T_n-T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}-(v+1)v^{-1}p_vt_v+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}v^{-1}P_vt_v+\left[\frac{n+1}{n}\right]\frac{p_n}{P_n}t_n$$

$$= t_{n1} + t_{n2} + t_{n3}, \qquad \text{(say)}$$

To prove the theorem, by Minkowski's inequality, it is enough if we show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|t_{n,r}\right|^k < \infty, \text{ for } r = 1,2,3. \qquad (3.8)$$

Now by using Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |t_{n1}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |t_{n1}|^k \qquad \text{(By (3.4))}$$
$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{\sum_{v=1}^{n-1} \frac{v+1}{v} p_v |t_v|\right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |t_v|^k \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} |t_v|^k$$
$$= O(1) \sum_{v=1}^{m} \frac{1}{v} |t_v|^k \qquad \text{(By (2.1))}$$
$$= O(1) \quad \text{as} \quad m \to \infty$$

Again consider,

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |t_{n2}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |t_{n2}|^k \qquad \text{(By (3.4))}$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v\, v^{-1} |t_v|\right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{\sum_{v=1}^{n-1} p_v |t_v|\right\}^k \qquad \text{(By (2.1)(ii))}$$
$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |t_v|^k \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} |t_v|^k$$
$$= O(1) \sum_{v=1}^{m} \frac{1}{v} |t_v|^k \qquad \text{(By (2.1))}$$
$$= O(1) \text{ as } m \to \infty.$$

Finally, we have

$$\sum_{n=1}^{m} \theta_n^{k-1} |t_{n3}|^k = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |t_{n3}|^k$$
$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} \left| \left[\frac{n+1}{n}\right] \frac{p_n}{P_n} t_n \right|^k$$
$$= O(1) \frac{p_n}{P_n} |t_n|^k$$
$$= O(1) \frac{1}{n} |t_n|^k \qquad \text{(By (2.1))}$$
$$= O(1) \quad as\ n \to \infty.$$

Therefore by (3.8), we get

$$\sum_{n=1}^{m} \theta_n^{k-1} |t_{nr}|^k = O(1) \ as\ m \to \infty, \quad \text{for } r = 1,2,3.$$

Which completes the proof of the theorem.

Remark. It may be noted that condition (2.1) is being discarded from the proof of the first part of the theorem.

Flett (cf. [7]) has shown that $|C,\alpha|_k \Rightarrow |C,\beta|_k,\ \alpha \geq 1, \beta \geq \alpha, \alpha > -1$. Following corollary is a direct consequence of a result due to Flett.

Corollary 3.1 $|\bar{N}, p_n, \theta_n|_k \subset |C,\beta|_k\,,\quad \beta \geq 1,\ \ k \geq 1.$

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