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**RESEARCH ARTICLE**

# Modified Variation Iteration method for fraction space- time partial differential heat and wave equations

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## *Abstract*

Variational iteration method (VIM) has been favorably applied to various kinds of differential equations. The main property of the method is in its flexibility and ability to solve differential equations accurately and conveniently. Recent trends and developments in the use of the method will be reviewed. The equations arising in various engineering applications will be surveyed. The confluence of modern mathematics and symbol computation has posed a challenge to developing technologies capable of handling strongly differential equations which cannot be successfully dealt with by classical methods. (VIM) is uniquely qualified to address this challenge. The flexibility and adaptation provided by the method have made the method a strong candidate for approximate analytical solutions. This paper outlines the basic conceptual framework of (VIM) with application to fractional order, space and time, partial differential equations, wave and heat with integral term, these kind of equations, with steps to finding multiply Lagrange and three methods to choose the start solution ( zeros solution), are discussed. Different examples will be solved and those figures will be given to show what happen to the equations with different fractional order. Matalab program was used to find these figures



## 1-Introductions:

The most important advantage of using fractional differential equations in these applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is a nonlocal operator. Numerical solution of differential equations of integer or fractional order has been a hot topic in numerical and computational mathematics for a long time. The solution of fractional differential equations has been recently studied by numerous authors. The (VIM) was originally proposed to solve ordinary differential equations and partial differential equations with integer and fractional order. He’s (VIM) [1], is based on the use of restricted variations and correction functional which has found a wide application for the solution of linear and nonlinear ordinary and partial differential equations. This method does not require the presence of small parameters in the differential equation, and provides exact solution (or an approximation solution) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. He’s (VIM) also based on a Lagrange multiplier technique. This method is, in fact, a modification of the general Lagrange multiplier method into an iteration method, which is called correction functional.

The method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems, generally one or two iterations lead to high accurate solutions differential equations and partial differential equations with integer and fractional order. In recent times, a new modified Riemann–Liouville left derivative is proposed by G. Jumarie[2] in fractional problems and fractional variational calculus. With Jumarie’s fractional derivative comparing with the classical Caputo derivative. The definition of the fractional derivative is not required to satisfy higher integer-order derivative than α. Second, αth derivative of a constant is zero. G. Jumarie's modified derivative was successfully applied in the stochastic fractional models. Y. Khana, N. Farazb [3,4], fractional functional for the (VIM) has been suggested to solve the linear and nonlinear time fractional order partial differential equations with time fractional order initial and boundary conditions by using the modified Riemann–Liouville. Olayiwola, M. O Akinpelu, F. O , Gbolagade, A .W [5], use Modified Variational Iteration Method solve of some partial differential equations with integer order of physical significance. Z. Odibat S. Momani [6], variational iteration method and the Adomian decomposition method, for solving nonlinear partial differential equations of fractional order. E. Salehpoora, H. Jafarib,[7], He's variational iteration method (VIM) to obtain solution nonlinear gas dynamics equation and Stefan equation. In this article modified variational iteration method to solve time and space fractional order partial differential equation with integral term, which is given by this form :

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + \int_{o}^{t} u(x,s)ds + q(x,t) \quad (1)$$

where: $0 < \alpha \leq 2$; $1 < \beta \leq 2$ ; $0 \leq x \leq l$ ; $0 \leq t$ with initial and boundary conditions given respectively: $u(x,0) = f_0(x)$, $u_t(x,0) = f_1(x)$ $\quad 0 \leq x \leq l$, $u(0, t) = g_0(t)$, $u(1, t) = g_1(t)$ $\quad 0 \leq t$.

## 2-material and method:

analytical solution isn't always available for a lot of this kind of equations, or the difficulty of finding those solutions, numerical methods were different genres, which gives the approximate solutions are identical or close acceptable analytical solutions.

He's variational iteration method for solving kinds of differential equations initial and initial-boundary, integer or fractional, order. This method is based on the use of Lagrange multipliers for identification of optimal value of a parameter in a functional. This technique provides a sequence of functions which converges to the exact solution of the problem. So, so many researchers used this method to finding approximate or exact solutions of more kinds of differential equations. In the past century, notable contributions have been made to both the theory and applications of the fractional differential equations. These equations are increasingly used to model problems in research areas as diverse as dynamical systems, mechanical systems, control, chaos, chaos synchronization, continuous time random walk, anomalous diffusive and sub diffusive systems, unification of diffusion and wave propagation phenomenon and others.

The most important advantage of using fractional differential equations in these applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is a nonlocal operator. Numerical solution of Differential equations of integer or fractional order has been a hot topic in numerical and computational mathematics for a long time.

The solution of fractional differential equations has been recently studied by numerous authors. The variational iteration method was originally proposed to solve ordinary. Lan Xu, [8], several integral equations solved by He's variational iteration method. Comparison with exact solution. E. J. Ali, [9], new technique is applied to modified treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations. M. Inc, [10], using variational iteration method to find numerical solution of the space- and time-fractional Burgers equations. Abdul-Majid Wazwaz [11], using variational iteration method for analytic treatment of the linear and nonlinear ordinary differential equations, homogeneous or inhomogeneous. A. Sevimlican [12], use He's variational iteration method is used for solving time fractional telegraph. Momani, S. Abuasad, Z. Odibat [13], use variational iteration method is implemented to give approximate and analytical solutions for a class of boundary value problems. Ji-Huan He [14], compares the classical variational iteration method with the fractional variational iteration method. The fractional complex transforms is introduced to convert a fractional differential equation to its differential partner. Guo-cheng Wu [15], use fractional variational iteration method to find approximately solutions of two fractional differential equations, S. Momania, Z. Odibatb [16], Comparison of the results obtained by the homotopy perturbation method with those obtained by the variational iteration method for solve time fractional partial differential equation. in this paper modified variational iteration method to solve space and time fractional heat and wave partial differential equation with integral term which is given by eq (1), Riemann-Liouvill fractional integral and derivatives and Caputo fractional derivative will be used, three methods to choose the start solution $u_0$ will be studied, a general classification to LaGrange's multiplier will be made which is important in the variational iteration method. Convergence of this method will be considered. Different examples will be solved. matlab programs will be used to do the figures to show the approximate and the exact solutions.

## 3-Theory and basic definitions:

In this section, formulas of fractional order of integral and derivatives with modify of its formulas will be given and some important properties of Riemann and Caputo formulas will be considered moreover fractional derivative of expansion functions. For more details [17-27].

**3-1 Riemann-Liouville**

Fractional Integral of order β > 0 given by the form [1-27]:

$$J_t^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} = \int_0^t (t-s)^{\beta-1} f(s)ds \\ J_t^0 = I \end{cases} \quad (2)$$

$$J_t^{\beta} J_t^{\alpha} = J_t^{\alpha} J_t^{\beta} = J_t^{\alpha+\beta} \quad \text{Where } \alpha \ge 0, \beta \ge 0 \quad (3)$$

**3-2 Riemann-Liouville**

Fractional derivative of order β, If m denotes a positive integer such that m-1< β ≤ m, then the fractional order derivative of **Riemann-Liouville is** given by the form:

$${}^{R}D_t^{\beta} = D_t^m J_t^{m-\beta} f(t) \qquad \text{This mean}$$

$${}^{R}D_t^{\beta} f(t) =$$

$$\begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\beta)} \int_0^t (t-s)^{m-\beta-1} f(s)ds\right] & m-1<\beta\le m \\ \dfrac{d^m}{dt^m} f(t) & \beta = m \end{cases} \quad (4)$$

$$D_t^m J_t^{\beta} = I, \quad \beta > 0 \quad where\, D_t^0 = I \quad (5)$$

**3-3 Caputo fractional order derivate:**

If m denotes a positive integer such that m-1< α ≤m, then **Caputo** fractional derivative of order α will be given by the form:

$$D_t^{\alpha} = J_t^{m-\alpha} D_t^m f(t) \qquad \text{This mean:}$$

$${}^{c}D_t^{\alpha} f(t) =$$

$$\begin{cases} \left[\dfrac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds\right] & m-1<\alpha\le m \\ \dfrac{d^m}{dt^m} f(t) & \alpha = m \end{cases} \quad (6)$$

Some properties of fractional derivatives $D_t^{\alpha}$ with $\alpha \in \Re$ as:

$$\frac{d^{\alpha}}{dt^{\alpha}}\left[t^{\beta}\right] \equiv D_t^{\alpha}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} \quad \text{where } \beta \ge \alpha \quad (7)$$

$$D_t^{\alpha}\left[t^{-m}\right] = (-1)^{\alpha} \frac{\Gamma(m+\alpha)}{\Gamma(m)} t^{-(m+\alpha)} \quad where \quad (-1)^{\alpha} = \pi^{i\alpha\pi} \quad (8)$$

Leibniz rule will be generalized to fractional derivatives as:

$$D_t^{\alpha}[f(t)\,g(t)] = \begin{cases} \sum_{k=0}^{\infty} \binom{\alpha}{k} D_t^{\alpha-k}[f(t)] D_t^{k}[f(t)] \\ D_t^{\alpha}[f(t)]C \qquad if \;\; g(t) = C \end{cases}$$ Where C is constant (9)

The fractional derivative of the exponential function:

$$D_t^{\alpha}\left[e^t\right] = D_t^{\alpha}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \equiv E_{\alpha}^{t} \tag{10}$$

for example negative integer

$\mu = -1, -2, \ldots$ for $\mu = -1$, equation (10) given :

$$E_{-1}^{t} = D_t^{-1}\left[e^t\right] = \sum_{k=0}^{\infty} \frac{t^{k+1}}{\Gamma(k+2)} = e^t - 1 = \int_0^t e^s\,ds$$

The fractional derivative of the negative exponential function, $e^{-t}$ , for a = -1 and $(-1)^{\alpha} = e^{i\pi\alpha}$.

$$D_t^{\alpha}\left[e^t\right] = D_t^{\alpha}\left[\sum_{k=0}^{\infty} \frac{(at)^k}{k!}\right] = a^{\alpha} \sum_{k=0}^{\infty} \frac{(at)^{k-\alpha}}{\Gamma(k+1-\alpha)} \equiv a^{\alpha} E_{\alpha}^{t} \tag{11}$$

If functions sin(t) and cos(t) are defined by : $\sin(t) = \left[\sum_{k=0}^{\infty} a_k \frac{t^k}{k!}\right]$ ; $\cos(t) = \left[\sum_{k=0}^{\infty} b_k \frac{t^k}{k!}\right]$

where $a_{2k+1} = (-1)^k \; and \; a_{2k} = 0 \;\; ; b_{2k+1} = 0 \; and \; b_{2k} = (-1)^k \quad k \in Z$

$$D_t^{\alpha} \sin(bt) = (b)^{\alpha} \sin(bt + \frac{b\alpha\pi}{4}) \;\; ;$$

$$D_t^{\alpha} \sin(bt) = (b)^{\alpha} \sin(bt + \frac{b\alpha\pi}{4}) \tag{12}$$

Where b is constant.

**3-4Jumarie fractional derivative and integral**:

Before definition of VIM, is appropriate to give definitions of Jumarie which allowed circulated VIM to using to solve many kinds equations of fractional order.

**3-4-1 Definition**:

Jumarie define fractional derivative as the following limit form:

$$D_t^{\alpha} u(x,t) = \lim_{h \to o} \frac{\Delta^{\alpha}\left[u(x,t) - u(x,0)\right]}{h^{\alpha}}$$

This definition is close to the standard definition of derivatives, and as a direct result, the αth derivative of a constant, 0 < α <1 is zero.

**3-4-2 Definition:**

Jumarie define **f**ractional derivative of compounded functions as:

$$d^{\alpha} u(x,t) \cong \Gamma(1+\alpha)\; df \qquad ; 0 < \alpha < 1.$$

**3-4-3 Definition:**

The integral with respect to $(dt)^{\alpha}$was given by Jumarie as the solution of the fractional differential equation:

$$du\,(x,t) \cong u\,(x,t)(dt)^{\alpha} \qquad ; \; t \geq 0 \;\; , \; u\,(x,0) = 0 \qquad ; \; 0 < \alpha < 1 \;\; .$$

**Lemma 3-4-1**:

Let u(x, t) denote a continuous function, then the solution of the Eq. (3-4-3) is defined as:

$$u(x,t) = \int_0^t u(x,s)(ds)^{\alpha} = \alpha \int_0^t (t-s)^{\alpha-1} u(x,s)\,ds \qquad ; 0 < \alpha < 1. \tag{13}$$

For example if u(x,t) = $t^p$ in equation then: $\int_0^t s^p \, (ds)^\alpha = \frac{\Gamma(\alpha+1)\,\Gamma(p+1)}{\Gamma(p+1+\alpha)}\, t^{p+\alpha} \quad ; 0<\alpha\le 1$ .

**3-4Variational Iteration method (He's method)**:

The variational iteration method (VIM) established by Ji-Huan He .is thoroughly used by mathematicians to handle a wide variety of scientific and engineering applications: linear and nonlinear, and homogeneous and inhomogeneous as well. It was shown that this method is effective and reliable for analytic and numerical purposes. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. The VIM does not require specific treatments for nonlinear problems as others method like Adomian method, perturbation techniques, etc. In what follows, we present the main steps of the method:

Consider the general nonlinear differential equation:

L u( x ,t ) + N u ( x ,t ) = g (x ,t ) (14)

Where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a **correction functional** of the form:

$$u_{n+1}(x,t) = u(x,t) + \int_0^t \lambda(t,s)\left\{Lu_n(x,s) + N\,\tilde{u}_n(x,s) - g(x,s)\right\} ds \qquad (15)$$

It is obvious that the successive approximations un, n ≥ 0 can be established by Determining λ,

a general Lagrange's multiplier, which can be identified optimally via the Variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\,\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}$ (x,t), n ≥ 0, of the solution u(x,t) will be readily obtained upon using the obtained Lagrange multiplier and by using Any selective function $u_0$ . The initial values are usually used for selecting the zero Approximation $u_0$, three different methods will be given to choose the start solution $u_0$. The first step to find the solutions, by using correctional function eq(15), is finding Lagrange multiplier, this methods will be done by using conditions of making correct function stationary and noting that $\delta\,\tilde{u}_n = 0$, and using integral by parts. Consider the stander differential equations which is given by the form, Lu(x,t) +R u(x,t) + N u(x,t) = g (x,t), where L and R, are linear operator which has partial derivatives with respect to t and x respectively , assume the solving will be to variable t.

Then for the following cases we have:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left\{Lu_n(x,s) + R\,\tilde{u}_n(x,s) + N\,\tilde{u}_n(x,s) - g(x,s)\right\} ds \;, \qquad (16)$$

With $R\,\tilde{u}_n$ *and* $N\,\tilde{u}_n$ are considered as restricted variations to making correct function stationary, i.e, $\delta R\,\tilde{u}_n \,,\, \delta N\,\tilde{u}_n = 0$ ,By taking δ for two sides of eq(16) yield:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(t,s)\left\{Lu_n(x,s) + R\,\tilde{u}_n(x,s) + N\,\tilde{u}_n(x,s) - g(x,s)\right\} ds \qquad (17)$$

**Case 1**: if L = $\frac{\partial}{\partial t}$ , then since $\delta R\,\tilde{u}_n \,,\, \delta N\,\tilde{u}_n = 0$ and using integral by parts eq(17) gives the following stationary conditions:

$\lambda' = 0 \;\; and \quad 1+\lambda\,|_{s=t} = 0$ Then λ= -1 put this value in the correctional function eq(16) yield:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{Lu_n(x,s) + R\,\tilde{u}_n(x,s) + N\,\tilde{u}_n(x,s) - g(x,s)\right\} ds \qquad (18)$$

Then several approximations $u_n$ (x,t), n ≥ 0, follows immediately. Consequently, the exact solution may be obtained by using, $u = \lim_{n\to\infty} u_n$ (19)

**Case 2:** L= $k\,\dfrac{\partial^2}{\partial t^2}$ where k is constant then by the same conditions and integral by parts eq(16) yield

$$\delta u_n + k\lambda(s)\delta u_n^{'}(s)|_{s=t} - k\lambda^{'}(s)\delta u_n(s)|_{s=t} + \int_0^t k\lambda^{''}\delta\, u_n(s)\, ds = 0$$

The following stationary conditions:

$$m\lambda^{''} = 0 \quad ; 1 - k\lambda^{'}|_{s=t} = 0 \quad ; \quad \lambda|_{s=t} = 0$$

these equations gives $\lambda = \dfrac{1}{k}(s-t)$ put this value in eq(17) yield

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{k}\int_0^t (s-t)\left\{Lu_n(x,s) + R\,\tilde{u}_n(x,s) + N\,\tilde{u}_n(x,s) - g(x,s)\right\} ds \qquad (20)$$

And by the same way Lagrange multiply will be had the values which are given by:

**Case 3**: if L = $k\,\dfrac{\partial^2}{\partial t^2} + c\,\dfrac{\partial}{\partial t}$ then $\lambda = \dfrac{1}{c}\left[e^{\frac{c}{k}(s-t)} - 1\right]$.

The next important step is how we choose the start zeros solution $u_0$.

$$\left.\begin{aligned} &I.C \quad u(x,0) = g_0(x); \quad u_t^{'}(x,0) = g_1(x); 0 \le x \le 1 \\ &B.C \; u(0,t) = f_0(t); \quad u(1,t) = f_1(t); t > 0 \end{aligned}\right\} \; (21)$$

1- For initial differential equations of order n the (n-1)th initial conditions will be used to find the zeros solution as:

$$u_{0\,(x,t)} = u(x,0) + t\,u_t^{'}(x,t) + \ldots \qquad (22)$$

**2-** for the initial-boundary differential equations eq(22) will be used, and we can use the initial and boundary conditions to find the zeros solution, consider the conditions eq(21) and $n \ge 0$, then :

$$u_n^{*}(x,t) = u_n(x,t) + (1 - x^m)\left[g_0(x) - u_n(0,t)\right] + x^m\left[g_1(x) - u_n(1,t)\right]; \qquad (23)$$

**3-** At this method, the approximate solution faster than method at 1 in one step at least.

$$u_n^{*}(x,t) = g_0(x) + t\,g_1(x) + t^2\,g_2(x) \quad \text{Where } g_2\,(x) \qquad (23a)$$

Can be founded by substituting $u_0$ at eq(22) in the general differential equation.

## 4-fractional variational iteration method

**:** consider the eq(1) correctional function and by the Jumarie definitions (3-4-1,2,3) and Lemma (3-4-1),the correctional function will be given by the forms :

Let $F(x,t) = \int_o^t u(x,s)ds + q(x,t)$ ; $L = \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ ; $N = \dfrac{\partial^\beta u(x,t)}{\partial x^\beta}$ ;

λ (x,t) given by cases(1-3)

$$u_{n+1}(x,s) =$$

$$\left\{\begin{array}{l} u_n(x,s)+\int_0^t \lambda(t,s)\{Lu_n(x,s)+N\,\tilde{u}_n(x,s)-F(x,s)\}\,ds \qquad where \quad \alpha=n \\ u_n(x,s)+\frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(t,s)\{Lu_n(x,s)+N\,\tilde{u}_n(x,s)-F(x,s)\}\,(ds)^{\alpha};\; n-1<\alpha\le n \end{array}\right\} \quad (24)$$

## 5- Convergence of variational iteration method

**(**VIM)**:** in this section convergence of VIM and important theorems will be given, to consider the convergence of this method. The VIM changes the differential equation to a recurrence sequence of functions. The limit of that sequence is considered as the solution of the partial differential equation. In the following form (more general form can be considered without loss of generality).

**For u (x, t), Assume** that F is the general form of D.E written as:

$$F\,(u,\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial t \partial x}) = 0 \quad (25)$$

with specified initial conditions.

The variational iteration method changes the partial differential equation to a correction functional in t-direction in the following correctional function form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda F\,(u_n, \frac{\partial u_n}{\partial s}, \frac{\partial \tilde{u}_n}{\partial x}, \frac{\partial^2 \tilde{u}_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial s^2}, \frac{\partial^2 \tilde{u}_n}{\partial s\,\partial x})\,ds \quad (26)$$

Where $\tilde{u}_n$ is considered as He's monographs i.e., $\delta(\tilde{u}_n) = 0$. To find the optimal value of λ, we make correction functional

Stationary in the following form:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda F\,(u_n, \frac{\partial u_n}{\partial s}, \frac{\partial \tilde{u}_n}{\partial x}, \frac{\partial^2 \tilde{u}_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial s^2}, \frac{\partial^2 \tilde{u}_n}{\partial s\,\partial x})\,ds = 0 \quad (27)$$

Which results the stationary conditions and consequently the optimal value of λ is obtained. In fact the solution of the differential equation is considered as the fixed point of the following functional under the suitable choice of the initial term u0(x, t):

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda F\,(u_n, \frac{\partial u_n}{\partial s}, \frac{\partial \tilde{u}_n}{\partial x}, \frac{\partial^2 \tilde{u}_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial s^2}, \frac{\partial^2 \tilde{u}_n}{\partial s\,\partial x})\,ds \quad (28)$$

**By** using the method using some concepts from the calculus of variations.

**Definition 1.5:** A variable quantity v is a functional dependent on a function u(x) if to each function u(x) of a certain class of functions u(x), there correspond a value v. The variation of a functional v[u(x)] is defined in the following form:

$$\delta\, v[u(x)] = \frac{\partial}{\partial s}\, v\,[u(x) + s\,\delta\, u]|_{s=0}$$

**Theorem 1.5:** If a functional v[u(x)] which has a variation achieves a maximum or a minimum at u = u0(x), where u(x) is an interior point of the domain of definition of the functional, then at u = u0(x), δ v = 0.

**Theorem 2.5**: (Banach's fixed point theorem). Assume that X be a Banach space and A: X → X is a nonlinear mapping, and suppose that,

$\|A[u] - A[\tilde{u}]\| \le \zeta \|u - \tilde{u}\|, \quad u, \tilde{u} \in X$ **,** For some constant ζ < 1. Then A has a unique fixed point. Furthermore, the sequence, $u_{n+1} = A[u_{n+1}]$.

With an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and

$\|u_k - u_l\| \le \|u_1 - u_0\| \sum_{j=l-1}^{k-2} \zeta^{j}$ **,** According to Theorem (2.5) for the nonlinear mapping

$$A[u] = u(x,t) + \int_0^t \lambda F\,(u, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial s^2}, \frac{\partial^2 u}{\partial s\,\partial x})\,ds \quad (29)$$

a sufficient condition for convergence of the variational iteration method is strictly contraction of A. Furthermore, the sequence (28) converges to the fixed point of A which also is the solution of the partial differential (25).

Consider the sequence (28) in the following form:

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t \lambda F\left(u_n, \frac{\partial u_n}{\partial s}, \frac{\partial \tilde{u}_n}{\partial x}, \frac{\partial^2 \tilde{u}_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial s^2}, \frac{\partial^2 \tilde{u}_n}{\partial s\, \partial x}\right) ds \qquad (30)$$

It is clear that the optimal value of $\lambda$ must be chosen such that extremities the residual functional

$$\int_0^t \lambda F\left(u_n, \frac{\partial u_n}{\partial s}, \frac{\partial \tilde{u}_n}{\partial x}, \frac{\partial^2 \tilde{u}_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial s^2}, \frac{\partial^2 \tilde{u}_n}{\partial s\, \partial x}\right) ds \qquad (31)$$

Which is equivalent to the extermination of A.? But in Theorem 2.5,

The necessary condition for minimization is given. As a direct result which shows the relation between He's variational technique and Adomian Decomposition method we have, For the time-dependent partial differential in the form of ut + F(u, $u_x$,$u_{xx}$) = 0,

With the properly given initial condition, the He's variational method and decomposition procedure of Adomian are equivalent.

**Numerical examples:**

Different examples will be solved at different orders:

**Example 1:** consider the equation (1) heat equation with $0< \alpha \le 1 < \beta \le 2$, $0 < x < 1$, $t > 0$.

$$\int_0^t u(x,s)ds = x(1+e^{-t}) \; ; \quad q(x,t) = (k+1)xe^{-t} - x,$$

Where $k = (-1)^{\alpha}$ .The initial condition are: $u(x, 0) = x$ ; $u_t(x, 0) = k\,x$, $t > 0$

**Solution:** consider correctional function (18) with $0 < \alpha \le 1$, and Lagrange

Multiply $\lambda = -1$, then eq (18) yield:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{D_s^{\alpha} u_n(x,s) - D_x^{\beta} u_n(x,s) - x(1+e^{-s}) - (k+1)xe^{-s} - x\right\} ds \qquad \text{(Ex1,1)}$$

Using eq(22) , $u_0(x, t) = x$, and eq (Ex1,1) to find approximate solutions $u_n$, $n > 0$, as the following:

$u_1(x, t) = (k + 1)x - k\,xe^{-t}$ .

$u_2(x, t) = (k^2 + 2k + 1)x - (k^2 + 2k)\,xe^{-t}$ .

$u_3(x, t) = (k^3 + 3k^2 + 3k + 1)x - (k^3 + 3k^2 + 3k)\,xe^{-t}$

$u_n(x, t) = (k^n + nk^{n-1} + \ldots + 1)x - (k^n + nk^{n-1} + \ldots + nk)\,xe^{-t}$

$\vdots$

if $\alpha = 1$ then $k = -1$ so that we can find $u_n = x\,e^{-t}$ $n > 1$. the exact solution. As soon as we can find $u = \lim_{n\to\infty} u_n(x,t) = x\,e^{-t}$ the exact solution.

Table1, shown the maximum error between approximate and exact solutions at different ( α and n ), then figure1 and

table1 shown the approximate solutions go to exact solution when the αth go to 1, and when n go to bigger value, the two solutions will be exact at α =1, for all value to n.

**Example2:** considerthe fractional space and time partial differential equation with integral term:

$$D_t^{\alpha} u(x,t) - k\,D_x^{\beta} u(x,t) - \int_0^t e^{x-s}\, ds = g(x,t) \qquad where \quad g(x,t) = e^x(1-e^{-t})$$

Where $0 < \alpha \le 1$; $1< \beta \le 2$, $0 < x < 1$, $t> 0$. With the initial condition: $u(x,0) = e^x$ , $k= (-1)^{\alpha}$.

**Solution:** since $\alpha \le 1$ then the Lagrange multiply $\lambda = -1$. Then the correction function will be given:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{D_s^{\alpha} u_n(x,s) - kD_x^{\beta} u_n(x,s) - \int_0^s e^{x-y}\, dy - e^x(1-e^{-s})\right\} ds$$

We choose the zeros solution $u_0(x,t) = u(x,0) = e^x$ , put it in correction function yield:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left\{0 - ke^x - e^x(e^{-s} - 1) - e^x(1-e^{-s})\right\} ds$$

$u_1(x,t) = e^x(1+kt)$ , by the same way we can get the approximate solutions:

$$u_2(x,t) = e^x\left(1 + 2kt + \frac{1}{2}t^2 - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\right)$$

$$u_3(x,t) = e^x\left(1 + 3kt + \frac{1}{2}k^2t^2 - \frac{3kt^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{k^3t^3}{6} - \frac{k^2t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{k^2t^{3-\alpha}}{\Gamma(3-\alpha)} + \frac{kt^{3-2\alpha}}{\Gamma(4-2\alpha)}\right)$$

**And so on…**we can see if α = 1, then k = -1and the approximate solutions yield:

$$u_0(x,t) = e^x$$

$$u_1(x,t) = e^x(1-t)$$

$$u_2(x,t) = e^x\left(1 - t + \frac{1}{2}t^2\right)$$

$$u_3(x,t) = e^x\left(1 - t + \frac{1}{2}t^2 - \frac{t^3}{6}\right)$$

so that if α = 1 then the solutions go to exact solution $u = \lim_{n\to\infty} u_n(x,t) = e^x e^{-t}$

figure2 shown the approximate and the exact so solutions at α = 1 and n=3 for the approximate solution.

**NOT1:** if we us equation (23a) to find the zeros solution, by putting $u_0(x,t) = u(x,0) = e^x$ in general equation then the zeros solution will be given by the form:

$$u_0^*(x,t) = g_0(x) + t\,g_1(x) \quad ; \quad u_0^* = u_0(x,t) + kt\,e^x = e^x - kt\,e^x$$

this method gives the second solution from the method which is given in eq(22), in zeros solution, and by the same way we can see this method is faster than other.

**NOT2:** considerexample1 with 1 < α , β ≤ 2 , with initial conditions:

$g_0(x) = u(x,0) = e^x \quad ; g_1(x) = u_t(x,0) = -e^x$, by the same way,

The start (zeros) solution will be chosen by two methods:

$$u_0(x,t) = g_0(x) + t\,g_1(x) = e^x(1-t)$$

$$u_0^*(x,t) = g_0(x) + t\,g_1(x) + t^2 g_2(x) = u_0(x,t) + t^2 g_2(x)$$

Where $g_2(x) = F(u_0)\big|_{t=0}$ where, F ($u_0$) from eq(23a), then $u_0^* = e^x(1 - t + kt^2 - kt^3)$.

Since 1 < α ≤ 2, then Lagrange multiply will be λ = (s-t), so that correction functional give by:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left\{D_s^\alpha u_n(x,s) - kD_x^\beta u_n(x,s) - \int_0^s e^{x-y}\,dy - e^x(1-e^{-s})\right\} ds$$

Now choosing of zero approximate solution $u_0(x,t)$ *and* $u_0^*(x,t)$ respectively, the approximate solutions are given as the fallowing respectively:

$$u_1(x,t) = e^x\left(1 - t + \frac{k}{2}t^2 - \frac{kt^3}{6}\right)$$

$$u_2(x,t) = e^x\left(1 - t + \frac{kt^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{kt^{5-\alpha}}{\Gamma(6-\alpha)} + \frac{kt^4}{24} - \frac{kt^5}{120}\right)$$

And

$$u_1^*(x,t) = e^x\left(1 - t + \frac{3kt^2}{2} - \frac{7kt^3}{6} + \frac{k^2t^4}{12} - \frac{k^2t^5}{20} - \frac{2kt^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{6kt^{5-\alpha}}{\Gamma(6-\alpha)}\right)$$

at α = 2 the approximate solutions yield :

$$u_1(x,t) = e^x\left(1 - t + \frac{1}{2}t^2 - \frac{t^3}{6}\right)$$

$$u_2(x,t) = e^x(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120})$$

$$u_1^*(x,t) = e^x(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{12} - \frac{t^5}{20})$$

Figures ( 1,2,3 and 4) shown the approximate solutions $U_1$ , $U_1$* at different values to α (1.3 ,1,5 , 1.8 ,2 ), figuer5 shown the exact solution, and the table 2 shows the maximum error ( MM1 and MM2) between exact solution ( u (x,t) = $e^x$ $e^{-t}$ **)** and the approximate solutions respectively.
AT all we can see easy the choice of start approximate solution $U^*$ is better than U To show the best choice of zero approximate solution from the three methods eq's(22,23, and 23a) , here example 3 give that case.

**Example 3**: consider the fractional heat equation with integral term given by

$$D_t^\alpha u(x,t) - AD_x^\beta u(x,t) - x^2 \int_0^t e^{-s}\, ds = g(x,t) where g(x,t) = x^2(e^{-t} - 1)$$

Where 0 < α ≤ 1 ;1 < β ≤ 2, 0 < x < 1, t > 0. with the initial and boundary conditions:
$u(x,0) = x^2$ , 0 < x < 1 and $u(0,t) = 0$ ; $u(1,t) = e^{-t}$ , t > 0 and k= $(-1)^\alpha$ ; $A = \frac{k\,\Gamma(2-\beta)}{2} x^\beta$;
since α ≤ 1 then the Lagrange multiply λ = -1. the correction function will be given by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ D_s^\alpha u_n(x,s) - AD_x^\beta u_n(x,s) - x^2 \int_0^s e^{-y}\, dy - x^2(e^{-s} - 1) \right\} ds$$

The three zeros approximate solutions by using eq's(22,23 and 23a) will be given by the following:

1- $u_0 = x^2$

2- $u_0^* = x^2(1 + kt)$ if α =1 then $u_0^* = x^2(1 - t)$

**3-** $u_n^* = x^2 e^{-t}$

Start with the first approximate solution and using correction function we get:

$u_1 = x^2(1 + kt) = u_0^*$

$u_2 = x^2(1 + \frac{kt^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{k}{2}t^2) = u_1^*$, if α =1 then $u_2 = u_1^* = x^2(1 - t + \frac{t^2}{2})$

And so on the approximate solution which is chosen by eq(23) go to the exact solution faster than other by using eq(22), the choice of zero approximate solution from eq(23a) is better than both others, it gives the exact solution in direct, $u_n^* = x^2 e^{-t}$ fro n= 0, 1, … ,this is exact solution.

## Conclusion

In this paper the (VIM) investigated and sufficient conditions which guarantee the convergence of method are presented. This technique is a very powerful tool for solving various partial differential equations fractional or integer order. The applications of the VIM for solving several examples are described. The main advantage of the VIM is so easy and faster than other method like decomposition procedure of Adomian, is that the former method provides the solution of the problem without calculating Adomian's polynomials.

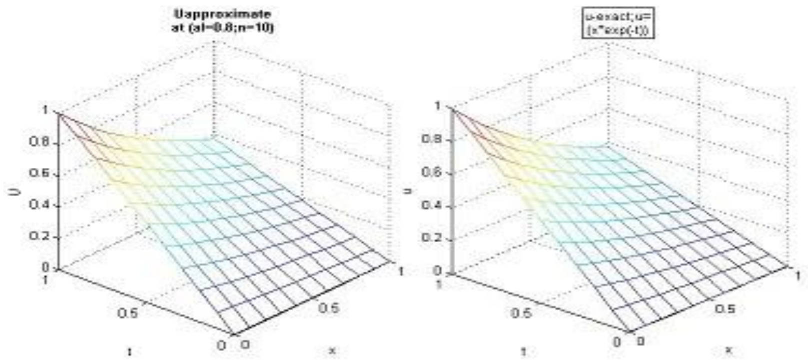



**Figure 1**

**Figure (1 , 2 ) shown the approximate solution U at α = 0.8 and at n = 10 and exact solution u.**

**Table1**

| n / α | 2 | 5 | 10 |
|---|---|---|---|
| **0.5** | **6.3212e-001** | **6.3212e-001** | **6.3212e-001** |
| **0.7** | **2.6057e-001** | **1.8251e-002** | **2.1723e-004** |
| **0.9** | **3.0938e-002** | **3.6273e-006** | **1.0129e-012** |
| **1** | **0** | **0** | **0** |

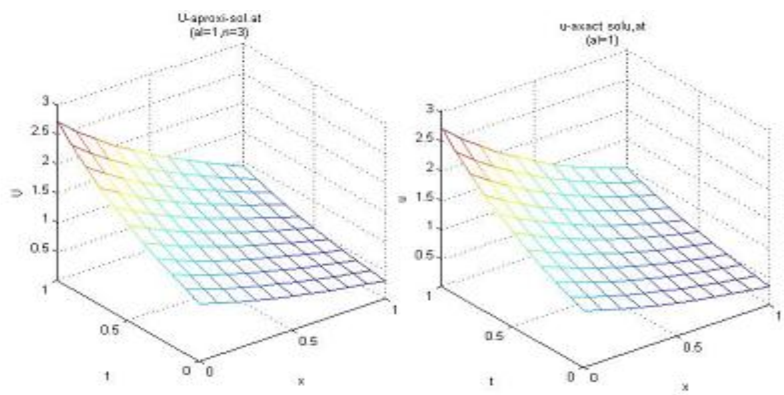



**Figure 2**

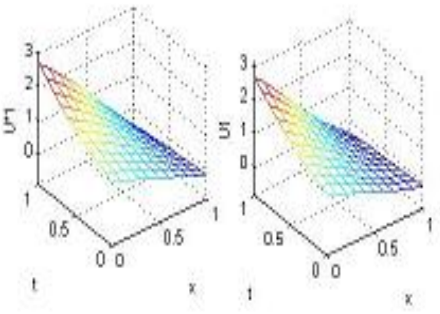


**Figure 1**

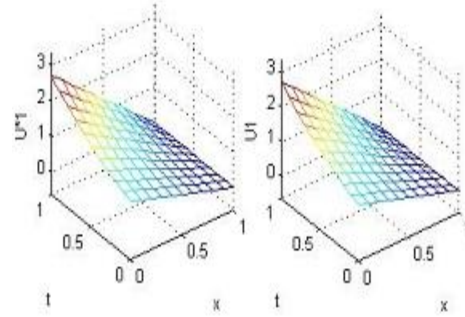
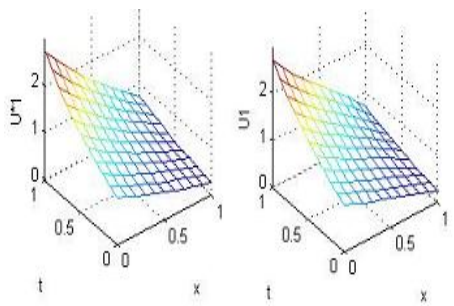


**Figure 2**



**Figuer 3**

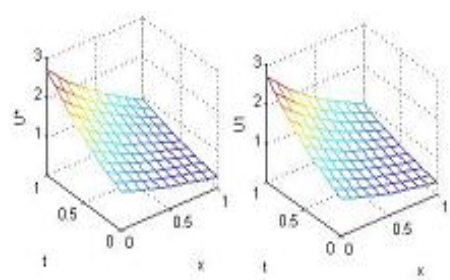


**Figure 4**

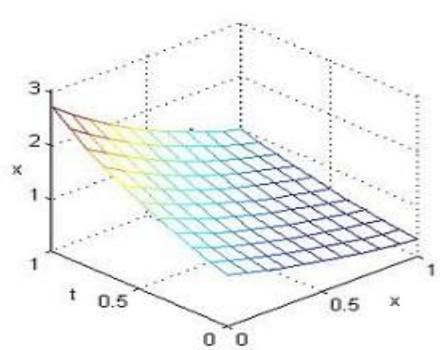


**Figure 5**

Table2

| α / error | 1.3 | 1.5 | 1.8 | 2 |
|---|---|---|---|---|
| **MM1** | **1.5326** | **1** | **0.267** | **0.0939** |
| **MM2** | **1.3563** | **1** | **0.3211** | **0.0084** |

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