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RESEARCH ARTICLE

SOME VERY SPECIAL PYTHAGOREAN TRIANGLES WITH THEIR PERIMETERS AS BOTH TRIANGULAR AND PENTAGONAL NUMBERS.

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Abstract

To find out the Special Pythagorean Triangles, where their perimeters are Triangular numbers and Pentagonal numbers both is the main objective of this paper. A few interesting results are observed.

Introduction:-

Figuret Numbers, first studied by Pythagoras, played very important role in Pythagorean science of numbers. Pythagorean Theorem, which gave the world, a powerful method of proof, ***reductio ad absurdum***, also called ***Proof by Contradiction***, continues to ignite the minds of those who love to play with numbers. Rana and Darbari [1] obtained special Pythagorean Triangles, with their legs to be consecutive, in terms of Triangular Numbers while Darbari [2] explored their perimeters as triangular numbers. Gopalan and Janaki [3] and Darbari [4] have studied special Pythagorean Triangles in terms of Pentagonal Numbers. Extending the problem further, existence of special Pythagorean triangles with their perimeters as Triangular Numbers and Pentagonal numbers both is explored in this paper.

Method of Analysis:-

The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2 \qquad (2.1)$$

is given by [5]

$$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2 \qquad (2.2)$$

for some integers ***m, n*** of opposite parity such that ***m*** > ***n*** > **0** and (*m*, *n*) = 1.

Perimeter is a Triangular number:-

Definition 2.1.1: A natural number *p* is called a Triangular number if it can be written in the form

$$\frac{\gamma(\gamma+1)}{2}, \gamma \in \mathrm{N}$$

Definition 2.1.2: A natural number *p* is called a Pentagonal number if it can be written in the form

$$\frac{\beta(3\beta-1)}{2}, \beta \in \mathrm{N}$$

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If the perimeter of the Pythagorean Triangle (X, Y, Z) is Triangular and Pentagonal number *p*, then

$$X + Y + Z = \frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2} = p \tag{2.3}$$

By virtue of equation (2.2), equation (2.3) becomes

$$2m^2 + 2mn = \frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2} = p, \gamma \,\&\, \beta \in N \tag{2.4}$$

Using *Mathematica* for $0 < \gamma < 10^{20}$, $0 < \beta < 10^{20}$, there are just 18 numbers which are both Triangular and Pentagonal numbers. They are as follows:

Table 2.1:- Numbers which are both Triangular and Pentagonal

S.N.	β	γ	$\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}$
1.	1	1	1
2.	12	20	210
3.	165	285	40755
4.	2296	3976	7906276
5.	31977	55385	1533776805
6.	445380	771420	297544793910
7.	6203341	10744501	57722156241751
8.	86401392	149651600	11197800766105800
9.	1203416145	2084377905	2172315626468283465
10.	16761424636	29031639076	421418033734080886426
11.	233456528757	404358569165	81752926228785223683195
12.	3251629977960	5631988329240	15859646270350599313653420
13.	45289363162681	78443478040201	3076689623521787481625080301
14.	630799454299572	1092576704233580	596861927316956420835951924990
15.	8785902997031325	15217630381229925	115788137209866023854693048367775
16.	122371842504138976	211954248632985376	22462301756786691671389615431423376
17.	1704419892060914337	2952141850480565345	4357570752679408318225730700647767185
18.	23739506646348661740	41118031658094929460	845346263718048427044120366310235410530

Since perimeter is even, investigating even such numbers from the Table 2.1 using software Mathematica, only the following six numbers have corresponding Pythagorean Triangles: 297544793910, 11197800766105800, 15859646270350599313653420, 596861927316956420835951924990, 22462301756786691671389615431423376 and 845346263718048427044120366310235410530.

Case 1: For γ = 771420 and β = 445380, $\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}$ = 297544793910, $0 < m < 10^{20}$, $0 < n < 10^{20}$, we get 07 (X, Y, Z) which are given below:

Table 2.2:- (X, Y, Z) with p = 297544793910 (Triangular and Pentagonal No.)

S.N.	m	n	X	Y	Z	X + Y + Z
1.	275793	263642	35209270941	128804961460	133530561509	297544793910
2.	288535	227078	80300244885	93781558468	123462990557	297544793910
3.	290465	221722	86262074885	88031908068	123250810957	297544793910
4.	319189	146906	95049293885	78940172148	123555327877	297544793910
5.	323661	135994	134369999309	26262379860	136912414741	297544793910
6.	330609	119386	6554674685	145421236212	145568883013	297544793910
7.	368295	35654	31688028141	131039901460	134816864309	297544793910

Table 2.3:- Verification of $X^2 + Y^2 = Z^2$

S.N.	X^2	Y^2	$X^2 + Y^2 = Z^2$
1.	42963760226179849225	21147335941426300108944	21190299701652479958169
2.	1004131127464807915881	17171455774646510131600	18175586902111318047481
3.	1239692760196747025481	16590718096712085331600	17830410856908832357081
4.	6448129328590968663225	8794980708686902507024	15243110037277871170249
5.	7441145563465347763225	7749616838092803492624	15190762401558151255849
6.	9034368268037098393225	6231550778755874933904	15265919046792973327129
7.	18055296714300660477481	689712595910933619600	18745009310211594097081

Case 2: For γ = 149651600 and β = 86401392, $\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}$ = 11197800766105800, $0 < m < 10^{20}$, $0 < n < 10^{20}$, we get 42 (X, Y, Z) out of which first five are given below:

Table 2.4:- (X, Y, Z) with p = 11197800766105800 (Triangular and Pentagonal No.)

S.N.	m	n	X	Y	Z
1.	52932221	52842679	9471296095800	5594160726120118	5594168743889882
2.	53815300	50223893	373647086014551	5405627737925800	5418525942165449
3.	54351525	48661231	586172867390264	5289624226454550	5322003672260986
4.	54404196	48508829	606710051455175	5278167681292968	5312923033357657
5.	54418764	48466711	612379800130175	5274997015530408	5310423950445217

Table 2.5:- Verification of $X^2 + Y^2 = Z^2$ with p = 11197800766105800

S.N.	X^2	Y^2	$X^2 + Y^2 = Z^2$
1.	89705449734316322777640000	31294634229664765872185764333924	31294723935114500188508541973924
2.	139612144887165273496983731601	29220811241032801487686305640000	29360423385919966761183289371601
3.	343598630464524024686081989696	27980124457094896460063215702500	28323723087559420484749297692196
4.	368097086536741096135034280625	27859054071845586219420246249024	28227151158382327315555280529649
5.	375009019607473080946945530625	27825593513854711458745572646464	28200602533462184539692518177089

Case 3: For γ = 5631988329240 and β = 3251629977960, $\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}$ = 15859646270350599313653420, we get 23 (X, Y, Z) out of which first five are given below:

Table 2.6: (X, Y, Z) with p = 15859646270350599313653420 **(Triangular and Pentagonal No.)**

S.N.	m	n	X	Y	Z
1.	2021846845270	1900222303403	477019863378048335992491	7683916938894096670907620	7698709468078454306753309
2.	2032160737355	1870002605047	632767519564651022523816	7600291745456164728861370	7626587005329783562268234
3.	2064809999165	1775651417809	1110502375084655418336744	7332765605047064712258970	7416378290218879183057706
4.	2075508496570	1745156727533	1262163515680572101498811	7244175231682075910123620	7353307522987951302030989
5.	2101164876678	1672847921267	1616473671297951917430395	7029858592380049417022052	7213314006672597979200973

Table 2.7:- X^2 and Y^2

S.N.	X^2	Y^2
1.	227547950057211900027952596044157366090008385081	590425795238236249523923551594632089400745740644 00
2.	400394733816001014565247114702198963754287201856	5776443461604911507145855358063846720825667827690 0
3.	1233215525068660711246312063387769132271380521536	5376945141856124503152580922459371964100034546090 0
4.	1593056740315141777140530505499837901792634413721	5247807478731605818902384785128340454800368190440 0
5.	2612987129999479100589174330360538919819669856025	4941891182885960978341856417072036958848785429070 4

Table 2.8:- $X^2 + Y^2 = Z^2$ and $X + Y + Z =$ 15859646270350599313653420

S.N.	$X^2 + Y^2 = Z^2$	X + Y + Z
1.	59270127473880836852420307755507366306164582449481	15859646270350599313653420
2.	58164829349865116086023800695340666172010965478756	15859646270350599313653420
3.	55002666943629905742772121287981488773271725982436	15859646270350599313653420
4.	54071131527631199966164378356783242449796316318121	15859646270350599313653420
5.	52031898958859088884007738501080908508307524146729	15859646270350599313653420

Case 4: For γ = 1092576704233580 and β = 630799454299572, $\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2} =$ 596861927316956420835951924990, $0 < m < 10^{26}$ and $0 < n < 10^{26}$, we get 50 (X, Y, Z) out of which first ten are given below:

Table 2.9:- (X, Y, Z) with p = (Triangular and Pentagonal No.)

S.N.	m	n	X	Y	Z
1.	387027573899585	384056927009662	2290619774493883266221817981	297281241399758964188625580540	297290066142703573381104526469
2.	388900007057443	378471960653522	8002190488358110294170693765	294375496338397990420948528492	294484240490200320120832702733
3.	392733138913605	367149168151214	19440806726804549316884422229	288383290515090496996643732940	289037830075061374522423769821
4.	398695204052845	349823866718126	36581128009118528407615642149	278946195847476996608846736940	281334603460360895819489545901
5.	398779965191097	349580006994238	36819279347721045980362862765	278811006041331350876495798172	281231641927904023979093264053

Table 2.10:- X^2 and Y^2

S.N.	X^2	Y^2
1.	5246938951302408627554725755668643721876815108694916361	88376136488181763109545570783716129636717530754052026691600
2.	64035052611969011723463457972078835464765370181409875225	86656932844478168665653143651541636111498725799580135794064
3.	377944966188969014618818061671633414963576493407149328441	83164922248311084608572697422486591664011006888578041043600
4.	1338178926419516109784395520830021602497785068541625338201	77810980177778992826348660996309110263321213562485560563600
5.	1355659331685517533155543348444117070786121538786223445225	77735577089779307036137831051492572047797468613171358541584

Table 2.11:- $X^2 + Y^2 = Z^2$ and X + Y + Z = 596861927316956420835951924990

S.N	$X^2 + Y^2 = Z^2$	X + Y + Z
1.	88381383427133065518173125509471798280439407569160721607961	596861927316956420835951924990
2.	86720967897090137677376607109513714946963491169761545669289	596861927316956420835951924990
3.	83542867214500053623191515484158225078974583381985190372041	596861927316956420835951924990
4.	79149159104198508936133056517139131865818998631027185901801	596861927316956420835951924990
5.	79091236421464824569293374399936689118583590151957581986809	596861927316956420835951924990

Case 5: For γ = 211954248632985376 and β = 122371842504138976, $\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}$ = 224651358482234633466179151142894071942298207292530339862362047627796736, $0 < m < 10^{30}$ and $0 < n < 10^{30}$, we get 13 (X, Y, Z) out of which first ten are given below:

Table 2.12:- (X, Y, Z) with p = (Triangular and Pentagonal No.)

S.N.	m	n	X	Y	Z
1.	78614658032641253	64248659819992643	2052374168923988806018990823284560	10101772841608439904112816636603358	10308154746254262961257807971535458
2.	79085050954136621	62928524846763707	2294446045028628434043080040915792	9953411187949920336172032364828094	10214444523808142901174503025679490
3.	81806709760355461	55482161914032547	3614067471160113748072494097215312	9077626233156616776757365286378334	9770608052469961146559756047829730
4.	87628440896593423	40539432039166633	6035298103909556635501069026300240	7104814448851158176629603097709518	9322189204025976859258943307413618

Table 2.13:- X^2 and Y^2

S.N.	X^2	Y^2
1.	4212239729266433738842834679924967966755835836639745386266734393600	102045814543457854678865189716222350170701983648689758136091416876164
2.	5264482653547514819530196306231113249103285991584762312822034987264	99070394276406644371529364038285219282605506403987514433554171672836
3.	13061483686097659618163016145492563671863199328749815107072887257344	82403298028893187411465656413944910957859028951673303872370184615556
4.	36424823203054289483448621970987815701069255757302270604822624057600	50478388352604186526427150307867709011929294298083756869857907792324
5.	37839109466418514255239437300552432114128022644963700347352765954304	48937666885429471943281080304371554256758238581181784328555042529796

Table 2.14:- $X^2 + Y^2 = Z^2$ and X + Y + Z = 22462301756786691671389615431423376

S.N.	$X^2 + Y^2 = Z^2$	X + Y + Z
1.	106258054272724288417708024396147318137457819485329503522358151269764	22462301756786691671389615431423376
2.	104334876929954159191059560344516332531708792395572276746376206660100	22462301756786691671389615431423376
3.	9546478171499084702962867255943747462972228280423118979443071872900	22462301756786691671389615431423376
4.	86903211555658476009875772278855524712998550055386027474680531849924	22462301756786691671389615431423376
5.	86776776351847986198520517604923986370886261226145484675907808484100	22462301756786691671389615431423376

Case 6: For γ = 41118031658094929460 and β = 23739506646348661740, $\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2} =$ 845346263718048427044120366310235410530, $0 < m < 10^{30}$ and $0 < n < 10^{30}$, we get 1194 (X, Y, Z) out of which first ten are given below: **+**

Table 2.15:- (X, Y, Z) with p = (Triangular and Pentagonal No.)

S.N	m	n	X	Y	Z
1.	1700707004 7924250415	7845719448 912824176	22768511794395321582 2273080167337593249	26686540048804409642 8916610630980066080	35079574528605111479 2930675511917751201
2.	1776493775 9082954231	6027610100 698743584	27926093005804577781 3182858666259636305	21416023654986583587 9863770185353807808	35192509711013681335 1073737458621966417
3.	1845860060 7691997615	4439837949 068880496	32100777538029522056 0863735378436482209	16390639092947365976 0036435691904034080	36043209740827954672 3220195239894894241
4.	1514716915 2872275295	1275726154 2606886672	66689011279248760219 228015989616101441	38647279702659762307 2695838569700736480	39218445541220204375 2196511750918572609
5.	1672132055 0497456663	8556176557 615200992	20639440366740465628 3194551083005311505	28614114181307129275 6600141447109219392	35281071823757247800 4325673780120879633

Table 2.16:- X^2 and Y^2

S.N.	X^2	Y^2
1.	518405129331518866650886685889121399347490 86909392487499286743954367770376001	712171419776441662604698457844791506264620347 63235489599355021390321166566400
2.	779866670568847357927349350643338427219869 37474752262854982822349270874053025	458646069190944906630902228093515495799154520 62335979637960666388925001764864
3.	103045991854606070335378694325477351851573 050128048304847215752296722773519681	268653049875254452066329610731702601943794044 45650373767294601976177801446400
4.	444742422540376836197278922365527144233379 1103610299361238167461283602276481	149361222841561724565620423677702648181947272 888786763299820469625854402790400
5.	425986498652235806314092518357488294012613 41535439639411437100248042085365025	818767530380881765085221266137634200169038719 34115466996680922609323588849664

Table 2.17:- $X^2 + Y^2 = Z^2$ and X + Y + Z = 845346263718048427044120366310235410530.

S.N.	$X^2 + Y^2 = Z^2$	X + Y + Z
1.	1230576549107960529255585143733912905612111216726279770986 41765344688936942401	8453462637180484270441203663102 35410530
2.	1238512739759792264558251578736853923019023895370882424929 43488738195875817889	8453462637180484270441203663102 35410530
3.	1299112968421315155420116553986476120459524545736986786145 10354272900574966081	8453462637180484270441203663102 35410530
4.	1538086470669654929275932129013579196242810639923970626610 58637087138005066881	8453462637180484270441203663102 35410530
5.	1244754029033117571399313784495122494181652134695551064081 18022857365674214689	8453462637180484270441203663102 35410530

Observations and Conclusion:-

We observe that

1. X + Y + Z = 0(mod 2).
2. Y + Z - X = 0(mod 2).
3. (X + Y + Z) (X + Y – Z) = 0(mod 8).
4. $(Y + Z – X)^2 = 2(Y + Z) (Z – X)$.
5. $(X + 2Y + Z)^2 = (Z – X)^2 + 4(X + Y) (Y + Z)$.

In conclusion, other special Pythagorean Triangle can be found which satisfy the conditions other than discussed in the above problem.

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