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Bayesian using Markov Chain Monte Carlo Estimation Based on Type I Censored Data

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Abstract

This paper describe the Bayesian Estimator using non informative prior and the Maximum Likelihood Estimation of the Weibull distribution with Type I censored data. The maximum likelihood method can't estimate the shape parameter in closed forms, although it can be solved by Newton Raphson methods. Moreover, the Bayesian estimates of the parameters, the survival function and hazard rate cannot be solved analytically. Hence Markov Chain Monte Carlo method is used, where the full conditional distribution for the parameters of Weibull distribution are obtained via Gibbs sampling and Metropolis-Hastings algorithm followed by the survival function and hazard rate estimates. The methods are compared to MLE counterparts and the comparisons are made with respect to the Mean Square Error (MSE) and absolute bias to determine the better method in scale and shape parameters, the survival function and the hazard rate.
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Introduction

The Weibull distribution observably has the widest variety of applications in many areas, including life testing, reliability theory and others. The most used methods, which are considered to be the traditional methods are maximum likelihood and the moment estimation (Cohen and Whitten, 1982). Sinha (1986) obtained Bayesian using Jeffreys prior to estimate the survival function and hazard rate of Weibull distribution under squared error loss function using Lindley's approximation method. Smith (1987) developed the maximum likelihood and Bayesian estimators and compared them using the three-parameter Weibull distribution. Singh *et al*. (2002) estimated Exponentiated Weibull shape parameters by maximum likelihood estimators and Bayesian estimator using Jeffreys prior. Hossain and Zimmer (2003) estimated the scale and shape parameters of Weibull distribution using complete and censored samples. Also, an array of methods ha been proposed for estimating the Weibull distribution parameters. These are maximum likelihood estimator and other two types of least squares method where the Mean Squared Error values were compared between these three estimators and the conclusion was that maximum likelihood estimator is the best compare to the others. Soliman *et al*. (2006) estimated Weibull distribution by using maximum likelihood estimator and Bayesian approach following by estimated the hazard and reliability functions were solved by taking the posterior for Bayesian estimator. Ibrahim *et al*. 1991 reported that generalized linear models have been proven suitable for modeling various kinds of data consisting of exponential family response variables with covariates. Green *et al*. (1994) applied the MCMC method for estimating the three - parameters Weibull distribution, and they showed in their work that, the MCMC method is better than the maximum likelihood method when given a proper prior distribution for the parameters. Berger and Sun, (1993) considered the Bayesian using Gibbs sampler for estimating the Poly Weibull distribution using informative priors, followed by the posterior moments, the marginal posterior probability density function and the reliability function. Assoudou and Essebbar

(2003) obtained Bayesian estimate by using Jeffreys prior information with Markov Chain, where the integral in the posterior density function under squared error loss could not be obtained in close form. For that, they proposed an approximation which was the Metropolis-Hasting algorithm, and Maximum likelihood estimation was also obtained for the study. Upadhyay *et al*. (2001) and Pang *et al*. (2007) estimated parameters of the three-parameter Weibull distribution by Markov Chain Monte Carlo methods and they have shown the flexibility of Markov Chain Monte Carlo methods over other traditional approaches. Joarder *et al.* (2011), considered the unknown parameters of a Weibull distribution under Type-I censored data where the maximum likelihood estimator and Bayesian approach using Gibbs sampling technique were used to construct the confidence intervals and the corresponding highest posterior density credible intervals of the unknown parameters were also obtained under fairly general priors on the unknown parameters. Gupta *et al.*, (2008) estimated Weibull extension model by Bayesian method using Markov Chain Monte Carlo (MCMC) simulation. MCMC methods are considered in two ways: the Gibbs sampler and the Metropolis-Hasting algorithms to simulate sample from the posterior, where by MCMC methods analyses the posterior to some graphs and tables that show the estimator for Weibull extension model correlation to MCMC output and in their study they showed that the MCMC was useful and a good estimator for the posterior distribution. Upadhyay and Gupta (2010) discussed some Bayes analysis of modified Weibull distribution using Markov Chain Monte Carlo technique for complete samples and independent vague priors for the unknown parameters.

Maximum Likelihood Estimation

Let t_i be the set of n random lifetimes from Weibull distribution with parameters θ and p. The PDF of Weibull distribution is given below,

$$f(t;\theta,p)=\frac{p}{\theta}t^{p-1}\exp\left(-\frac{t^p}{\theta}\right)$$

For Type I censored data, the likelihood function as in Klein and Moeschberger, (2003) is

$$L(\theta,p;t)=\prod_{i=1}^{n}[f(t_i;\theta,p)]^{\delta_i}\,[S(t_i;\theta,p)]^{1-\delta_i}\,, \tag{1}$$

where $\delta_i=1$ for failure and $\delta_i=0$ for censored observation, and $S(.)$ is the survival function.

Taking the logarithm of equation (1), we have

$$\ln L(\theta,p;t)=\sum_{i=1}^{n}\left[\delta_i\left(\ln p-\ln\theta+(p-1)\ln t_i\right)-\frac{t_i^p}{\theta}\right].$$

To obtain the equations for the unknown parameters, we differentiate equation above partially with respect to the parameters θ and p and equate them to zero. The resulting equations are given respectively as, see for example Al omari *et al* (2012)

$$U(\theta)=\frac{\partial\ln L(\theta,p;t)}{\partial\theta}=-\frac{\sum_{i=1}^{n}\delta_i}{\theta}+\frac{\sum_{i=1}^{n}t_i^p}{\theta^2}$$

$$U(p)=\frac{\partial\ln L(\theta,p;t)}{\partial p}=\frac{\sum_{i=1}^{n}\delta_i}{p}+\sum_{i=1}^{n}\delta_i\ln(t_i)-\frac{\sum_{i=1}^{n}t_i^p\ln(t_i)}{\theta}.$$

Let $U(\theta)$ equals to zero, then the maximum likelihood estimator for the scale parameter of Weibull distribution is,

$$\hat{\theta}_M=\frac{\sum_{i=1}^{n}t_i^p}{\sum_{i=1}^{n}\delta_i}\,, \tag{2}$$

The shape parameter p as in $U(p)$ usually cannot be solved analytically, and for that we need to employ a numerical approach which in most cases is determined by Newton-Raphson method.

Then the estimates of the survival function and hazard rate of Weibull distribution are

$$\hat{S}_M(t) = \exp\left(-\frac{t^{\hat{p}_M}}{\hat{\theta}_M} \right) \quad (3)$$

$$\hat{h}_M(t) = \frac{\hat{p}_M}{\hat{\theta}_M} t^{\hat{p}_M - 1} \quad (4)$$

Bayesian Estimation using Markov Chain Monte Carlo

The posterior probability density function of θ and p is

$$\Pi_1(\theta, p \mid t) = \frac{1}{J_1 \theta^{\sum_{i=1}^{n}\delta_i + 1}} p^{\sum_{i=1}^{n}\delta_i - 1} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right), \quad (5)$$

where J_1 is the marginal likelihood estimation of scale and shape parameters of the Weibull distribution,

$$J_1 = \int_0^{\infty}\int_0^{\infty} \frac{1}{\theta^{\sum_{i=1}^{n}\delta_i + 1}} p^{\sum_{i=1}^{n}\delta_i - 1} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right) d\theta\, dp .$$

This posterior cannot be computed explicitly. We propose to use the Gibbs sampling technique to generate MCMC samples for the estimation of the scale parameter, and then use Metropolis- Hastings Algorithm for estimation of the shape parameter.
From equation (5) we can get the conditional posterior of the scale parameter θ as follows

$$\Pi_1(\theta \mid p, t) \propto \frac{\left(\sum_{i=1}^{n} t_i^p \right)^{\sum_{i=1}^{n}\delta_i}}{\Gamma\left(\sum_{i=1}^{n}\delta_i \right)} \theta^{-\sum_{i=1}^{n}\delta_i - 1} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right) . \quad (6)$$

The conditional posterior of the scale parameter θ follows inverse gamma density function with scale and shape parameters $\sum_{i=1}^{n} \delta_i$ and $\sum_{i=1}^{n} t_i^p$ respectively. We propose to use Gibbs sampling technique to generate MCMC sample as shown in Algorithm.

The conditional posterior of the shape parameter p is given by

$$\Pi_1(p \mid \theta, t) \propto p^{\sum_{i=1}^{n}\delta_i - 1} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right) . \quad (7)$$

As shown in the conditional posterior of the shape parameter, p does not follow any close distribution. Here we propose to use the Metropolis- Hastings algorithm to generate MCMC sample as shown in Algorithm follow by estimate the survival function and hazard rate.

Algorithm:

1. Start with initial value p_0, where the value of p_i in the first step will be the initial value p_0.
2. Generate the scale parameter θ from inverse gamma $\left(\sum_{i=1}^{n}\delta_i, \sum_{i=1}^{n} t_i^p\right)$.
3. Generate candidate value p^* from arbitrary distribution Uniform (0, 1)
4. Take the ratio at the candidate value p^* and current value p_i $\alpha = \min\left\{1, \dfrac{\Pi(p^* \mid \theta, t)}{\Pi(p_i \mid \theta, t)}\right\}$.
5. The next value for the p_i is given as $p_{i+1} = \begin{cases} p^* \text{with probability } \alpha \\ p_i \text{with probability } 1\text{-}\alpha \end{cases}$.
6. Generate u from Uniform (0, 1).
7. Accept p^* if $u < \alpha$ and return to step 2, otherwise accept p_i and return to step 2.
8. The Bayesian estimation of the scale and shape parameters θ and p under the squared error loss function is given as

$$\hat{E}_1(\theta \mid p, t) = \frac{1}{N}\sum_{i=1}^{N}\theta_i \,.$$

$$\hat{E}_1(p \mid \theta, t) = \frac{1}{N}\sum_{i=1}^{N} p_i \,.$$

9. Obtain the posterior variance of p for Bayesian by using Jeffreys prior

$$\hat{V}_1(\theta \mid p, t) = \frac{1}{N}\sum_{i=1}^{N}\left(\theta_i - \hat{E}(\theta \mid p, t)\right)^2 .$$

$$\hat{V}_1(p \mid \theta, t) = \frac{1}{N}\sum_{i=1}^{N}\left(p_i - \hat{E}(p \mid \theta, t)\right)^2 .$$

Extension of Jeffreys Prior Estimation using Markov Chain

The posterior by using extension of Jeffreys is see Al omari *et al* (2010)

$$\Pi_2(\theta, p \mid t) = \frac{1}{J_2 \theta^{\sum_{i=1}^{n}\delta_i + 2c}} p^{\sum_{i=1}^{n}\delta_i - 2c} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right)$$

where,

$$J_2 = \int_0^\infty\int_0^\infty \frac{1}{\theta^{\sum_{i=1}^{n}\delta_i + 2c}} p^{\sum_{i=1}^{n}\delta_i - 2c} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right) d\theta dp \,.$$

From equation (7) we obtain the conditional posterior by using extension of Jeffreys prior of the scale parameter θ as follows

$$\Pi_2(\theta \mid p, t) \propto \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i + 2c - 1}}{\Gamma\left(\sum_{i=1}^{n}\delta_i + 2c - 1\right)} \theta^{-\sum_{i=1}^{n}\delta_i - 2c} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right) . \qquad (8)$$

As shown, the conditional posterior with extension of Jeffreys prior of the scale parameter θ follows inverse gamma density function with scale and shape parameters $\left(\sum_{i=1}^{n}\delta_i + 2c - 1\right)$ and $\sum_{i=1}^{n} t_i^p$ respectively. We propose to use Gibbs sampling technique to generate MCMC sample.

From equation (7) we can get the conditional posterior by using extension of Jeffreys prior of the shape parameter p as given below,

$$\Pi_2(p\,|\,\theta,t) \propto p^{\sum_{i=1}^{n}\delta_i - 2c} \prod_{i=1}^{n} t_i^{\delta_i(p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right). \tag{9}$$

As shown in the conditional posterior for extension of Jeffreys of the shape parameter p does not follow any close distribution. For that, we propose to use the Metropolis- Hastings algorithm to generate the MCMC samples follows by estimate the survival function and hazard rate.

Simulation Study

In our simulation study, we have chosen n=25, 50 and 100, censored data was 20%. The value of parameters chosen were θ = 0.5 and 1.5 and p = 0.8 and 1.2. The two values of extension of Jeffreys were 1.5 and 3. The process was repeated 10,000 times. For each repetition the mean squared error (MSE) and absolute bias of scale and shape parameters and survival function and hazard rate for the three methods above.
The results are presented in Tables 1 and 2 for different selections of the parameters and c extension of Jeffreys prior.

TABLE (1). Estimated scale parameter with MSE (parentheses) of Weibull distribution censored data by maximum likelihood (MLE) and Bayesian approach using Gibbs sampler.

Size	Estimators	θ=0.8		θ=1.2	
		p=0.5	p=1.5	p=0.5	p=1.5
25	MLE	0.9083 (0.0512)	0.7432 (0.0491)	1.1370 (0.1369)	1.3166 (0.1743)
	BJ	0.9336 (0.0613)	0.8718 (0.0536)	1.2880 (0.2345)	1.3011 (0.0970)
	BE(c=1.5)	0.8237 (0.0310)	0.7645 (0.0416)	1.1568 (0.1201)	1.2634 (0.1295)
	BE(c=3)	0.8637 (0.0433)	0.7882 (0.0319)	1.0782 (0.1550)	1.2415 (0.0622)
50	MLE	0.9001 (0.0321)	0.7757 (0.0217)	1.1434 (0.0418)	1.3078 (0.0757)
	BJ	0.9602 (0.0538)	0.7657 (0.0274)	1.1207 (0.0420)	1.2790 (0.0761)
	BE(c=1.5)	0.8410 (0.0225)	0.7690 (0.0164)	1.1753 (0.0394)	1.2481 (0.0873)
	BE(c=3)	0.8104 (0.0140)	0.8195 (0.0153)	1.0925 (0.0462)	1.2399 (0.0572)
100	MLE	0.8994 (0.0194)	0.7763 (0.0139)	1.1537 (0.0250)	1.2854 (0.0416)
	BJ	0.9373 (0.0267)	0.7552 (0.0233)	1.1428 (0.0259)	1.2730 (0.0444)
	BE(c=1.5)	0.9135 (0.0207)	0.7858 (0.0133)	1.2142 (0.0206)	1.2383 (0.0537)
	BE(c=3)	0.8353 (0.0117)	0.7931 (0.0087)	1.2625 (0.0334)	1.2102 (0.0274)

TABLE (2). Estimated shape parameter with MSE (parentheses) of Weibull distribution censored data by maximum likelihood (MLE), Bayesian approach using Metropolis- Hastings Algorithm.

Size	Estimators	θ =0.8		θ =1.2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.5380	1.6022	0.5380	1.6039
		(0.0109)	(0.1005)	(0.0111)	(0.1011)
	BJ	0.5937	1.3105	0.6045	1.6578
		(0.0177)	(0.3728)	(0.0191)	(0.3492)
	BE(*c*=1.5)	0.5674	1.3109	0.5357	1.6516
		(0.0144)	(0.3694)	(0.0095)	(0.3693)
	BE(*c*=3)	0.4723	1.3648	0.5745	1.6199
		(0.0095)	(0.4046)	(0.0103)	(0.3262)
50	MLE	0.5213	1.5442	0.5244	1.5451
		(0.0048)	(0.0395)	(0.0049)	(0.0415)
	BJ	0.5750	1.3596	0.6161	1.6239
		(0.0058)	(0.3156)	(0.0094)	(0.2906)
	BE(*c*=1.5)	0.5701	0.3623	0.6021	1.6150
		(0.0063)	(0.3120)	(0.0086)	(0.2954)
	BE(*c*=3)	0.5112	1.3310	0.5438	1.6417
		(0.0045)	(0.3229)	(0.0063)	(0.3053)
100	MLE	0.5168	1.5224	0.5056	1.5224
		(0.0021)	(0.0180)	(0.0021)	(0.0185)
	BJ	0.5587	1.3743	0.6082	1.5824
		(0.0049)	(0.2753)	(0.0038)	(0.2717)
	BE(*c*=1.5)	0.5428	1.3712	0.5862	1.5779
		(0.0034)	(0.2755)	(0.0031)	(0.2713)
	BE(*c*=3)	0.5061	1.3725	0.5259	1.5756
		(0.0019)	(0.2863)	(0.0024)	(0.2727)

CONCLUSION

Bayesian using extension of Jefferys prior using Gibbs sampling for estimate the scale parameter is better than Maximum Likelihood for all cases. The Maximum Likelihood estimate of shape parameter is more efficient than their Bayesian models using the Metropolis- Hastings algorithm. However the extension of Jeffreys is better than MLE for certain value of extension of Jeffreys prior.

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