



Journal Homepage: - www.journalijar.com
**INTERNATIONAL JOURNAL OF
 ADVANCED RESEARCH (IJAR)**

Article DOI: 10.21474/IJAR01/3318
 DOI URL: <http://dx.doi.org/10.21474/IJAR01/3318>



RESEARCH ARTICLE

WEYL FRACTIONAL DERIVATIVE OF THE PRODUCT MULTIVARIABLES POLYNOMIALS AND I –FUNCTION.

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Manuscript Info

Manuscript History

Received: 20 December 2016
 Final Accepted: 28 January 2017
 Published: February 2017

Abstract

In this research work, we establish a theorem on Weyl fractional derivative of the product multivariable polynomials and I –function. Certain special cases of our theorem have been discussed.

Mathematics Subject classification - 26A33, 33C 60, 44A15.

Key words:-

Weyl fractional derivative operator,
 multivariable polynomials and I-function

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Introduction:-

The I- function of the one variable is defined by Saxena (1982) and we will represent here in the following manner:

$$(1.1) \quad I[z] = I_{p_i, q_i; r}^{m, n}[z] = I_{p_i, q_i; r}^{m, n} \left[z \left| \begin{matrix} \dots, \dots \\ \dots, \dots \end{matrix} \right. \right] = I_{p_i, q_i; r}^{m, n} \left[z \left| \begin{matrix} (a_j, e_j)_{1, n}; & (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1, m}; & (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right]$$

$$(1.2) \quad = \frac{1}{2\pi i} \int_L \theta(s) z^s ds,$$

where $i = \sqrt{-1}$, $z (\neq 0)$ is a complex variable and (1.2) $z^s = \exp[s \{ \log |z| + i \arg z \}]$. In which $\log |z|$ represent the natural logarithm of $|z|$ and $\arg |z|$ is not necessarily the principle value. An empty product is interpreted as unity. Also,

$$(1.3) \quad \theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1 - a_j + e_j s)}{\sum_{i=1}^r \left[\prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + f_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - e_{ji} s) \right]}, \quad m, n, p_i \text{ and } q_i \forall i \in (1, \dots, r) \text{ are non-negative}$$

integers satisfying $0 \leq n \leq p_i$, $0 \leq m \leq q_i$; $\forall i \in \{1, \dots, r\}$, $e_{ji}, (j = 1, \dots, p_i; i = 1, \dots, r)$ and

$f_{ji}, (j = 1, \dots, q_i; i = 1, \dots, r)$ are assumed to be positive quantities for standardization purpose. Also $a_{ji}, (j = 1, \dots, p_i; i = 1, \dots, r)$ and $b_{ji}, (j = 1, \dots, q_i; i = 1, \dots, r)$ are complex numbers such that none of the points.

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$$(1.4) \quad S = \{(b_n + v) | f_h\}, h=1, \dots, m; v=0, 1, 2, \dots,$$

which are the poles of $\Gamma(b_h - f_h s), h=1, \dots, m$ and the points

$$(1.5) \quad S = \{(a_l - n - 1) | e_l\} l=1, \dots, n; \eta=0, 1, 2, \dots,$$

Which poles are of $\Gamma(1 - a_l + e_l s)$ coincide with one another, i.e. with

$$(1.6) \quad e_l(b_n + v) \neq b_n(a_l - \eta - 1),$$

for $v, \eta=0, 1, 2, \dots; h=1, \dots, m; l=1, \dots, n$.

Further, the contour L runs from $-i_\infty$ to $+i_\infty$. Such that the poles of $\Gamma(b_n - s), h=1, \dots, m$; lie to the right of L and the poles $\Gamma(1 - a_l + e_l s), l=1, \dots, n$ lie to the left of L. The integral converges, if $|\arg z| < \frac{1}{2} B\pi, B > 0, A \leq 0$, where

$$(1.7) \quad A = \sum_{j=1}^{p_i} e_{ji} - \sum_{j=1}^{q_i} f_{ji} \text{ and}$$

$$(1.8) \quad B = \sum_{j=1}^n e_j - \sum_{j=n+1}^{p_i} e_{ji} + \sum_{j=1}^m f_j - \sum_{j=m+1}^{q_i} f_{ji} \quad \forall i \in (1, \dots, r)$$

Let A denote a class of good functions. By good function f, we mean Miller [1975, p.82] a function which is everywhere differentiable any number of times and if all of its derivatives are $O(x^{-\nu})$, for all ν as x increases without limit. We define the Weyl fractional derivatives of a function $g(z)$ as follows:-

Let $g \in A$, then for $q < 0$,

$$(1.9) \quad {}_z W_\infty^q g(z) = \frac{(-1)^q}{\Gamma(-q)} \int_z^\infty (u-z)^{-q-1} g(u) du.$$

For $q \geq 0$

$$(1.10) \quad {}_z W_\infty^q g(z) = \frac{d^n}{dz^n} ({}_z W_\infty^{q-n} g(z)),$$

n being positive integer, such that $n > q$.

The general class of multivariable polynomials is defined by Srivastava and Garg [1987]:

$$(1.11) \quad S_L^{h_1, \dots, h_r} [x_1, \dots, x_r] = \sum_{k_1, \dots, k_r=0}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L)_{h_1 k_1 + \dots + h_r k_r} A(L; k_1, \dots, k_r) \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!},$$

where h_1, \dots, h_r are positive integers and the co-efficient $A(L; k_1, \dots, k_r)$, ($L; h_i \in \mathbb{N}; i=1, \dots, r$) are arbitrary constant, real or complex.

Evidently the case $r=1$ of the polynomials (1.11).

Would correspond the polynomials given by Shrivastava [1972]

$$(3.1.2) \quad S_L^h [x] = \sum_{k=0}^{[L, h]} \frac{(-L)_{hk}}{k!} A_{L, k} x^k \quad \{L \in \mathbb{N} = (0, 1, 2, \dots)\},$$

where h is arbitrary positive integers and the co-efficient $A_{L, k} (L, k \geq 0)$ are arbitrary constant, real or complex.

Mathematical pre-requisites:-

To establish the main result, we need the following integral of the H-function by Saigö [1992]:

$$(2.1) \quad \int_x^\infty t^{\rho-1} (t-x)^{\sigma-1} I_{p_i, q_i; r}^{m, n} \left[z t^\mu (t-x)^\nu \left| \begin{matrix} (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right] dt$$

$$= x^{\rho+\sigma-1} I_{p_i+2, q_i+1; r}^{m+1, n+1} \left[z x^{\mu+\nu} \left| \begin{matrix} (1-\sigma, \nu), (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho, \mu) \\ (1-\rho-\sigma, \mu+\nu), (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right],$$

where

(i) ρ, σ are complex numbers and μ, ν are positive real numbers,

(ii) $|\arg z| < \frac{1}{2} A\pi$, A defined as

$$A = \sum_{j=1}^{p_i} e_{ji} - \sum_{j=1}^{q_i} f_{ji},$$

$$(iii) \quad \min \left[\operatorname{Re} \left(\frac{1-\rho-\sigma}{\mu-\nu} \right), \min_{1 \leq j \leq m} \left[\operatorname{Re} \left(\frac{b_j}{f_j} \right) \right] \right] >$$

$$\max \left[-\operatorname{Re} \left(\frac{\sigma}{\nu} \right), \max_{1 \leq j \leq N} \left[\operatorname{Re} \left(\frac{a_j-1}{e_j} \right) \right] \right].$$

Weyl Fractional Derivatives Of The Product Of Multivariable Polynomials And I-Function.

Theorem.

Let m, n, p_i and q_i be non-negative integers such that $0 \leq n \leq p_i, 0 \leq m \leq q_i$ and $\sum_{j=1}^n e_j - \sum_{j=n+1}^{p_i} e_{ji} + \sum_{j=1}^m f_j - \sum_{j=m+1}^{q_i} f_{ji} > 0$ together with the set of conditions (i) – (iii) given with equation (2.1).

Then, for all value of q ,

$$(3.1) \quad {}_z W_\infty^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^{h_1, \dots, h_r} \left[c_1 x^{\delta_1}, \dots, c_r x^{\delta_r} \right] \right.$$

$$\left. \times I_{p_i, q_i; r}^{m, n} \left[y x^\mu (x-z)^\nu \left| \begin{matrix} (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right] \right\}$$

$$= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-\sum_{i=1}^r k_i \delta_i - 2} \sum_{\substack{k_1, \dots, k_r=0 \\ h_1 k_1 + \dots + h_r k_r}}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L) A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!}$$

$$\times I_{p_i+2, q_i+1; r}^{m+1, n+1} \left[y z^{\mu+\nu} \left| \begin{matrix} (2-\sigma+q, \nu), (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho-\sum_{i=1}^r k_i \delta_i, \mu) \\ (2+q-\rho-\sigma-\sum_{i=1}^r k_i \delta_i, \mu+\nu), (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right].$$

Proof : Taking left hand side of equation (3.1) and using equation (1.11), we get

$$(3.2) \quad {}_z W_\infty^q \left\{ x^{\rho+\sum_{i=1}^r k_i \delta_i - 1} (z-x)^{\sigma-1} \sum_{\substack{k_1, \dots, k_r=0 \\ h_1 k_1 + \dots + h_r k_r}}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L) A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \right.$$

$$\left. \times I_{p_i, q_i; r}^{m, n} \left[y x^\mu (x-z)^\nu \left| \begin{matrix} (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right] \right\},$$

Now using equation (1.9) and definition of I – function, easily we can find the proof of equation (3.1).

For $q \geq 0$ invoking the definition (1.10) the relation (3.2) further reduces to

$$= \sum_{\substack{h_1 k_1 + \dots + h_r k_r \leq L \\ k_1, \dots, k_r = 0}} (-L) A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \frac{(-1)^{q+r+\sigma-1}}{\Gamma(r-q)} \frac{d^r}{dz^r} \left\{ z^{\rho+\sigma-q+r-\sum_{i=1}^r k_i \delta_i - 2} \right. \\ \times I_{p_i+2, q_i+1; r}^{m+1, n+1} \left[yz^{\mu+\nu} \left| \begin{array}{l} (2-\sigma-h_1 k_1+\dots-h_r k_r, (a_j, e_j)_{1,n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho-\sum_{i=1}^r k_i \delta_i, \mu) \\ (2+q-\rho-\sigma-\sum_{i=1}^r k_i \delta_i, \mu+\nu), (b_j, f_j)_{1,m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right] \right\}$$

In replacing of $(q-r)$ by q , we may obtain again

$$= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-\sum_{i=1}^r k_i \delta_i - 2} \sum_{\substack{h_1 k_1 + \dots + h_r k_r \leq L \\ k_1, \dots, k_r = 0}} (-L) A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \\ \times I_{p_i+2, q_i+1; r}^{m+1, n+1} \left[yz^{\mu+\nu} \left| \begin{array}{l} (2-\sigma+q, \nu), (a_j, e_j)_{1,n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho-\sum_{i=1}^r k_i \delta_i, \mu) \\ (2+q-\rho-\sigma-\sum_{i=1}^r k_i \delta_i, \mu+\nu), (b_j, f_j)_{1,m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right]$$

Special Case : If we put $r=1$ in the general call of multivariable polynomials given by Srivastava and Garg [1987] reduces to the polynomials given by Srivastava [1972] and I- function reduces into Fax's H – function as follows :

$$(4.1) \quad {}_z W_{\infty}^q \left\{ {}_z W_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{\nu} \left| \begin{array}{l} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{array} \right. \right] \right\} \right\} \\ = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,k]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+2, q+1}^{m+1, n+1} \left[yz^{\mu+\nu} \left| \begin{array}{l} (2-\sigma+q, \nu), (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (2+q-\rho-\sigma-k, \mu+\nu), (b_j, f_j)_{1,q} \end{array} \right. \right]$$

Replacing ν by $-\nu$ equation (3.3) correspond to the following result according as $\mu > \nu$, $\mu < \nu$ and $\mu = \nu$, i.e. for $\mu > \nu$

$$(4.2) \quad {}_z W_{\infty}^q \left\{ {}_z W_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{-\nu} \left| \begin{array}{l} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{array} \right. \right] \right\} \right\} \\ = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\ \times H_{p+1, q+2}^{m+2, n} \left[yz^{\mu-\nu} \left| \begin{array}{l} (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (2+q-\rho-\sigma-k, \mu-\nu), (\sigma-q-1, \nu), (b_j, f_j)_{1,q} \end{array} \right. \right]$$

For $\mu < \nu$

$$(4.3) \quad {}_z W_{\infty}^q \left\{ {}_z W_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{-\nu} \left| \begin{array}{l} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{array} \right. \right] \right\} \right\} \\ = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\ \times H_{p+2, q+1}^{m+1, n+1} \left[yz^{\mu-\nu} \left| \begin{array}{l} (\rho+\sigma-q+k-1, \nu-\mu), (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (\sigma-q-1, \nu), (b_j, f_j)_{1,q} \end{array} \right. \right]$$

and $\mu = \nu$

$$\begin{aligned}
 (4.5) \quad & {}_z W_\infty^q \left\{ {}_z W_\infty^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^\mu (x-z)^{-\nu} \left| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right. \right] \right\} \right. \\
 &= \frac{(-1)^{q+\sigma-1} \Gamma(2-q-\rho-k)}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\
 &\times H_{p+1,q+1}^{m+1,n} \left[y \left| \begin{matrix} (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (\sigma-q-1, \nu), (b_j, f_j)_{1,q} \end{matrix} \right. \right].
 \end{aligned}$$

Finally writing $-\mu$ instead of μ , equation (3.3) yields the following results according as $\mu > \nu$, $\mu < \nu$ and $\mu = \nu$ respectively.

For $\mu > \nu$

$$\begin{aligned}
 (4.6) \quad & {}_z W_\infty^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{-\mu} (x-z)^{\nu} \left| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right. \right] \right\} \\
 &= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\
 &\times H_{p+2,q+1}^{m,n+2} \left[yz^{-\mu+\nu} \left| \begin{matrix} (q-\rho-\sigma-k+1, \mu-\nu), (2+q-\sigma, \nu), (a_j, e_j)_{1,p} \\ (b_j, f_j)_{1,q}, (\rho+k, \mu) \end{matrix} \right. \right]
 \end{aligned}$$

for $\mu < \nu$

$$\begin{aligned}
 (4.7) \quad & {}_z W_\infty^q \left\{ {}_z W_\infty^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{-\mu} (x-z)^{\nu} \left| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right. \right] \right\} \right. \\
 &= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\
 &\times H_{p+1,q+1}^{m+1,n+1} \left[yz^{-\mu+\nu} \left| \begin{matrix} (2-\sigma+q, \nu), (a_j, e_j)_{1,p} \\ (2+q-\rho-\sigma-k, \nu-\mu), (b_j, f_j)_{1,q}, (\rho+k, \mu) \end{matrix} \right. \right]
 \end{aligned}$$

and for $\mu = \nu$

$$\begin{aligned}
 (4.8) \quad & {}_z W_\infty^q \left\{ {}_z W_\infty^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{-\mu} (x-z)^{\nu} \left| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right. \right] \right\} \right. \\
 &= \frac{\Gamma(2-\rho-k-\sigma+q)}{\Gamma(-q)} \frac{(-1)^{q+\sigma-1}}{k!} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\
 &\times H_{p+2,q+1}^{m+1,n+1} \left[yz^{\mu+\nu} \left| \begin{matrix} (2-\sigma+q, \nu), (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (2+q-\rho-\sigma-k, \mu+\nu), (b_j, f_j)_{1,q} \end{matrix} \right. \right].
 \end{aligned}$$

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