WEYL FRACTIONAL DERIVATIVE OF THE PRODUCT MULTIVARIABLES POLYNOMIALS AND I–FUNCTION.

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Abstract
In this research work, we establish a theorem on Weyl fractional derivative of the product multivariable polynomials and I–function. Certain special cases of our theorem have been discussed.

Mathematics Subject classification - 26A33,33C 60 , 44A15.

Introduction:
The I–function of the one variable is defined by Saxena (1982) and we will represent here in the following manner:

\[ \phi(z) = \prod_{j=1}^{m} \frac{\Gamma(b_j - f_j s)}{\Gamma(b_j - f_j s)} \prod_{i=1}^{n} \frac{\Gamma(1 - a_j + e_j s)}{\Gamma(1 - a_j + e_j s)} \]

where \( i = \sqrt{-1} \), \( z \neq 0 \) is a complex variable and

\[ z' = \exp[s \{ \log |z| + i \arg z \}] \]

In which \( \log |z| \) represent the natural logarithm of \( |z| \) and \( \arg |z| \) is not necessarily the principle value. An empty product is interpreted as unity. Also,

\[ \theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s)}{\prod_{j=m+1}^{r} \Gamma(1 - b_j - f_j s)} \frac{\prod_{i=1}^{p} \Gamma(1 - a_j - e_j s)}{\prod_{i=p+1}^{q} \Gamma(1 - a_j - e_j s)} \]

\[ \gamma_i, \forall i \in \{1, \ldots, r\} \text{ are non-negative integers satisfying } 0 \leq n \leq p_i, \quad 0 \leq m \leq q_i, \quad \forall i \in \{1, \ldots, r\}, e_{ji}, (j = 1, \ldots, p_i; i = 1, \ldots, r) \text{ and } f_{ji}, (j = 1, \ldots, q_i; i = 1, \ldots, r) \text{ are assumed to be positive quantities for standardization purpose. Also } a_{ji}, (j = 1, \ldots, p_i; i = 1, \ldots, r) \text{ and } b_{ji}, (j = 1, \ldots, q_i; i = 1, \ldots, r) \text{ are complex numbers such that none of the points.}

Key words:-
Weyl fractional derivative operator, multivariable polynomials and I–function.

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Mathematical pre-requisites:-
To establish the main result, we need the following integral of the H-function by Saigo[1992]:

(1.4) \[ S = \{ (b_n + v) \mid f_h \}, h = 1, \ldots, m; v = 0,1,2,\ldots, \]
which are the poles of \( \Gamma(b_n - f_h s), h = 1, \ldots, m \) and the points

(1.5) \[ S = \{ (a_i - n - 1) \mid e_i \}, l = 1, \ldots, n; \eta = 0,1,2,\ldots, \]
Which poles are of \( \Gamma(1 - a_i + e_i s) \) coincide with one another, i.e. with

(1.6) \[ e_i (b_n + v) \neq b_n (a_i - \eta - 1). \]
for \( v, \eta = 0,1,2,\ldots; h = 1, \ldots, m; l = 1, \ldots, n. \)

Further, the contour L runs from \(-i_\infty\) to \(+i_\infty\). Such that the poles of \( \Gamma(b_n - s), h = 1, \ldots, m \) lie to the right of L and the poles \( \Gamma(1 - a_i + e_i s), l = 1, \ldots, n \) lie to the left of L. The integral converges, if \(|\arg z| < \frac{1}{2} B \pi, B > 0, A \leq 0, \infty\), where

(1.7) \[ A = \sum_{j=1}^{p_j} e_{ji} - \sum_{j=1}^{q_j} f_{ji} \]
and

(1.8) \[ B = \sum_{j=1}^{n} e_j - \sum_{j=1}^{p_j} e_{ji} + \sum_{j=m+1}^{m} f_j - \sum_{j=m+1}^{q_j} f_{ji} \quad \forall i \in (1, \ldots, r) \]

Let A denote a class of good functions. By good function \( f \), we mean Miller [1975, p.82] a function which is everywhere differentiable any number of times and if it is all of its derivatives are \( O(x^{-u}) \), for all \( u \) as \( x \) in increases without limit. We define the Weyl fractional derivatives of a function \( g(z) \) as follows:-

Let \( g \in A \), then for \( q < 0 \),

(1.9) \[ z^q W_{\infty} g(z) = \frac{(-1)^q}{\Gamma(-q)} \int_{-\infty}^{\infty} (u - z)^{-q-1} g(u) \, du. \]

For \( q \geq 0 \)

(1.10) \[ z^q W_{\infty} g(z) = \frac{d^n}{dz^n} (z^q W_{\infty} W_{\infty}^{-n} g(z)), \]
n being positive integer, such that \( n > q \).

The general class of multivariable polynomials is defined by Srivastava and Garg [1987]:

(1.11) \[ S^{h_1,\ldots,h_r}_{L} [x_1, \ldots, x_r] = \sum_{k_1, \ldots, k_r = 0} (-L)_{h_1,\ldots,+,h_r,\ldots,+} A(L; k_1, \ldots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}, \]
where \( h_1, \ldots, h_r \) are positive integers and the co-efficient \( A(L; k_1, \ldots, k_r), (L; h_1 \in N; i = 1, \ldots, r) \) are arbitrary constant, real or complex.

Evidently the case \( r = 1 \) of the polynomials (1.11).

Would correspond the polynomials given by Shrivastava [1972]

(3.1.2) \[ S^h_{L} [x] = \sum_{k=0}^{[L,h]} (-L)^{h} A_{L,k} x^k \{ L \in N = (0,1,2,\ldots) \}, \]
where \( h \) is arbitrary positive integers and the co-efficient \( A_{L,k} (L, k \geq 0) \) are arbitrary constant, real or complex.

\textbf{Mathematical pre-requisites:-}
To establish the main result, we need the following integral of the H-function by Saigo[1992]:
where \( \rho, \sigma \) are complex numbers and \( \mu, \nu \) are positive real numbers,

\[
\begin{align*}
\text{(ii)} \quad & \quad \arg \ z < \frac{\pi}{2} A \pi, \\
A &= \sum_{j=1}^{p} e_{ji} - \sum_{j=1}^{q} f_{ji},
\end{align*}
\]

iii) \[
\begin{align*}
\min \left[ \Re \left( \frac{1 - \rho - \sigma}{\mu - \nu} \right) , \min 1 \leq j \leq m \left[ \Re \left( \frac{b_j}{f_j} \right) \right] \right] > \\
\max \left[ - \Re \left( \frac{\sigma}{\nu} \right) , \max 1 \leq j \leq N \left[ \Re \left( \frac{a_j - 1}{e_j} \right) \right] \right] .
\end{align*}
\]

Weyl Fractional Derivatives Of The Product Of Multivariable Polynomials And I-Function.

Theorem.

Let \( m, n, p_i \) and \( q_i \) be non-negative integers such that \( 0 \leq n \leq p_i, 0 \leq m \leq q_i \) and \( \sum_{j=1}^{n} e_{j} - \sum_{j=n+1}^{m} e_{ji} + \sum_{j=1}^{m} f_{j} - \sum_{j=m+1}^{q_i} f_{ji} > 0 \) together with the set of conditions (i) – (iii) given with equation (2.1).

Then, for all value of \( q \),

\[
\begin{align*}
& \quad \int_{z}^{W_{\infty}} \left( z-x \right)^{1/2} S_{L}^{h_{1}, \ldots, h_{r}} \left[ c_{1} x^{\delta_{1}}, \ldots, c_{r} x^{\delta_{r}} \right] \\
& \quad \times I_{m, n}^{1, 1, \ldots, 1} \left\{ \begin{array}{l}
\left( a_{j, e_{j}} \right)_{1, n} \left( a_{ji, e_{ji}} \right)_{n+1, p_i} \\
\left( b_{j, f_{j}} \right)_{1, m} \left( b_{ji, f_{ji}} \right)_{m+1, q_i}
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\end{array} \right\}
\end{align*}
\]
Now using equation (1.9) and definition of I – function, easily we can find the proof of equation (3.1). For q ≥ 0 invoking the definition (1.10) the relation (3.2) further reduces to

\[
= \sum_{k_0=0}^{h_k} \cdots \sum_{k_{s}=0}^{h_{s}} (-L)^{\sum_{i=1}^{s} k_{i} + k_{s}} A(L; k_0, \ldots, k_{s}) \frac{c_{1}^{r_1}}{k!} \cdots \frac{c_{r}^{r_s}}{k!} \left( \begin{array}{l}
\rho \sum_{i=1}^{s} q_{i} \delta_{i} + \sum_{i=1}^{s} k_{i} \\
\end{array} \right)
\]

In replacing of (q - r) by q, we may obtain again

\[
= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} \sum_{k=0}^{\infty} \frac{(-L)^{h_k} k!}{k!} A_{L,k} \times H_{p+1,q+1}^{m+1,n+1} \left[ y_{\mu-\nu} \left(2 - \sigma + q, \nu\right), (a_{j}, e_{j})_{1,p}, (1 - \rho - k, \mu) \right] \]

Special Case: If we put r=1 in the general call of multivariable polynomials given by Srivastava and Garg [1987] reduces to the polynomials given by Srivastava [1972] and I- function reduces into Fax ’s H- function as follows:

\[
W_{\infty}^{q} \left[ W_{\infty}^{q} \left[ x^\rho (z - x)^{\sigma-1} S_{L}^{h} \left[ x^{k} \right] \times H_{p,q}^{m,n} \left[ y_{\mu} (x - z)^{-\nu} \left( a_{j}, e_{j} \right)_{1,p}, (b_{j}, f_{j})_{1,q} \right] \right] \right] \]

(4.1)

\[
= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} \sum_{k=0}^{\infty} \frac{(-L)^{h_k} k!}{k!} A_{L,k} \times H_{p+1,q+1}^{m+1,n+1} \left[ y_{\mu-\nu} \left(2 - \sigma + q, \nu\right), (a_{j}, e_{j})_{1,p}, (1 - \rho - k, \mu) \right] \]

Replacing u by −u equation (3.3) correspond to the following result according as μ > ν, μ < ν and μ = ν, i.e. for μ > ν

\[
\text{For } \mu < \nu
\]

\[
= \todo{\text{expression}}
\]

(4.3)

\[
= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} \sum_{k=0}^{\infty} \frac{(-L)^{h_k} k!}{k!} A_{L,k} \times H_{p+1,q+1}^{m+1,n+1} \left[ y_{\mu-\nu} \left(\rho + \sigma - q + k, \nu - \mu\right), (a_{j}, e_{j})_{1,p}, (1 - \rho - k, \mu) \right] \]

\[
\text{and } \mu = \nu
\]
\[ W^q_{\infty} \left\{ z W^q_{\infty} \{ x^{\rho-1} (z-x)^{\sigma-1} S^h_L [x^k] \} \times H^{m,n}_{p,q} \right\} \times \left( H^{m,n}_{p,q} \right) \]
\[ = \frac{(-1)^{q+\sigma-1} \Gamma(2-q-\rho-k)}{\Gamma(-q)} \sum_{k=0}^{[L_h]} (-L)_{hk} A_{l,k} \]
\[ \times H^{m+1,n}_{p+1,q+1} \left\{ y X^{-\mu} (x-z)^{\mu} \right\} \]

Finally writing \(-\mu\) instead of \(\mu\), equation (3.3) yields the following results according as \(\mu > \nu, \mu < \nu\) and \(\mu = \nu\) respectively.

For \(\mu > \nu\)

\[ z W^q_{\infty} \left\{ x^{\rho-1} (z-x)^{\sigma-1} S^h_L [x^k] \right\} \times H^{m,n}_{p,q} \left( y X^{-\mu} (x-z)^{\mu} \right) \]

For \(\mu < \nu\)

\[ z W^q_{\infty} \left\{ x^{\rho-1} (z-x)^{\sigma-1} S^h_L [x^k] \right\} \times H^{m,n}_{p,q} \left( y X^{-\mu} (x-z)^{\mu} \right) \]

And for \(\mu = \nu\)

\[ z W^q_{\infty} \left\{ x^{\rho-1} (z-x)^{\sigma-1} S^h_L [x^k] \right\} \times H^{m,n}_{p,q} \left( y X^{-\mu} (x-z)^{\mu} \right) \]

Reference:


