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RESEARCH ARTICLE

UNIQUENESS OF FRACTIONAL DIFFERENTIAL EQUATIONS USING GRONWALL TYPE INEQUALITIES.

G. U. Pawar[1] and J. N. Salunke[2].

1. Department of Mathematics, Mahatma Jyotiba Fule Jr. College,Pishor-431104,Dist.Aurangadad (M.S.), India.
2. School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded (M.S.), India.

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Abstract

In this paper we study the boundedness and uniqueness of solution of initial value problem by using Gronwall type integral inequalities.

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Introduction:-

Fractional calculus deals with the investigations of derivatives and integrals of any arbitrary real or complex order (non-integer order). The fractional order differentiation and integration represents a rapidly growing field both in theory and applications of real world problems. The class of fractional differential equations of various types plays important roles in physics, control systems, dynamical systems and engineering, porous media [1,3,7] [5].In the recent years, there has been a significant development in fractional calculus and fractional differential equations [1,5,8] Different aspects of solutions of differential equations can be resolved by using differential and integral inequalities. Perhaps Gronwall type inequalities and their generalizations is the most useable tool in the literature. The integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations [16]. As an application of Generalized Gronwall -type inequalities, we prove the uniqueness and boundedness of solutions of the R-L fractional differential equation of order $0 < \alpha < 1$.

In this paper we consider the following initial value problem

(1.1) $\quad \mathrm{D}_x^\alpha u(x) = f\big(x, u(x)\big) , 0 \leq x \leq X.$

(1.2) $\quad$ *with initial condition* $\quad \mathrm{D}_x^{\alpha-1} u(x)|_{x=0} = \delta$

Where $0 < \alpha < 1 , f \epsilon\, C(R \times R , R)$, D_x^α denotes the Riemann-Liouville fractional derivative defined by

$$\mathrm{D}_x^\alpha v(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} v(t) dt$$

In this paper, we apply some new generalized Gronwall-type inequalities suitable for the qualitative and quantitative analysis of the solutions to fractional differential equations. In section 2 we introduce some preliminary definitions and results. In section 3 we investigate a certain fractional differential equation , in which the boundedness and

Corresponding Author:- G. U. Pawar
Address:- Department of Mathematics, Mahatma Jyotiba Fule Jr. College, Pishor -431104,Dist.Aurangadad (M.S.),

uniqueness on initial data for the solution to the fractional differential equation are investigated by the use of the generalized Gronwall-type inequalities.

Preliminaries:-

The definitions of Riemann-Liouville fractional derivative and integral are given in [5].

$\boldsymbol{Definition(2.1)}$[5]: $Let\ f\epsilon\ L_1[a,b], \alpha\epsilon\mathbb{R}^+, the\ Riemann-Liouville\ fractional\ integral\ of\ order\ \alpha\ is\ defined\ as$

(2.1) $$\mathrm{I}_x^\alpha v(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{-\alpha} v(s)ds \qquad for\ all\ x\epsilon\ [a,b].$$

The author in [4] studied IVP (1.1)-(1.2) for existence and uniqueness. The equivalent integral equation corresponding to IVP(1.1)-(1.2) is

(2.2) $$u(x) = \frac{\delta}{\Gamma(\alpha)} x^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t,u(t)\big)dt$$

Now we present some generalized Gronwall-type inequalities studied by [4] which are required to prove the boundedness and uniqueness of IVP (1.1)-(1.2).

$\boldsymbol{Lemma\ (2.1)[2]}$: $Assume\ that\ a \geq 0, p \geq q \geq 0\ with\ p \neq 0. Then, for\ any\ K > 0, we\ have$

(2.3) $$a^{\frac{p}{q}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}$$

$\boldsymbol{Theorem(2.2)[2]}$: $Suppose\ that\ \alpha > 0, p \geq 1\ are\ constants, L\epsilon C(R_+ \times R_+, R_+) with$

$$0 \leq L(t,u) - L(t,v) \leq T(u-v) \qquad for\ u \geq v \geq 0,$$

$where\ T\ is\ the\ Lipschitz\ constant\ u, a, h\ are\ nonnegative\ functions\ locally\ integrable\ on\ [0,X)$ $with\ h\ nondecreasing\ and\ bounded\ by\ M, where\ M\ is\ a\ positive\ constant. If\ the\ following\ inequality\ is\ satisfied:$

(2.4) $$u^p(x) = a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L\big(t,u(t)\big)dt, 0 \leq x < X.$$

$then\ we\ have\ the\ following\ explicit\ estimate\ for\ u$:

(2.5) $$u(x) \leq \left\{\tilde{a}(x) + \int_0^x \left[\sum_{n=1}^\infty \left(\frac{T}{P} K^{\frac{1-p}{p}}\right)^n \frac{h^n(x)}{\Gamma(n\alpha)} (x-t)^{n\alpha-1} \tilde{a}(t)\right] dt\right\}^{\frac{1}{p}}$$

(2.6) Where $\tilde{a}(x) = a(x) + \frac{1}{\Gamma(\alpha)} h(x)\ and\ K > 0\ is\ a\ constant.$

Main Result:-

We prove boundedness and uniqueness of solution of IVP (1.1)-(1.2) in this section.

$\boldsymbol{Theorem(3.1)}$: $For\ IVP(\,), if\ |f(x,u)| \leq L(x,|u|), where\ L\ is\ defined\ as\ in\ the\ Theorem(2.2),$ $then\ we\ have\ the\ following\ estimate:$

(3.1) $$u(x) \leq \frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \left[\sum_{n=1}^\infty \left(\frac{T}{P} K^{\frac{-2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}}{\Gamma(n+1)\alpha}|\delta|\right], 0 \leq x < X.$$

$where\ K > 0\ is\ a\ constant, and\ T\ is\ defined\ as\ in\ Theorem(2.1).$

Proof: The integral equation corresponding to IVP (1.1)-(1.2) is

(3.2) $$u(x) = \frac{\delta}{\Gamma(\alpha)} x^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t,u(t)\big)dt$$

Therefore,

$$|u(x)| \leq \left|\frac{\delta}{\Gamma(\alpha)} x^{\alpha-1}\right| + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|f\big(t,u(t)\big)\right| dt$$
$$\leq \frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} L(t,|u(t)|)dt \quad, 0 \leq x < X.$$

Then application of Theorem (2.2) gives

$$u(x) \leq \frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \int_0^x \left[\sum_{n=1}^{\infty}\left(\frac{T}{P}K^{\frac{-2}{3}}\right)^n \frac{(x-t)^{n\alpha-1}}{\Gamma(n\alpha)}\frac{t^{\alpha-1}}{\Gamma(\alpha)}|\delta|\right]dt$$

$$= \frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \left[\sum_{n=1}^{\infty}\left(\frac{T}{3}K^{\frac{-2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}B(\alpha,n\alpha)}{\Gamma(n\alpha)\Gamma(\alpha)}|\delta|\right]$$

(3.3) $$= \frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \left[\sum_{n=1}^{\infty}\left(\frac{T}{3}K^{\frac{-2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}}{\Gamma(n+1)\alpha}|\delta|\right], \qquad 0 \leq x < X.$$

which is the required result.

***Theorem*(3.2)**: *If* $|f(x,u) - f(x,v)| \leq L(x, |u-v|)$, *where L is defined as in the Theorem*(2.2) , *and* $L(t,0) \equiv 0$, *then* $IVP(1.1) - (1.2)$ *has a unique solution.*

Proof: Suppose that IVP (1.1)-(1.2) has two solutions $u_1(x)$ *and* $u_2(x)$.

Then we have

(3.4) $$u_1(x)\ \frac{\delta}{\Gamma(\alpha)}x^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t, u_1(t))dt$$

(3.5) $$u_2(x)\ \frac{\delta}{\Gamma(\alpha)}x^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t, u_2(t))dt$$

Now

$$u_1(x) - u_2(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[f(t,u_1(t)) - f(t,u_2(t))\right]dt$$

$$|u_1(x) - u_2(x)| \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left|f(t,u_1(t)) - f(t,u_2(t))\right|dt$$

(3.6) $$\leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} L(t, |u_1(t) - u_2(t)|)dt$$

Consider $|u_1(t) - u_2(t)|$ as independent function and applying Theorem (2.2) to (3.6) , we get

$$|u_1(x) - u_2(x)| \leq 0$$

which implies $u_1(x) = u_2(x)$.

This completes the proof.

Conclusions:-

In this paper, by using generalized inequalities of Gronwall type we establish the boundedness and uniqueness aspects of solution of fractional differential equations. While proving the uniqueness of solution of fractional differential equations integral inequalities proves a powerful technique.

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