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RESEARCH ARTICLE

PROJECTIVELY RELATED EINSTEIN RANDERS SPACE.

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Abstract

The projectively related Einstein Finsler Spaces is used to establish a new approach to study of projective geometry in Finsler spaces. In this paper, first we found the formula for Weyl projective curvature of Randers metric and using this formula, we study the projectively related Einstein Randers metric. Further, we show that two projectively related complete Einstein Randers spaces with constant negative scalar curvature are homothetic.

Introduction:-

One important study in projective geometry is to determine a relationship among geometric structures with common geodesics (as point sets). Two regular metrics on an n-dimensional manifold are said to be pointwise projectively related if they have the same geodesics as point sets. Two regular matric spaces are said to be projectively related if there is a diffeomorphism between them such that the pull back metric is pointwise projectively related to another one. In the Riemannian geometry, two spaces of constant sectional curvature are always projectively related according to the Bltrami theorem. In Finsler geometry, there are many Finsler metrics on a strongly convex subset $\Omega \in \mathbb{R}^n$ which are pointwise projectively related to the standard Euclidean metric. These are simply called to be projectively flat. Projectively flat Finsler metrics are characterized by vanishing Douglas tensor and Weyl projective curvature tensor. These are some quantities in the projective Finsler geometry which are projective invariant. One of the most important of them is the Weyl curvature tensor. The Finsler metrics with Weyl curvature $W_k^i = 0$ are called Weyl metrics. It is well known metrics that a Finsler metric is Weyl metric if and only if it is of scalar flag curvature. There are many new advances (cf. [9],[8],[14],[16],[19],[20],[21]).

The Ricci curvature plays an important role in the projective geometry of Riemannian Finsler manifolds. The Ricci tensor was introduced by G. Ricci in 1904. Then, Ricci's work was used to formulate Einstein theory of gravitation [3]. Hence, Finsler metric is said to be Einstein if the Ricci scalar Ric is a function of x alone. Equivalently,

$$Ric_{ij} = Ric(x) g_{ij.}$$

In Riemannian space if g and $\bar{g}$ are pointwise projectively related Riemannian metric on manifold M of dimensional $n \geq 3$, then g is of constant curvature if and only if $\bar{g}$ is of constant curvature. Moreover, the authors , Z.Shen, N.Sadeghzadeh, A.Razavi and B.Razaei were studied the projectively related Einstein Finsler metrics ([13], [22]).

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In 2006, the author Y.Shen and L.Zhao were proved that the Randers metrics is projectively flat if and only if α is projectively flat and β is closed with constant flag curvature.

In[22], Z.Shen found out that two pointwise projectively equivalent Einstein Finsler metric F and $\bar{F}$ on a n-dimensional compact manifold M have same sign Einstein constants. In addition, if two pointwise projectively related Einstein metrics are complete with negative Einstein constants then one of them is a multiple of the other. And the authors Z.Shen, Yibing and Yaoyoug were got the results, the two Einstein Randers metrics are projectively related then α and $\bar{\alpha}$ are Einstein metrics with non positive scalar curvature and F and $\bar{F}$ have non positive Ricci curvature[13]. So, it is natural to study projectively related Einstein Randers metrics, which is just the purpose of this paper. The main results of the present paper is stated as follows: follows:

Theorem 1: Let $F = \alpha + \beta$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be two Einstein Randers metrics. If F is projectively related to $\bar{F}$ of non zero Ricci scalar. Then,

(i) F is Einstein if and only if it is a constant co-efficients of $\bar{F}$, when $\bar{F}$ is not projectively flat.

(ii) F is Einstein if and only if it is a constant Ricci scalar, when $\bar{F}$ is projectively flat.

Theorem 2:- Let $(M, F = \alpha + \beta)$and $\left(M, \bar{F} = \bar{\alpha} + \bar{\beta}\right)$ be two complete Einstein Randers metrics of constant non-positive Ricci curvature. If F and $\bar{F}$ are projectively related then they are homothetic.

Preliminaries:-

Let M be an n-dimensional C^∞ manifold. Denote by T_xM be the tangent space at $x = M$ and $TM = U_{x\in M} T_xM$ be the tangent bundle of M. Each element of TM has the form (x, y) where $x \in M$and $y \in T_xM$. Let $TM_0 = TM\backslash\{0\}$. The natural projection $\pi: TM \to M$ is given by, $\pi(x, y) = x$. The pull-back tangent bundle $\pi * TM$ is a vector bundle over TM_0 whose fiber $\pi_v^* TM$ at $v\epsilon TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \left\{(x, y, v) \,\middle|\, y \in T_xM_0, v \in T_xM\right\}.$$

A Finsler metric on a manifold M is a function $F: TM \to [0, \infty)$ which has the following properties[12]:

(i)FisC^∞ on $TM_{0,}$

(ii)$F(x, \lambda y) = \lambda F(x, y), \lambda > 0;$

(iii)For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij} = \left[\frac{1}{2}F^2\right]_{y^i y^j},$$

is positive definite.

The symmetric tensor C defined by,

$$C(U, V, W) = C_{ijk}(y) U^i V^j W^k,$$

Where $U = U^i \frac{\partial}{\partial x^i}, V = V^i \frac{\partial}{\partial x^i}, W = W^i \frac{\partial}{\partial x^i}$ and $C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}(y).$

C is called the $Cartan$ tensor. It is well known that $C = 0$ if and only if F is Riemannian. Every Finsler metric F including a spray:

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i} ,$$

where

$$G^i(x, y) = \frac{1}{4} g^{il}(x, y) \left\{2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\right\} y^j y^k = \frac{1}{4} g^{il}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\right\}.$$

Where the matrix (g^{ij}) means the inverse of matrix $\left(g_{ij}\right)$ and the coefficients G^i_j, G^i_{jk} and hv-curvature G^i_{jkl} of the Berwald connection can be derived from the spray G^i as follows [10];

$$G^i_j = \frac{\partial G^i}{\partial y^j},\ \ G^i_{jk} = \frac{\partial G^i_j}{\partial y^k},\ \ G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}.$$

The Riemannian curvature $R_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i} \Big| p :\ T_pM \to T_pM$ is defined by

$$R_k^i(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.2}$$

The Riemannian Curvature has the following properties. For any non zero vector $y \in T_pM$,

$$R_y(y) = 0, g_y\big(R_y(u), v\big) = g_y\Big(u, R_y(v)\Big), \quad u, v \in T_pM,$$

and

$$R_{kl}^i = \frac{1}{3}\left\{\frac{\partial R_k^i}{\partial y_l} - \frac{\partial R_l^i}{\partial y_k}\right\}. \tag{2.3}$$

Given the orbitary plane $P = span\{y, u\} \subset T_pM$, the flag curvature of the flag P with the flag pole y is defined by [2];

$$K(P, y) = \frac{g_y\big(u, R_y(u)\big)}{g_y(y,y)g_y(u,u) - g_y(y,u)g_y(y,u)}.$$

F is said to be scalar curvature $K = \lambda(y)$, if for any non-zero tangent vector $y \in T_pM$, the flag curvature $K(P, y) = \lambda(y)$ is independent of the plane P, in this case equivalent to the following system in a local coordinate system (x^i, y^i) in TM,

$$R_k^i = \lambda F^2\{\delta_k^i = F^{-1}F_{y^k}y^i\}. \tag{2.4}$$

If λ is a constant, then F is said to be of constant curvature. The Ricci scalar function of F is given by,

$$\rho = \frac{1}{F^2}R_i^i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means that $\rho(x, y)$ depends on the direction of the flag pole y but not its length. The Ricci tensor of a Finsler metric is defined by

$$Ric_{ij} = \left\{\frac{1}{2}R_k^k\right\}_{y^i y^j}$$

If (M, F) is a Finsler space with constant curvature λ ,then (2.4) becomes

$$Ric = (n-1)\lambda, \quad Ric_{ij} = (n-1)\lambda F^2.$$

Ricci flat manifolds are Riemannian manifolds whose Ricci tensor vanishes. In physics they are important because they represent vacuum solution to Einstein's equations.

Definition 2.1. A Finsler metric is said to be an Einstein metric if the Ricci scalar function is a function of x alone, equivalently

$$Ric = \rho(x)g_{ij,} \quad \text{or} \quad Ric_{00} = \rho(x)F^2.$$

Definition 2.2. A Finsler space F^n is projective to another Finsler space $\bar{F}^n$, if and only if there exists a one positive homogeneous scalar field $P(x, y)$ on TM satifying

$$\bar{G}^i(x, y) = G^i(x, y) + P(x, y)y^i.$$

The scalar field $P = P(x, y)$ is called the projective factor of the projective change.
Let G^i and $\bar{G}^i = G^i + Py^i$ be sprays on n-dimensional manifold M. The Riemannian curvature are related by [7]

$$\bar{R}_k^i = R_k^i + E\delta_k^i + T_k y^i,$$

Where

$$E = P^2 - P_{|k}y^k,$$
$$T_k = 3\big(P_{|k} - PP_{y^k}\big) + E_{y^k}.$$

In 1961, A.Rapcsak [6] proved the following:-

Lemma 2.1. Let $F^n = (M, F)$ and $\bar{F}^n = (M, \bar{F})$ be two Finsler space on a common underlying manifold M of dimension n . A Finsler metric F is pointwise projective to $\bar{F}$ if and only if

$$\frac{\partial \bar{F}_{|k}}{\partial y^l} y^k - \bar{F}_{|l} = 0$$

Then,

$$\bar{G}^i = G^i + Py^i,$$

where

$$P = \frac{\bar{F}_{|k} y^k}{2\bar{F}}.$$

The study of Weyl curvature of spray as an important projective invariant. The Weyl's projective invariant is constructed from the Riemannian curvature. Define

$$W_k^i(y) = R_k^i - R\delta_k^i - \frac{1}{n+1}\frac{\partial}{\partial y^m}(R_k^m - R\delta_k^m)y^i, \tag{2.7}$$

Where $R = \frac{1}{n+1} Ric = \frac{1}{n+1} R_m^m$. $W_y : T_x M \to T_x M$ is a linear transformation satisfying $W_y(y) = 0$. We call $W = W_y, y \in TM_0$ the Weyl curvature W is a projective invariant under projective transformation. By this direction the author Z.Shen has proved that [17].

Theorem 2.1. A Finsler metric is of scalar curvature if and only if $W = 0$.

Two Finsler metrics F and $\bar{F}$ are said to be homothetic if there is a constant λ such that $F = \lambda\bar{F}$. As a result of Busemann-Mayer theorem ; the authors M.Sepasi and B.Bidabada were proved the following [14],

Corollary2.1. Let (M, F) and $(M, \bar{F})$ be two complete Einstein Finsler spaces with $Ric_{ij} = -c^2 g_{ij}$ and $\overline{Ric_{ij}} = -\bar{c}^2 \bar{g}_{ij}$ respectively, F and $\bar{F}$ are projectively related then they are homothetic.

Weyl projective curvature of randers metric:-

In general it is much more difficult to compute the Weyl projective curvature tensor. In what follows the lemma shows that the formula for Weyl projective curvature of a Randers metric $F = \alpha + \beta$, Where β is killing form with constant length.

Lemma 3.2. For a Randers metric $F = \alpha + \beta$, the Weyl projective curvature is given by

$$\begin{aligned} W_k^i = \bar{W}_k^i &+ \alpha b_{i|k|p} y^p - 2\alpha b_m \hat{R}_{ikp}^m y^p + \alpha^{-1} b_m \hat{R}_i^m y^k \\ &-\alpha^2 b_{i|m} b_{m|k} + 3\left(b_{i|p} y^p\right)\left(b_{k|q} y^q\right) + \left(b_{i|m} b_{m|p} y^p\right) y^k \\ &\left(\alpha b_m \hat{R}_{mp} y^p + \frac{1}{n+1}\left\{\alpha^2\left(b_{m|p}\right) + 2\left(b_{m|p} y^p\right)^2\right\}\right) \\ &-\frac{1}{n+1}\left(\alpha b_m \hat{R}_{mp} y^p y_k - \frac{a_{ij} y^j}{2\alpha} b_m \bar{R}_{mp} y^p - \alpha b_m \bar{R}_{mp|k} y^p - \alpha b_m \bar{R}_{mp}\right) \\ &-\frac{1}{n^2-1}\left(\alpha^2\left(b_{m|p}\right)^2 - 2\left(b_{m|p} y^p\right)^2 y_k + 4 b_{m|p} y^p\right), \end{aligned}$$

Where $\widehat{W}_k^i$ denote the Weyl curvature tensor of α.

Proof. The geodesic coefficients G^i and F are given by

$$G^i = \hat{G}^i + Py^i + Q^i, \tag{3.1}$$

Where $\hat{G}^i$ denote the spray coefficients of α and

$$P = \frac{y^i y^j + \alpha y^i}{2\alpha F}, \quad Q^i = \alpha a^{ir} S_{rl} y^l.$$

We assume that β is killing form with constant length and since Weyl curvature tensor W is a projective invariant. Consider the another spray as

$$\bar{G} = y^i \frac{\partial}{\partial x^i} - 2\bar{G}^i \frac{\partial}{\partial y^i},$$

Where

$$\bar{G}^i = \hat{G}^i + Q^i, \quad \text{with} \quad Q^i = \alpha a^{ir} b_{r|l} y^l.$$

Here, we see that G and $\bar{G}$ are projectively equivalent. Thus, they have the same Weyl curvature tensor,

$$W_k^i = \bar{W}_k^i.$$

Now,we compute the Weyl curvature tensor W_k^i using $\bar{G}^i$. First we have

$$\bar{R}_k^i = \hat{R}_k^i + T_k^i, \quad \bar{R} = \hat{R} + T, \tag{3.2}$$

Where $\bar{R} = \frac{1}{n-1}\bar{R}_m^m$, $\hat{R} = \frac{1}{n-1}\hat{R}_m^m$ and

$$T_k^i = 2Q^i_{|k} - \left(Q^i_{|j}\right)_{y^k} y^j + 2Q^j (Q^i)_{y^j y^k} - (Q^i)_{y^j} Q^j_{y^k} \ ,$$

Where $Q^i_{|j}$ denote the covariant derivatives of Q^i with respect to α .
Using equation (3.2) in (3.1) we have,

$$W_k^i = \widehat{W}_k^i + \ominus_k^i \ , \tag{3.3}$$

Where $\widehat{W}_k^i$ denote the Weyl curvature tensor of α and

$$\ominus_k^i = R_k^i - R\delta_k^i - \frac{1}{n+1}\frac{\partial}{\partial y^m}(R_k^m - R\delta_k^m)y^i \tag{3.4}$$

Using maple codes we get values of T_k^i as

$$\begin{aligned}
Q^i_{|k} &= \alpha\alpha b_{i_p|k} y^p, \\
Q^i &= \alpha b_{i|p} y^p, \\
\left(Q^i_{|j}\right)_{y^k} y^j &= \alpha^{-1}\left(b_{i|p|j} y^j y^p\right) + \alpha b_{i|k|p} y^p, \\
(Q^i)_{y^j} &= \alpha^{-1}\left(b_{i|p} y^p\right) y^j + \alpha b_{i|j}, \\
(Q^i)_{y^j y^k} &= \alpha^{-1}\left(b_{i|p} y^p\right)\delta_{jk} - \alpha^{-3}\left(b_{i|p} y^p\right) y^j y^k + \alpha^{-1}\left(b_{i|j} y^k + b_{i|k} y^j\right).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
R_k^i = {} & \hat{R}_k^i + \alpha b_{i|k|p} y^p - 2\alpha b_m \hat{R}_{ikp}^m y^p + \alpha^{-1} b_m \hat{R}_i^m y^k \\
& -\alpha^2 b_{i|m} b_{m|k} + 3\left(b_{i|p} y^p\right)\left(b_{k|q} y^q\right) + \left(b_{i|m} b_{m|p} y^p\right) y^k \quad ,
\end{aligned} \tag{3.5}$$

and

$$R = \hat{R} + \alpha b_m \hat{R}_{mp} y^p + \frac{1}{n-1}\left\{\alpha^2\left(b_{m|p}\right)^2 + 2\left(b_{m|p} y^p\right)^2\right\}. \tag{3.6}$$

Substitute (3.5) and (3.6) in (3.4), we get

$$\begin{aligned}
\ominus_k^i = {} & \alpha b_{i|k|p} y^p - 2\alpha b_m \hat{R}_{ikp}^m y^p + \alpha^{-1} b_m \hat{R}_i^m y^k \\
& -\alpha^2 b_{i|m} b_{m|k} + 3\left(b_{i|p} y^p\right)\left(b_{k|q} y^q\right) + \left(b_{i|m} b_{m|p} y^p\right) y^k \\
& \left(\alpha b_m \hat{R}_{mp} y^p + \frac{1}{n+1}\left\{\alpha^2\left(b_{m|p}\right)^2 + 2\left(b_{m|p} y^p\right)^2\right\}\right) \\
& -\frac{1}{n+1}\left(\alpha b_m \hat{R}_{mp} y^p y_k - \frac{a_{ij} y^j}{2\alpha} b_m \hat{R}_{mp} y^p - \alpha b_m \hat{R}_{mp|k} y^p - \alpha b_m \hat{R}_{mp}\right) \\
& -\frac{1}{n^2-1}\left(\alpha^2\left(b_{m|p}\right)^2 - 2\left(b_{m|p} y^p\right)^2 y_k + 4 b_{m|p} y^p\right).
\end{aligned}$$

This proves the lemma.

Remark: We know that, every Finsler metric is of scalar curvature if and only if its spray is isotropic and also Weyl curvature $W = 0$.
A Spray is isotropic , the equation (3.4) can be written as

$$R_k^i = R\delta_k^i + \xi_k y^i, \tag{3.7}$$

With $\xi_k y^k = -R$;where R is given in (3.6). Then the equqtion (3.3) can be written as

$$W_k^i = R_k^i - R\delta_k^i - \zeta_k y^i,$$

Where R_k^i, R given in (3.5), (3.6) respectively and

$$\zeta_k = \frac{1}{n+1}\left(\alpha b_m \hat{R}_{mp} y^p y_k - \frac{a_{ij} y^j}{2\alpha} b_m \hat{R}_{mp} y^p - \alpha b_m \hat{R}_{mp|k} y^p - \alpha b_m \hat{R}_{mp}\right).$$

Differentiate ζ_k with respect y^k, we get

$$\zeta_k y^k = -\alpha b_m \hat{R}_{mp} y^p + \frac{1}{n-1}\left\{\alpha^2 \left(b_{m|p}\right)^2 + 2\left(b_{m|p} y^p\right)^2 = -R.\right.$$

Thus, if $W_k^i = 0$ then the equation (3.7) holds, by taking $\xi_k = \zeta_k$.

Assume that (3.7) holds. By homogeneity of ξ_k we obtain, $\xi_k = \zeta_{k.}$ And obviously $\bar{\xi}_k = \bar{\zeta}_k$, which implies that $\hat{W}_k^i = 0$ Thus, $W_k^i = 0$. We obtain the following:

Lemma 3.3. A Randers metric F is of scalar curvature if and only if $W = 0$.

Proof of Main Theorem:-

Proof of Theorem 1: To prove theorem 1, in the following.

Proposition 4.1. Let $F = \alpha + \beta$ be a Randers metric of dimension $n > 2$. If F is Einstein Randers metric if and only if

$$y_i V_k^i = -\frac{3(n-1)}{n+1}\left\{\hat{R}_k y^k + \frac{a_{ij} F^2}{\alpha} b_m \hat{R}_{mp} y^p + \alpha b_m F \alpha^2 \right.$$
$$\left. \hat{R}_{mpk} + \frac{a_{ij} F^2}{n-1}\left(b_{m|p}\right)^2\right\},$$

where $V_k^i - R_k^i - \frac{3}{n+1} Ric_{0k} y^i$.

$Proof.$ Assume that F is Einstein .
By definition of Weyl curvature tensor (2.7), we have

$$y_i W_k^i - y_i R_k^i = -R y_k - \frac{n-2}{n+1} R_k F^2 + \frac{3F^2}{n+1} Ric_{0k}.$$

Then ,

$$y_i V_k^i = -\left(\hat{R} + \frac{n-2}{n+1}\hat{R}_k F^2\right) - b_m \hat{R}_{mp} y^p \left(\alpha y_k + \frac{a_{ij} F^2}{\alpha}\right)$$
$$-\alpha b_m \hat{R}_{mpk} y^p F^2 - \frac{\left(b_{m|p}\right)^2}{n-1}\left(\alpha^2 y_k + 2(y^p)^2 y_k + a_{ij} F^2 y_i\right). \tag{4.1}$$

Since F is Einstein, $2\hat{R} y_k = \hat{R}_k F^2$.

Therefore,

$$y_i V_k^i = -\frac{3(n-1)}{n+1}\left\{\hat{R}_k y^k + \frac{a_{ij} F^2}{\alpha} b_m \hat{R}_{mp} y^p + \alpha b_m F \alpha^2 \right.$$
$$\left. \hat{R}_{mpk} + \frac{a_{ij} F^2}{n-1}\left(b_{m|p}\right)^2\right\}. \tag{4.2}$$

Conversly, if (4.2) holds.
Again by definition of Weyl curvature tensor, we have

$$-y_i V_k^i = y_i \left(R\delta_k^i + \frac{n-2}{n+1} R_k y_i\right).$$

Using (4.1) and (4.2) and since β is closed , it is concluded that

$$2\hat{R} y_k = \hat{R}_k F^2,$$

Which implies that $\hat{R}_k = 0 \Rightarrow \hat{R} = 0$. It shows that α is Einstein.
Therefore, F is Einstein.

Proposition 4.2. Let F be Einstein Randers metric projectively related to another Einstein Randers metric $\bar{F}$ with projective factor P. If F is Einstein then $\left(\frac{E}{F^2}\right).k = 0$, where $E = P^2 - P_{|k} y^k$.

$Proof.$ Let W and $\bar{W}$ be the Weyl curvature tensor of F and $\bar{F}$ respectively. Foe Einstein Randers metric F, we have

$$y_i V_k^i = -\frac{3(n-1)}{n+1}\Big\{\hat{R}_k y^k + \frac{a_{ij} F^2}{\alpha} b_m \hat{R}_{mp} y^p + \alpha b_m F \alpha^2$$
$$\hat{R}_{mpk} + \frac{a_{ij} F^2}{n-1}\left(b_{m|p}\right)^2\Big\}.$$

Therefore

$$\begin{aligned} y_i W_k^i = y_i \widehat{W}_k^i - \frac{3(n-1)}{n+1}\left[\frac{a_{ij} F^2}{\alpha} b_m \hat{R}_{mp} y^p + \alpha b_m F^2 \hat{R}_{mpk} + \frac{a_{ij} F^2}{n-1}\left(b_{m|p}\right)^2\right] \\ +\alpha b_{i|k|p} y^p y_i - 2\alpha b_m \hat{R}_{ikp}^m y^p y_i + \alpha^{-1} b_m \hat{R}_i^m y^k y_i - \alpha^2 b_{i|m} b_{m|k} \\ +3\left(b_{i|p} y^p\right)\left(b_{k|q} y^q\right) y^i + \left(b_{i|m} b_{m|p} y^p\right) y^k y_i + \frac{3}{n+1}[\alpha b_{0|k|p} y^p - 2\alpha b_{m|\hat{k}p} y^p \\ +\alpha^{-1} b_m \hat{R}_0^m y^k - \alpha^2 b_{0|m} b_{m|k} + 3\left(b_{0|p} y^p\right)\left(b_{k|q} y^q\right) + \left(b_{0|m} b_{m|p} y^p\right) y^k], \end{aligned}$$

Where $\widehat{W}_k^i$ denote the Weyl curvature of α.
But $\bar{W}_k^i$ is invariant under projective transformation , then

$$y_i W_k^i = y_i \bar{W}_k^i = y_i \left(\bar{R}_k^i + \frac{3\overline{Ric}_{0k}}{n+1} y_i\right) - \frac{3(n-1)}{n+1} \bar{R} y_k.$$

Therefore

$$y_i\left(R_k^i - \bar{R}_k^i\right) + \frac{3y_i}{n+1}\left(Ric_{0k} y^i - \overline{Ric}_{0k} y^i\right) - \frac{3(n-1)}{n+1}\left(R y_k - \bar{R} y_k\right) = 0.$$

From equation (2.3) and (2.4) we have,

$$Ric = \widehat{Ric} + (n-1)E.$$

Which implies that

$$R = \hat{R} + E \tag{4.3}$$

Also, from equations (2.4) and (2.5), it becomes

$$3R_{kl}^i = 3\hat{R}_{kl}^i + (E_{.l} + T_l)\delta_k^i - (E_{.k} - T_k)\delta_k^i + (T_{k.l} - T_{l.k}) y^i, \tag{4.4}$$

where

$$\begin{aligned} E = \left(1/4\alpha^3(\alpha+\beta)\right)[\alpha(y^i y^j)^2 + \alpha^3 (y^i)^2 + 2\alpha^2 (y^i)^2 y^j \\ -\{2(y^i)^2 y^j (\alpha^3 + \alpha^2\beta) - 2(\alpha(y^i y^j)^2 + \alpha^2 (y^i)^2 y^j + \beta(y^i y^j)^2 \\ + \alpha\beta(y^i)^2 + \alpha^2 (y^i)^2 y^j + \alpha^3 (y^i)^2) y^k], \end{aligned}$$

and

$$T_l = 4\alpha(\alpha+\beta)\Xi_0 - \Xi_1 + 4\alpha(\alpha+\beta)\Xi_2,$$

where

$$\Xi_0 = 6\alpha^3 (y^i)^2 y^j + 6\alpha^2\beta(y^2) y^j - 6\alpha(y^i)^2 (y^j)^2 - 6\alpha(y^i)^2 (y^j)^2$$

$$-2\beta(y^i)^2(y^j)^2 - 12\alpha^2(y^i)^2y^j - 6\alpha\beta(y^i)^2 - 6\alpha^3(y^i)^2,$$

$$\begin{aligned}\Xi_1 &= 2\alpha^3(y^i)^2y^j + 2\alpha^3(y^iy^j)^2 + 2\alpha^4a_{ij}(y^i)^2y^j + 2\alpha^4(y^i)^2 + 2\alpha^4y^iy^j \\ &+2\alpha^5a_{ij}(y^i)^2 + 2\alpha^2\beta(y^i)^2y^j + \alpha^2\beta y^i(y^j)^2 + 2\alpha^3\beta a_{ij}(y^i)^2y^j + 2\alpha^3\beta(y^i)^2 \\ &+2\alpha^3\beta y^iy^j + 2\alpha^4\beta a_{ij}(y^i)^2 + \alpha a_{ij}(y^i)^2(y^i)^2 + \alpha^2a_{ij}(y^i)^2y^j + \alpha^2a_{ij}(y^i)^2y^i \\ &+\alpha^3a_{ij}(y^i)^2 + \beta a_{ij}(y^i)^2(y^j)^2 + \alpha\beta a_{ij}(y^i)^2y^j + \alpha\beta a_{ij}(y^i)^2y^j + \alpha^2\beta a_{ij}(y^i)^2 \\ &+\alpha\beta y^i(y^j)^2 + 2\alpha^3\beta y^iy^j + \alpha^4\beta y^i,\end{aligned}$$

$$\begin{aligned}\Xi_2 &= \alpha F^2y^i(y^j)^2 + \alpha^3F^2y^i + 2\alpha^2y^iy^jF^2 - 2(\alpha^3 + \alpha^2\beta)(y^i)^2y^j + 2\alpha(y^iy^j)^2 \\ &-2\alpha^2(y^i)^2y^j - 2\beta(y^iy^j)^2 - 2\alpha\beta(y^i)^2y^j - 2\alpha^2(y^2) - 2\alpha^3(y^i)^2.\end{aligned}$$

Equation (4.4) implies that

$$3Ric_{0l} = 3\,\widehat{Ric_{0l}} + (n-2)E_{.l} + (n+1)\mathrm{T}_l. \tag{4.5}$$

By substituting (4.5) in (2.5). Using Maple codes and values of E and T_l, we get the following

$$\frac{n-2}{n+1}(E_{.k}F^2 - 2Ey_k) = 0.$$

Thus, since $n > 2$, $\left(\frac{E}{F^2}\right)_{.k} = 0.$

Lemma 4.4. Let $F = \alpha + \beta$ be Einstein Randers metric $(n > 2)$ projectively related to another Einstein Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ of non zero Ricci scalar then,

(a) F is Einstein if and only if $\left(\frac{F}{F}\right)_{.k} = 0,$ when $\bar{F}$ is not projectively flat.

(b) F is Einstein if and only if it is of constant Ricci scalar, when $\bar{F}$ is projectively flat.

Proof. (b) Assume that F is Einstein

If $\bar{F}$ is projectivelyflat , then F is also projectively flat, since they are projectively related to each other. According $([4],[11])$, we obtain F which is of constant flag curvature. And since it is Einstein then F is of constant flag curvature . Therefore, it conclude that it has constant Ricci scalar.

(a) Let $\bar{F}$ is not projectively flat. Since F is Einstein then it is Ricci flat $(\frac{R}{F^2})_{.k} = 0$ and by proposition 4.2, we see that $\left(\frac{E}{F^2}\right)_{.k} = 0$. Then, there exists a function $\eta(x)$ which is defined by $\eta(x) = \frac{R-E}{F^2}$. F is projectively related to $\bar{F}$ by (4.3), we have $\frac{R}{F^2} = \frac{\hat{R}+E}{F^2}$. But $\bar{F}$ is Einstein Randers metric of non-zero function $\sigma(x)$ such that $\bar{R} = \sigma(c)\bar{F}^2$. Therefore, it can be concluded that $\left(\frac{F}{F}\right)_{.k} = 0.$

Conversly, F is projectively relatedto $\bar{F}$, which is defined by

$$G^i = \bar{G}^i + Py^i, \tag{4.6}$$

Where

$$P = \frac{y^iy^j + \alpha y^i}{2\alpha(\alpha+\beta)}.$$

From proposition 4.2 there is a function of x only, where $F = f(x)\bar{F}$ then,

$$P = \frac{f_{|k}y^k}{2f} = \frac{\bar{\alpha}(\bar{\alpha}+\bar{\beta})(y^iy^j + \alpha y^i)}{2\alpha(\alpha+\beta)(\bar{y}^i\bar{y}^j + \bar{\alpha}\bar{y}^i)}.$$

Using the formula of G^i in (2.1), which yields

$$\begin{aligned}G^i &= \frac{\bar{g}^{il}}{4f}\left[\left(f[\bar{F}^2]_{x^ky^l}y^k - [\bar{F}^2]_{x^l}\right) + \left(f_{x^k}y^k[\bar{F}^2]_{\bar{y}^l}f_{x^l}\bar{F}^2\right)\right], \\ &= \bar{G}^i + \left(\frac{y^iy^j + \alpha y^i}{2\alpha(\alpha+\beta)}\right)y^i + \frac{f_{x^l}F^2}{4f}\left(\frac{\alpha}{F}a^{il} - \frac{\alpha}{F^2}(b^il^l + b^ll^i) + \left(\frac{b^2\alpha+\beta}{F^3}\right)y^iy^j\right).\end{aligned}$$

By equation (4.6) implies that, f must be constant. Thus, it has the constant coefficients .

Proof of Theorem 2: Since F and $\bar{F}$ are projectively related. From [*16*], we clear that F and $\bar{F}$ be two Einstein Randers metrics, $\alpha \neq \lambda\bar{\alpha}$. If F is projectively related to $\bar{F}$ then they have negative Ricci curvature. i.e., they have negative Einstein constants. Also, by corollary 2.1 we conclude that they are homothetic.

Conclusion:-

In projective geometry there are two important projectively invariant tensors, namely Douglas tensor and Weyl tensor. These tensors provide the more information about the projective properties of Finsler metric. One of the most important them is the Weyl tensor. It is well known fact that in Finsler geometry: Everey Finsler metric is of scalar curvature if and only if Weyl curvature tensor is equal to zero.

As we know, in general Einstein metrics is said to Ricci tensor is proportionality of metric tensor, i.e., $Ric \propto g_{ij}$ which are a natural extension of those in Riemannian geometry and they have good properties in Riemann geometry for some class of Finsler metrics. Some research have been progressed to projectively related Finsler spaces and also the curvatureproperties.

Especially in this study, we find the formula for Weyl projective curvature of Randers metric and using this formula we studied the projectively related Einstein Randers metric.

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