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RESEARCH ARTICLE

A Modified Fractional Kinetic Equation

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Abstract

In this present paper, we derive the solution of generalized fractional kinetic equation using the Wright function. The result obtained here is fairly general in nature and competent results

1. Introduction:

The Sun which is a big star is assumed to be in thermal equilibrium and hydrostatic equilibrium. To describe a model, we consider it is a spherical symmetric, self-gravitating non-rotating. The features of its are mass, luminosity, diameter, effective surface temperature, central temperature and density. The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star like sun (Clayton [2] and Perdang [17]). Energy in such stars being produced by the process of chemical reactions.

Consider an arbitrary reaction characterized by a time dependent quantity $N = N(t)$. The rate of change dN/dt is equal to the destruction rate d and the production rate p of N, that is $\frac{dN}{dt} = -d + p$. Normally, destruction and production depend on the quantity N itself: $d = d(N)\ or\ p = p(N)$. This dependence is very difficult because the destruction or production at time t depends not only on N(t) but also on the past history N(τ), τ < t, of the variable N.

From (Haubold and Mathai [7])

$$dN\ /dt\ = -d\ (Nt\ )\ +\ p(Nt), \tag{1}$$

where Nt denotes the function defined by $Nt(t\,*) = N(t\ - t\,*), t * > 0.$

Haubold and Mathai [7] studied a special case of this equation, when spatial fluctuations or inhomogenities in quantity $N(t)$ are neglected, is given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t) \tag{2}$$

with the initial condition that $Ni\ (t\ =\ 0)\ =\ N_0$ is the number density of species i at time $t\ =\ 0$; constant $c_i > 0$, called as standard kinetic equation.
Thus, the solution of the equation (2) is given by

$$Ni(t)\ = N_0\ e^{-c_i t} \tag{3}$$

An alternative form of the same equation can be obtained on integration:

$$N(t) - N_0 = c\,{}_0D_t^{-1} N(t) \tag{4}$$

where ${}_0D_t^{-1}$ is the standard integral operator. Haubold and Mathai [7] have given the fractional generalization of the standard kinetic equation (2) as

$$N(t) - N_0 = c^v\,{}_0D_t^{-v} N(t) \tag{5}$$

where ${}_0D_t^{-v}$ is the well known Riemann-Liouville fractional integral operator (Oldham and Spanier [16]) given by

$${}_0D_t^{-v} N(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u)\,du, \quad R(v) > 0, \tag{6}$$

The solution of the fractional kinetic equation (6) is given by (see Haubold and Mathai [7])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^{vk}}{\Gamma(vk+1)} (ct)^{vk} \tag{7}$$

Also, Saxena, Mathai and Haubold [20] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which extension of the work of Haubold and Mathai [7].

In the present work, we establish the generalized fractional kinetic equation. The fractional kinetic equation and its solution, discussed in terms of the Wright function, are written in compact form.

2. The Wright function:

This function introduced by the authors is defined as follows:

$$W(t;\, \alpha, \beta) = \sum_{k=o}^{\infty} \frac{t^k}{n!\,\Gamma(\alpha k + \beta)} \tag{8}$$

Here, $\alpha, \beta \epsilon C$, $R(\alpha) > 0$ and $R(\beta) > 0$.
And Laplace transform of (8), we have

$$L\{W(t;\, \alpha, \beta; s)\} = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{1}{s^{k+1}}$$

Or

$$L\{W(t;\, \alpha, \beta; s)\} = s^{-1} E_{\alpha,\beta}(s^{-1})$$

This is Mittag-Leffler function.

3. Generalized Fractional Kinetic Equations:

In this section we investigate the solution of generalized fractional kinetic equations. The results are obtained in a compact form in terms of Wright function and are suitable for computation. The results are presented in the form of two theorems as follows:

Theorem 1:

If $v > 0,\ c > 0,\ d > 0,\ \mu > 0,\ Re(s) > |d|^{v/\alpha}$, $c \neq d$ then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1} W(-d^v t^v;\, v, \mu) = -c^v\,{}_0D_t^{-v} N(t) \tag{9}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (ct)^{vk}\ W(-d^v t^v;\, v, vk+\mu) \tag{10}$$

Proof. Applying the Laplace transform both the sides of equation (9), we get

$$L\{N(t)\} - L\{N_0 t^{\mu-1} W(-d^v t^v;\, v, \mu)\} = L\{-c^v\,{}_0D_t^{-v} N(t)\}$$

$$N(s) - N_0 \sum_{k=o}^{\infty} \frac{(-d^v)^k}{k!\ s^{vk+\mu}} = -c^v \frac{N(s)}{s^v} \tag{11}$$

Solving for$N(s)$, it gives

$$N(s) = \frac{N_0}{(1 + c^v s^{-v})} \sum_{k=o}^{\infty} \frac{(-d^v)^k}{k!\, s^{vk+\mu}} \tag{12}$$

Now, taking inverse Laplace transform both the sides of (12), we get

$$L^{-1}\{N(s)\} = L^{-1}\left\{\frac{N_0}{(1 + c^v s^{-v})} \sum_{k=o}^{\infty} \frac{(-d^v)^k}{k!\, s^{vk+\mu}}\right\} \tag{13}$$

$$N(t) = L^{-1}\left\{N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (c^v s^{-v})^k (1)_k}{k!} \sum_{k=o}^{\infty} \frac{(-d^v)^k}{k!\, s^{vk+\mu}}\right\} \tag{14}$$

$$N(t) = N_0 \sum_{k=o}^{\infty} (-1)^k (c^v)^k \sum_{k=o}^{\infty} (-d^v)^k \frac{t^{vk+vk+\mu-1}}{k!\,\Gamma(vk + vk + \mu)} \tag{15}$$

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (ct)^{vk} \sum_{k=o}^{\infty} (-d^v)^k \frac{t^{vk}}{k!\,\Gamma(vk + vk + \mu)} \tag{16}$$

Or

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (ct)^{vk}\ W\,(-d^v t^v;\ v\,, vk + \mu) \tag{17}$$

This is complete proof of the statement (9).

When $\mu = -vk + 1$, theorem reduces to

Corollary: If $v > 0,\ c > 0, d > 0,\ \mu > 0,\ Re(s) > |d|^{v/\alpha}$, $c \neq d$ then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{-vk} W\,(-d^v t^v;\ v\,, -vk + 1) = -c^v\, {}_0D_t^{-v}\, N(t) \tag{18}$$

there holds the formula

$$N(t) = N_0 t^{-vk} \sum_{k=o}^{\infty} (-1)^k (ct)^{vk}\ W\,(-d^v t^v;\ v\,, vk + \mu) \tag{19}$$

When $d = 1$ in (9), we get

Corollary: If $v > 0,\ c > 0, d > 0,\ \mu > 0,\ Re(s) > |d|^{v/\alpha}$, $c \neq d$ then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{\mu-1} W\,(-t^v;\ v\,, \mu) = -c^v\, {}_0D_t^{-v}\, N(t) \tag{20}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (ct)^{vk}\ W\,(-t^v;\ v\,, vk + \mu) \tag{21}$$

When $\mu = v + 1$, then the solution of

$$N(t) - N_0 t^v W\,(-t^v;\ v\,, v + 1) = -c^v\, {}_0D_t^{-v}\, N(t) \tag{22}$$

There holds the result

$$N(t) = N_0 t^{v} \sum_{k=o}^{\infty} (-1)^k (ct)^{vk}\ W\,(-t^{v};\ v\,,(k+1)v+1) \tag{23}$$

Theorem 2:

If $v > 0,\ c > 0, d > 0,\ \mu > 0,\ Re(s) > |d|^{v/\alpha}$, $c = d$ then for the solution of the generalized fractional kinetic equation

$$N(t) – N_0 t^{\mu-1} W\,(-d^v t^v;\ v\,,\mu) = -d^v\,{}_0D_t^{-v}\,N(t) \tag{24}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (-1)^{vk}\ W\,(d^{2v} t^v;\ v\,,vk+\mu) \tag{25}$$

Proof. We solve similarly to previous theorem (1) and taking the Laplace transforms both the sides of equation (24), we get

$$L\{N(t)\} – L\{\,N_0 t^{\mu-1} W\,(-d^v t^v;\ v\,,\mu)\} = L\{\,-d^v\,{}_0D_t^{-v}\,N(t)\}$$

$$N(s) - N_0 \sum_{k=o}^{\infty} \frac{(-d^v)^k}{\text{k!}\ s^{vk+\mu}} = -d^v\,\frac{N(s)}{s^v}$$

Solving for$N(s)$, it gives

$$N(s) = \frac{N_0}{(1+d^v s^{-v})} \sum_{k=o}^{\infty} \frac{(-d^v)^k}{\text{k!}\, s^{vk+\mu}}$$

Now, taking inverse Laplace transform both the sides of (26), we get

$$L^{-1}\{N(s)\} = L^{-1}\left\{\frac{N_0}{(1+d^v s^{-v})} \sum_{k=o}^{\infty} \frac{(-d^v)^k}{k!\, s^{vk+\mu}}\right\}$$

$$N(t) = L^{-1}\left\{N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (d^v s^{-v})^k (1)_k}{k!} \sum_{k=o}^{\infty} \frac{(-d^v)^k}{k!\, s^{vk+\mu}}\right\}$$

$$N(t) = N_0 \sum_{k=o}^{\infty} (-1)^k (d^v)^k \sum_{k=o}^{\infty} (-d^v)^k\,\frac{t^{vk+vk+\mu-1}}{k!\,\Gamma(vk+vk+\mu)}$$

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (dt)^{vk} \sum_{k=o}^{\infty} (-d^v)^k\,\frac{t^{vk}}{k!\,\Gamma(vk+vk+\mu)}$$

Or

$$N(t) = N_0 t^{\mu-1} \sum_{k=o}^{\infty} (-1)^k (-1)^{vk}\ W\,(d^{2v} t^v;\ v\,,vk+\mu)$$

This is complete proof of the theorem (2).

4. Conclusion:

In this paper we have introduced an extended fractional generalization of the standard kinetic equation and established solution for the same. Fractional kinetic equation can be used to compute the particle reaction rate and describes the statistical mechanics associated with the particle distribution function. The generalized fractional kinetic equation discussed in this paper involving Wright function.

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