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RESEARCH ARTICLE

MATHEMATICAL ANALYSIS OF MULTICOMPARTMENT EPIDEMIC MODEL.

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Abstract

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In this paper, we study a nonlinear mathematical model in population with variable size. Size N(t) at time t, is divided into eight sub classes, with $N(t) = S(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + I_4(t) + Q(t) + R(t)$; where S(t), I(t), and Q(t) denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively. The stability of a disease-free status equilibrium and the existence of endemic equilibrium can be determined by the ratio called the basic reproductive number. This paper study the equilibrium, local stability and and the stochastic stability of the free disease equilibrium under certain conditions .

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Introduction:-

This paper considers the following epidemic model with temporary immunity:

$$
\begin{cases}
\dot{S}(t) = l + n - (\mu + d)S(t) + \dfrac{s\,I(t)}{N(t)} - a\dfrac{S(t)I(t)}{N(t)} + gS(t)I(t - t)e^{-\mu_6 t}, \\
\dot{I}(t) = a\dfrac{S(t)I(t)}{N(t)} - \dfrac{s\,I(t)}{N(t)} - (m_0 + d + b)I(t), \\
\dot{I}_1(t) = b_1 I(t) - (m_1 + d + g_1)I_1(t), \\
\dot{I}_2(t) = b_2 I(t) - (m_2 + d + g_2)I_2(t), \\
\dot{I}_3(t) = b_3 I(t) - (m_3 + d + g_3)I_3(t), \\
\dot{I}_4(t) = b_4 I(t) - (m_4 + d + g_4)I_4(t), \\
\dot{Q}(t) = g_1 I_1(t) + g_2 I_2(t) + g_3 I_3(t) + g_4 I_4(t) - (m_5 + d + d)Q(t), \\
\dot{R}(t) = dQ(t) - (\mu_6 + d)R(t) - gS(t)I(t - t)e^{-\mu_6 t}
\end{cases}
$$

Consider a population of size N(t) at time t, this population is divided into for sub-classes, with $N(t) = S(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + I_4(t) + Q(t) + R(t)$.

Where S(t), I(t), $I_1(t)$, $I_2(t)$, $I_3(t)$, $I_4(t)$, Q(t) and R(t) denote the sizes of the population susceptible to disease, infectious members, quarantine members with the possibility of infection through temporary immunity, and who were removed from the possibility of infection respectively. The positive constants μ_1, μ_2, μ_3, μ4, μ5 and μ6 represent the death rates of susceptible, infectious, quarantine and removed. Biologically, It is natural to assume that $\mu \leq \min\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$.

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The positive constant d is natural mortality rate. The positive constant β = β1+β2+β3+β4 is the average numbers of contacts. The positive constants λ represent the incidence rate of the population. The positive constants $\gamma_1,\gamma_2,\gamma_3$, and γ_4 is the numbers of transfer or conversion of infected people quarantined. d the number of transfer or conversion of Q to R. ν the positive constant is the parameter of immigration. α the positive constant is the parameter of emigration The term $gS(t)I(t-t)e^{-\mu_3 t}$ indicates that an individual has quarantined in a pool recovery before becoming susceptible again, where τ is the length of immunity period.

The initial condition of (1) is given as:

$$S(h) = F_1(h),\ I(h) = F_2(h),\quad Q(h) = F_3(h),\ R(h) = F_4(h),\quad -t \le h \le 0,$$

where, $= (\ _1,\ _2,\ _3,\ _4)^T$ C such that:

S ( ) = $_1$( ) = $_1$(0) 0, I ( ) = $_2$( ) = $_2$(0) 0, Q( ) = $_3$( ) = $_3$(0) 0, R( ) = $_4$( ) = $_4$(0) 0.

Let C denote the Banach space C ([- , 0], $\mathbb{R}^4$) of continuous functions mapping the interval [- , 0] into $\mathbb{R}^4$. With a biological meaning, we further assume that:

$_i$( ) = $_i$(0) 0 for i = 1, 2, 3,4.

With the initial condition in (2) which becomes:-

$$S(h) = F_1(h),\ I(h) = F_2(h),\quad Q(h) = F_3(h),\ R(h) = F_4(h),\quad -t \le h \le 0.$$

Where,

$_1$(0) 0, $_2$(0) 0, $_3$(0) 0, $_4$(0) 0, - 0.

The region $W = \left\{ \begin{array}{l} (S(t), I(t), I_1(t), I_2(t), I_3(t), I_4(t), Q(t), R(t)) \in \mathbb{R}^8_+, \\ S(t)+I(t)+I_1(t)+I_2(t)+I_3(t)+I_4(t)+Q(t)+R(t) \le N < \dfrac{l+n}{m+d} \end{array} \right\}$ is positively invariant.

Hence system (1) can be rewritten as

$$\begin{cases} \dot{S}(t) = l + n - (\mu + d)S + \dfrac{s}{N(t)} - a\dfrac{S(t)I(t)}{N(t)} + gS(t)I(t-t)e^{-\mu_6 t}, \\ \dot{I}(t) = a\dfrac{S(t)I(t)}{N(t)} - \dfrac{s\,I(t)}{N(t)} - (m_0 + d + b)I(t), \\ \dot{I}_1(t) = b_1 I(t) - (m_1 + d + g_1)I_1(t), \\ \dot{I}_2(t) = b_2 I(t) - (m_2 + d + g_2)I_2(t), \\ \dot{I}_3(t) = b_3 I(t) - (m_3 + d + g_3)I_3(t), \\ \dot{I}_4(t) = b_4 I(t) - (m_4 + d + g_4)I_4(t), \\ \dot{Q}(t) = g_1 I_1(t) + g_2 I_2(t) + g_3 I_3(t) + g_4 I_4(t) - (m_5 + d + d)Q(t), \\ \dot{R}(t) = dQ(t) - (\mu_6 + d)R(t) - gS(t)I(t-t)e^{-\mu_6 t} \end{cases}$$

Equilibrium Points:-

An equilibrium point of system (4)

satisfies:
$$\begin{cases} l + n - (\mu + d)S + \frac{s\,I}{N} - a\frac{SI}{N} + g\,SI(t - t)e^{-\mu_6 t} = 0,\\ a\frac{SI}{N} - \frac{s\,I}{N} - (m_0 + d + b)I = 0,\\ b_1 I - (m_1 + d + g_1)I_1 = 0,\\ b_2 I - (m_2 + d + g_2)I_2 = 0,\\ b_3 I - (m_3 + d + g_3)I_3 = 0,\\ b_4 I - (m_4 + d + g_4)I_4 = 0,\\ g_1 I_1 + g_2 I_2 + g_3 I_3 + g_4 I_4 - (m_5 + d + d)Q = 0,\\ dQ - (\mu_6 + d)R - g\,SI(t - t)e^{-\mu_6 t} = 0 \end{cases}$$

We calculate the points of equilibrium in the absence and presence of infection.
In the absence of infection, the system (5) has a disease-free equilibrium E_0:

$$E_0 = \left(\hat{S}, \hat{I}, \hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_4, \hat{Q}, \hat{R}\right)^T = \left(\frac{l + n - a}{m + d}, 0, 0, 0, 0, 0, 0, 0\right)^T.$$

The eigenvalues can be determined by solving the characteristic equation of the linearization of (4) near E_0. So, the eigenvalues are:

$$\begin{aligned} &A_1 = -(\mu + d), A_2 = a - \frac{s}{N^*} - (\mu_0 + d + b),\\ &A_3 = -(\mu_1 + d + g_1), A_4 = -(\mu_2 + d + g_2)\\ &A_5 = -(\mu_3 + d + g_3), A_6 = -(\mu_4 + d + g_4)\\ &A_7 = -(\mu_6 + d + g_6), A_8 = -(\mu_6 + d)\end{aligned}$$

In order to A_2, will be negative, then we define the basic reproduction number of the infection R_0 as follows:

$$R_0 = \frac{a(l + n)}{s(m + d) + (l + n)(m_0 + d + b)}$$

In the presence of infection, substituting in the system, Ω also contains a unique positive, endemic equilibrium

$$E_t^* = \left(S_t^*, I_t^*, I_{1t}^*, I_{2t}^*, I_{3t}^*, I_{4t}^*, Q_t^*, R_t^*\right)^T$$
$$E_t^* = \left(S_t^*, I_{it}^*, Q_t^*, R_t^*\right)^T, \forall\ i = 1, 2, 3, 4.$$

Where:-

$$\begin{cases} S_t^* = \frac{1}{a}\left[s - (m_0 + d + b)N^*\right]\\ I_t^* = \frac{1}{g e^{-m_3 t}}\left[(m + d) - \frac{m + l}{S_t^*}\right]\\ I_{it}^* = \frac{b_i}{m_i + d + g_i}, \forall\ i = 1, 2, 3, 4,\\ Q_t^* = \frac{1}{\mu_5 + d + d}\left[\sum_{i=1}^{4}\frac{g_i b_i}{m_i + d + g_i}\right]\\ R_t^* = \frac{1}{\mu_6 + d}\left[\frac{d}{\mu_5 + d + d}\left(\sum_{i=1}^{4}\frac{g_i b_i}{m_i + d + g_i}\right) - (m + d)S_t^* + l + u\right] \end{cases}$$

Proposition

Let $(S, I, I_1, I_2, I_3, I_4, Q, R)$, the solution of the system (4) is defined in $(0, \infty]$ and

$$\lim_{t \to \infty} \sup N(t) \le \frac{\mu + \lambda}{\mu + d}.$$

Proof

We have $\dot{N} = \nu + \lambda - \mu S - \mu_0 I - \mu_1 I_1 - \mu_2 I_2 - \mu_3 I_3 - \mu_4 I_4 - \mu_5 Q - \mu_6 R - dN$,

$$\dot{N} \le \nu + \lambda - (\mu + d)$$

By integration,

$N(t) \le \frac{\nu + \lambda}{\mu + d}\left(1 - e^{-(\mu + d)t}\right)$, for every $t \in (0, T]$,

$$N(t) \le 2.\frac{\mu + \lambda}{\mu + d}$$

The solutions of sub-populations are bounded in the interval $(0, T]$.

Then we have $N(t) \le \frac{\nu + \lambda}{\mu + d}\left(1 - e^{-(\mu + d)t}\right)$, for every $t \in (0, \infty]$

Finally $\lim_{t \to \infty} \sup N(t) \le \frac{\mu + \lambda}{\mu + d}$.

The local stability of the free-disease equilibrium :-

Theorem 1:-

The disease-free equilibrium E_0 is locally asymptotically stable if and only if

$$\alpha < \frac{\sigma(\nu + \lambda)}{\mu + d} - (\mu_0 + d + \beta)$$

Proof:- *Let*

$$x = S - \frac{\nu + \lambda}{\mu + d}, y = I, y_i = I_i, i = 1, 2, 3, 4,$$

$$z = Q, u = R, w = N - \frac{\nu + \lambda}{\mu + d}.$$

With the changement, the system (4) becomes

$$\dot{x} = \left[-(\mu + d)\right]x + \left[\frac{\sigma}{w + N} + \left(\gamma e^{-\mu_6 \tau} - \frac{\alpha(\mu + d)}{w(\mu + d) + (\nu + \lambda)}\right)S\right]y,$$

$$\dot{y} = \left[\frac{\alpha S - \sigma}{w + N} - (\mu_0 + d + \beta)\right]y,$$

$$\dot{y}_i = \beta_i y - (\mu_i + d + \gamma_i)y_i, \forall i = 1, 2, 3, 4,$$

$$\dot{z} = \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3 + \gamma_4 y_4 - (\mu_5 + d + \delta)z,$$

$$\dot{u} = \left[-\gamma e^{-\mu_6 \tau} S\right]y + \delta z - (\mu_6 + d)u.$$

With the linearized of system (7) at the point (0,0,0,0,0,0,0,0) , we obtain the eigenvalues,

$$A_i = -(\mu_i + d + \gamma_i), \forall i = 1,2,3,4,$$
$$A_5 = -(\mu + d),$$
$$A_6 = \alpha - \frac{\sigma}{N} - (\mu_0 + d + \beta),$$
$$A_7 = -(\mu_5 + d + \delta),$$
$$A_8 = -(\mu_6 + d).$$

The eigenvalues have a negative real part, so;

E_0 is locally asymptotically stable if and only if $\alpha < \frac{\sigma(\nu+\lambda)}{\mu+d} - (\mu_0 + d + \beta)$.

Stochastic stability of the free-disease equilibrium:-

We limit ourselves here to perturbing only the contact rate so we replace k by a + a b(t), where b(t) is white noise (Brownian motion). The system (4) is transformed to the following Itô stochastic differential equations:

$$\begin{cases} dS = \left[d + n - (\mu + d)S + \frac{sI}{N} + g e^{-\mu_6 t} SI(t - t) - a\frac{SI}{N} \right] dt - aSI\, db, \\ dI = \left[a\frac{SI}{N} - \frac{sI}{N} - (m_0 + d + b) \right] dt + aSI\, db, \\ dI_1 = \left[b_1 I - (m_1 + d + g_1) I_1 \right] dt, \\ dI_2 = \left[b_2 I - (m_2 + d + g_2) I_2 \right] dt, \\ dI_3 = \left[b_3 I - (m_3 + d + g_3) I_3 \right] dt, \\ dI_4 = \left[b_4 I - (m_4 + d + g_4) I_4 \right] dt, \\ dQ = \left[g_1 I_1 + g_2 I_2 + g_3 I_3 + g_4 I_4 - (m_5 + d + d) Q \right] dt, \\ dR = \left[dQ - (\mu_6 + d)R - g e^{-\mu_6 t} SI(t - t) \right] dt \end{cases}$$

Theorem 2:- *If $R_0<1$, I(t) and R(t) are exponentially almost surely stable.*

Proof:-

Let w such that $\frac{\sigma(\mu+d)}{\lambda+\nu} + (\mu_0 + d + \beta) - \left(\frac{\alpha(\mu+d)}{\lambda+\nu} - w\,\gamma e^{-\mu_6 \tau} \right)\left(\frac{\lambda+\nu}{\mu+d} \right) > 0$

Withe Itô's formula, we obtain

$$d\log(I + wR) = \frac{1}{I + wR} \begin{bmatrix} \left(\frac{\alpha}{N} - w\,\gamma e^{-\mu_6 \tau} \right) SI \\ -\left(\frac{\sigma}{N} + (\mu_0 + d + \beta) \right) I + w\,\delta Q \\ -w\,(\mu_6 + d)R - \frac{a^2 (SI)^2}{2(I + wR)} \end{bmatrix} dt + \frac{aSI}{I + wR} db$$

$$d\log(I + wR) \le \frac{-1}{(I + wR)} \begin{bmatrix} \begin{bmatrix} \left(\frac{(\lambda+\nu) w\,\gamma e^{-\mu_6 \tau}}{\mu+d} \right) - \alpha + \frac{\sigma(\mu+d)}{\lambda+\nu} \\ +(\mu_0 + d + \beta) \end{bmatrix} I \\ +\left(w\,(\mu_6 + d) \right) R \end{bmatrix} dt + \frac{aSI}{I + wR} db$$

We suppose that

$$M = \min\left\{\left(\left(\frac{(\lambda+\nu)w\,\gamma e^{-\mu_6\tau}}{\mu+d}\right)-\alpha+\frac{\sigma(\mu+d)}{\lambda+\nu}+(\mu_0+d+\beta)\right), w\,(\mu_6+d)\right\}$$

Then

$$d\log(I+wR)\le -Mdt+\frac{aSI}{I+wR}db$$

With integration , we obtain

$$\log(I+wR)\le -Mdt+a\int_0^t \frac{S(v)I(v)}{(I(v)+wR(v))}db(v)\cdot$$

We have

$\left(\frac{S(v)I(v)}{(I(v)+wR(v))}\right)^2$ *is bounded. Then*

$$\lim_{t\to\infty}\int_0^t \frac{S(v)I(v)}{(I(v)+wR(v))}db(v)=0$$ *almost surely.*

The following form from Doob's martingale inequality combined with Itô isometry see [17].

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(I+wR)\le -M$$ *almost surely*

Then we have

$$\lim_{t\to\infty}\sup\frac{1}{t}\log I\le -M$$ *, so I is almost surely*

$$\lim_{t\to\infty}\sup\frac{1}{t}\log R\le -M$$ *, so R is almost surely.*

Theorem 3.

If $\left(\frac{\alpha(\mu+d)}{\lambda+\nu}-\gamma e^{-\mu_6 t}\right)^2-2(\mu+d)<0$, *S(t) converge exponentially almost surely to* $\frac{\lambda+\nu}{\mu+d}$.

Proof:

Applying Itô formula to the first equation in system (8), we obtain

$$d\log\left|S-\frac{\lambda+\nu}{\mu+d}\right| = \left[(\mu+d)+\frac{\sigma I/N}{S-\frac{\lambda+\nu}{\mu+d}}\;\frac{\left(\gamma e^{-\mu_6 t}+\frac{\sigma(\mu+d)}{\lambda+\nu}\right)SI}{S-\frac{\lambda+\nu}{\mu+d}}-\frac{1}{2}\frac{a^2(SI)^2}{\left(S-\frac{\lambda+\nu}{\mu+d}\right)^2}\right]dt-a\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}}db,$$

$$d\log\left|S-\frac{\lambda+\nu}{\mu+d}\right| \le \left[(\mu+d)-\left(\frac{\alpha(\mu+d)}{\lambda+\nu}-\gamma e^{-\mu_6 t}\right)\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}}-\frac{1}{2}\alpha^2\left(\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}}\right)^2\right]dt-\alpha\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}}db,$$

We suppose that

$$F(x)=(\mu+d)-\left(\frac{\alpha(\mu+d)}{\lambda+\nu}-\gamma e^{-\mu_6 t}\right)x-\frac{1}{2}\alpha^2x^2 \quad , \textit{ with } \quad x=\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}};$$

$$d\log\left|S-\frac{\lambda+\nu}{\mu+d}\right| = F(x)dt-\alpha\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}}db,$$

If the determinant of the equation is negative, then for all x.

$$F(x)\le\frac{\Delta}{\alpha^2}<0, \quad \textit{with} \quad \Delta=\left(\frac{\alpha(\mu+d)}{\lambda+\nu}-\gamma e^{-\mu_6 t}\right)^2-2(\mu+d)$$

We have

$$d\log\left|S-\frac{\lambda+\nu}{\mu+d}\right| \le \frac{\Delta}{\alpha^2}dt-\alpha\frac{SI}{S-\frac{\lambda+\nu}{\mu+d}}db,$$

With integration , we obtain

$$\log\left|S-\frac{\lambda+\nu}{\mu+d}\right| \le \frac{\Delta}{\alpha^2}t-\alpha\int_0^t\frac{S(v)I(v)}{S(v)-\frac{\lambda+\nu}{\mu+d}}db(v)$$

Since

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{S(v)I(v)}{S(v)-\frac{\lambda+\nu}{\mu+d}}db(v)=0 \textit{ almost surely}$$

$$\lim_{t\to\infty}\sup\frac{1}{t}\log\left|S-\frac{\lambda+\nu}{\mu+d}\right|\le\frac{\Delta}{\alpha^2}.$$

Conclusion:-

In this paper, the epidemic model has a disease free equilibrium $E_{0,}$ witch is locally asymptotically stable if and only if $\alpha<\frac{\sigma(\nu+\lambda)}{\mu+d}-(\mu_0+d+\beta)$ and the endemic equilibrium E_t^*.

We proof *I(t) and R(t) are exponentially almost surely stable if If $R_0<1$, Finally S(t) converge exponentially almost surely to* $\frac{\lambda+\nu}{\mu+d}$ *If* $\left(\frac{\alpha(\mu+d)}{\lambda+\nu}-\gamma e^{-\mu_6 t}\right)^2-2(\mu+d)<0$.

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