RESEARCH ARTICLE

MATHEMATICAL ANALYSIS OF MULTICOMPARTMENT EPIDEMIC MODEL.

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Abstract

In this paper, we study a nonlinear mathematical model in population with variable size. Size \( N(t) \) at time \( t \), is divided into eight sub classes, with \( N(t) = S(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + I_4(t) + Q(t) + R(t) \); where \( S(t) \), \( I(t) \), and \( Q(t) \) denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively. The stability of a disease-free status equilibrium and the existence of endemic equilibrium can be determined by the ratio called the basic reproductive number. This paper study the equilibrium, local stability and the stochastic stability of the disease-free equilibrium under certain conditions.

Introduction:

This paper considers the following epidemic model with temporary immunity:

\[
\begin{align*}
S(t) &= I + n - (\mu + d)S(t) - \frac{sI(t)}{N(t)} - aS(t)I(t) + gS(t)I(t - r)e^{-\beta}\, dt, \\
I(t) &= \frac{S(t)I(t)}{N(t)} - \frac{sI(t)}{N(t)} - (n_0 + d + b)I(t), \\
I_1(t) &= b_1I(t) - (n_0 + d + g_1)I_1(t), \\
I_2(t) &= b_2I(t) - (n_0 + d + g_2)I_2(t), \\
I_3(t) &= b_3I(t) - (n_0 + d + g_3)I_3(t), \\
I_4(t) &= b_4I(t) - (n_0 + d + g_4)I_4(t), \\
Q(t) &= g_1I_1(t) + g_2I_2(t) + g_3I_3(t) + g_4I_4(t) - (n_0 + d + d)Q(t), \\
R(t) &= Q(t) - aQ(t) - gS(t)I(t - r)e^{-\beta}\, dt.
\end{align*}
\]

Consider a population of size \( N(t) \) at time \( t \), this population is divided into sub-classes, with \( N(t) = S(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + I_4(t) + Q(t) + R(t) \).

Where \( S(t), I(t), I_1(t), I_2(t), I_3(t), I_4(t), Q(t) \) and \( R(t) \) denote the sizes of the population susceptible to disease, infectious members, quarantine members with the possibility of infection through temporary immunity, and who were removed from the possibility of infection respectively. The positive constants \( \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \) and \( \mu_6 \) represent the death rates of susceptible, infectious, quarantine and removed. Biologically, It is natural to assume that \( \mu \leq \min \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\} \).
The positive constant \( d \) is natural mortality rate. The positive constant \( \beta = \beta_1 + \beta_2 + \beta_3 + \beta_4 \) is the average numbers of contacts. The positive constants \( \lambda \) represent the incidence rate of the population. The positive constants \( \gamma_1, \gamma_2, \gamma_3 \), and \( \gamma_4 \) is the numbers of transfer or conversion of infected people quarantined. \( d \) the number of transfer or conversion of Q to R. \( \nu \) the positive constant is the parameter of immigration. \( \alpha \) the positive constant is the parameter of emigration.

The term \( gS ( t ) ( t - t ) e^{\nu ( t - t )} \) indicates that an individual has quarantined in a pool recovery before becoming susceptible again, where \( \tau \) is the length of immunity period.

The initial condition of (1) is given as:

\[
S ( h ) = F_1 ( h ), \quad I ( h ) = F_2 ( h ), \\
Q ( h ) = F_3 ( h ), \quad R ( h ) = F_4 ( h ),
\]

where, \( \mathbb{h} = ( 1, 2, 3, 4 )^T \) \( \mathbb{C} \) such that:

\[
S ( ) = _1 ( ) = _1 ( 0 ) \quad 0, \quad I ( ) = _2 ( ) = _2 ( 0 ) \quad 0, \quad Q ( ) = _3 ( ) = _3 ( 0 ) \quad 0, \quad R ( ) = _4 ( ) = _4 ( 0 ) \quad 0.
\]

Let \( C \) denote the Banach space \( C ([ - \tau, 0 ] \mathbb{R}^4) \) of continuous functions mapping the interval \([ - \tau, 0 ]\) into \( \mathbb{R}^4 \). With a biological meaning, we further assume that:

\[
_i ( ) = _i ( 0 ) \quad 0 \quad \text{for } i = 1, 2, 3, 4.
\]

With the initial condition in (2) which becomes:

\[
S ( h ) = F_1 ( h ), \quad I ( h ) = F_2 ( h ), \\
Q ( h ) = F_3 ( h ), \quad R ( h ) = F_4 ( h ),
\]

\[
- \tau \leq h \leq 0.
\]

Where, \( 1 ( 0 ) \quad 0, \quad 2 ( 0 ) \quad 0, \quad 3 ( 0 ) \quad 0, \quad 4 ( 0 ) \quad 0, \quad - \tau \quad 0. \)

The region \( W = \left\{ \left( S ( t ), I ( t ), I_1 ( t ), I_2 ( t ), I_3 ( t ), I_4 ( t ), Q ( t ), R ( t ) \right) \right\} \) is positively invariant.

Hence system (1) can be rewritten as

\[
\begin{aligned}
S ( t ) &= l + n - ( \mu + d ) S + \frac{s}{N(t)} S - a \frac{S(t)I(t)}{N(t)} + g S(t)I(t - t) e^{\mu t}, \\
I ( t ) &= a \frac{S(t)I(t)}{N(t)} - \frac{s I(t)}{N(t)} - ( m_2 + d + b ) I ( t ), \\
I_1 ( t ) &= b_1 I ( t ) - ( m_2 + d + g_1 ) I_1 ( t ), \\
I_2 ( t ) &= b_2 I ( t ) - ( m_3 + d + g_2 ) I_2 ( t ), \\
I_3 ( t ) &= b_3 I ( t ) - ( m_4 + d + g_3 ) I_3 ( t ), \\
I_4 ( t ) &= b_4 I ( t ) - ( m_4 + d + g_4 ) I_4 ( t ), \\
Q ( t ) &= g_1 I_1 ( t ) + g_2 I_2 ( t ) + g_3 I_3 ( t ) + g_4 I_4 ( t ) - ( m_2 + d + d ) Q ( t ), \\
R ( t ) &= d Q ( t ) - ( \mu_0 + d ) R ( t ) - g S(t)I(t - t) e^{\mu t}.
\end{aligned}
\]

Equilibrium Points:--

An equilibrium point of system (4)
We calculate the points of equilibrium in the absence and presence of infection. 

In the absence of infection, the system (5) has a disease-free equilibrium $E_0$:

$$E_0 = \left( \frac{m_1}{m + d}, \frac{m_2}{m + d}, \frac{m_3}{m + d}, \frac{m_4}{m + d}, 0, 0, 0, 0, 0, 0 \right)^T.$$ 

The eigenvalues can be determined by solving the characteristic equation of the linearization of (4) near $E_0$. So, the eigenvalues are:

$$A_1 = - \left( \mu + d \right), A_2 = a - \frac{s}{N} - \left( \mu_0 + d + b \right),$$

$$A_3 = - \left( \mu_1 + d + g_1 \right), A_4 = - \left( \mu_2 + d + g_2 \right),$$

$$A_5 = - \left( \mu_3 + d + g_3 \right), A_6 = - \left( \mu_4 + d + g_4 \right),$$

$$A_7 = - \left( \mu_6 + d + g_6 \right).$$

In order to $A_2$, will be negative, then we define the basic reproduction number of the infection $R_0$ as follows:

$$R_0 = \frac{a \left( l + n \right)}{s \left( m + d \right) + \left( l + n \right) \left( m_0 + d + b \right)}.$$ 

In the presence of infection, substituting in the system, $\Omega$ also contains a unique positive, endemic equilibrium

$$E_{i}^* = \left( S_i^*, I_1^*, I_{1i}^*, I_2^*, I_{2i}^*, I_3^*, I_{3i}^*, I_4^*, Q_i^*, R_i^* \right)^T, \quad \forall i = 1, 2, 3, 4.$$ 

Where:

$$S_i^* = \frac{1}{a} \frac{\xi}{\xi} - (m_0 + d + b) N_i^* \frac{\dot{u}}{\dot{u}}$$

$$I_i^* = \frac{1}{g e^{-m_i \tau}} \frac{\xi}{\xi} (m + d) - \frac{m + l}{S_i^*} \frac{\dot{u}}{\dot{u}}$$

$$I_{ii}^* = \frac{b_i}{m_i + d + g_i}, \quad \forall i = 1, 2, 3, 4,$$

$$Q_i^* = \frac{1}{\mu_5 + d + g_i} \sum_{j=1}^{4} \frac{g_j b_i}{m_j + d + g_i} \frac{\dot{u}}{\dot{u}}$$

$$R_i^* = \frac{1}{\mu_6 + d + g_i} \sum_{j=1}^{4} \frac{g_j b_i}{m_j + d + g_i} \frac{\dot{u}}{\dot{u}} (m + d) S_i^* + \frac{l + u}{\dot{u}}.$$ 

715
Proposition
Let \((S, I, I_1, I_2, I_3, I_4, Q, R)\), the solution of the system (4) is defined in \((0, \infty)\) and
\[
\limsup_{t \to \infty} N(t) \leq \frac{\mu + \lambda}{\mu + d}.
\]

Proof
We have
\[
\dot{N} = v + \lambda - \mu S - \mu_0 I - \mu_1 T_1 - \mu_2 T_2 - \mu_3 T_3 - \mu_4 T_4 - \mu_5 R - dN.
\]

By integration,
\[
N(t) \leq \frac{v + \lambda}{\mu + d} \left(1 - e^{-\frac{\mu + d}{\mu + d}t}\right), \quad t \in (0, T],
\]

\[
N(t) \leq 2 \frac{\mu + \lambda}{\mu + d}.
\]

The solutions of sub-populations are bounded in the interval \((0, T]\).

Then we have
\[
N(t) \leq \frac{v + \lambda}{\mu + d} \left(1 - e^{-\frac{\mu + d}{\mu + d}t}\right), \quad t \in (0, \infty)
\]

Finally
\[
\limsup_{t \to \infty} N(t) \leq \frac{\mu + \lambda}{\mu + d}.
\]

The local stability of the free-disease equilibrium:

Theorem 1:
The disease-free equilibrium \(E_0\) is locally asymptotically stable if and only if
\[
\alpha < \frac{\sigma(v + \lambda)}{\mu + d} - \left(\mu_0 + d + \beta\right).
\]

Proof: Let
\[
x = S - \frac{v + \lambda}{\mu + d}, \quad y = I, \quad y_i = I_i, \quad i = 1, 2, 3, 4,
\]
\[
z = R, \quad u = N - \frac{v + \lambda}{\mu + d}.
\]

With the chage of the system (4) becomes
\[
\dot{x} = \left[-(\mu + d)\right]x + \left[\frac{\sigma}{w + N} + \left(\gamma e^{-\mu_0 r} - \frac{\alpha(\mu + d)}{w(\mu + d) + (v + \lambda)}\right)S\right]y,
\]
\[
\dot{y} = \left[\frac{\sigma S - \sigma}{w + N} - (\mu_0 + d + \beta)\right]y,
\]
\[
\dot{y}_i = \beta_y y - (\mu_i + d + \gamma_i) y_i, \quad \forall i = 1, 2, 3, 4,
\]
\[
\dot{z} = \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3 + \gamma_4 y_4 - (\mu_5 + d + \delta) z,
\]
\[
\dot{u} = \left[-\gamma e^{-\mu_0 r} S\right]y + \delta z - (\mu_6 + d) u.
\]
With the linearized of system (7) at the point (0,0,0,0,0,0,0,0), we obtain the eigenvalues,

\[ A_i = -\left( \mu_i + d + \gamma_i \right), \forall i = 1, 2, 3, 4, \]
\[ A_5 = -\left( \mu + d \right), \]
\[ A_6 = \alpha - \frac{\sigma}{N} - \left( \mu_0 + d + \beta \right), \]
\[ A_7 = -\left( \mu_5 + d + \delta \right), \]
\[ A_8 = -\left( \mu_6 + d \right). \]

The eigenvalues have a negative real part, so;
\[ E_0 \text{ is locally asymptotically stable if and only if } \alpha < \frac{\sigma (\nu + \lambda)}{\mu + d} - (\mu_0 + d + \beta). \]

**Stochastic stability of the free-disease equilibrium:**

We limit ourselves here to perturbing only the contact rate so we replace \( k \) by \( a + a b(t) \), where \( b(t) \) is white noise (Brownian motion). The system (4) is transformed to the following Itô stochastic differential equations:

\[
\begin{align*}
    dS &= \frac{\sigma}{N} n - (\mu + d) S + \frac{\sigma}{N} I - g e^{\nu x} S I(t - t) - \frac{\sigma}{N} S I dt - a S I db, \\
    dI &= \frac{\sigma}{N} S I - \frac{\sigma}{N} I - (m_0 + d + b) I dt + a S I db, \\
    dI_1 &= \Phi_1-I_1 - (m_0 + d + g_1) I_1 dt, \\
    dI_2 &= \Phi_1-I_2 - (m_0 + d + g_2) I_2 dt, \\
    dI_3 &= \Phi_1-I_3 - (m_0 + d + g_3) I_3 dt, \\
    dI_4 &= \Phi_1-I_4 - (m_0 + d + g_4) I_4 dt, \\
    dQ &= \Phi_1+1 + g_2 I_2 + g_3 I_3 + g_4 I_4 - \frac{\sigma}{N} S I(t - t) dt, \\
    dR &= \Phi_1 - (\mu_6 + d) R - g e^{\nu x} S I(t - t) dt.
\end{align*}
\]

**Theorem 2:** If \( R_0 < 1 \), \( I(t) \) and \( R(t) \) are exponentially almost surely stable.

**Proof:**

Let \( w \) such that
\[
\frac{\sigma (\mu + d)}{\lambda + \nu} + (\mu_0 + d + \beta) - \left( \frac{\alpha (\mu + d)}{\lambda + \nu} - w \gamma e^{\nu x} \right) \left( \frac{\lambda + \nu}{\mu + d} \right) > 0
\]

With Itô’s formula, we obtain

\[
\begin{align*}
    d \log (I + w R) &= \frac{1}{I + w R} \left[ \left( \frac{\alpha}{N} - w \gamma e^{\nu x} \right) S I \
    - \left( \frac{\sigma}{N} + (\mu_0 + d + \beta) \right) I + w \delta Q \
    - w (\mu_5 + d) R - \frac{\sigma^2 (S I)^2}{2 (I + w R)} \right] dt + \frac{a S I}{I + w R} db \\
    d \log (I + w R) &\leq \frac{-1}{I + w R} \left[ \left( \frac{(\lambda + \nu) w \gamma e^{\nu x}}{\mu + d} \right) - \frac{\alpha (\mu + d)}{\lambda + \nu} \right] I + w (\mu_5 + d) R dt + \frac{a S I}{I + w R} db
\end{align*}
\]
We suppose that
\[ M = \min \left\{ \left( \frac{\lambda + \nu}{\mu + d} \right) - \alpha + \frac{\sigma (\mu + d)}{\lambda + \nu} + (\mu_0 + d + \beta) \right\} \cdot \omega (\mu_0 + d) \right\} \]

Then
\[ d \log (I + wR) \leq -Mdt + \frac{aSI}{I + wR} db \]

With integration, we obtain
\[ \log (I + wR) \leq -Mdt + a \int_0^t S(v)I(v) \frac{1}{I(v) + wR(v)} db(v) \]

We have
\[ \left( \frac{S(v)I(v)}{(I(v) + wR(v))} \right)^2 \]

is bounded. Then
\[ \lim_{t \to \infty} \int_0^t S(v)I(v) \frac{1}{(I(v) + wR(v))} db(v) = 0 \quad \text{almost surely.} \]

The following form from Doob's martingale inequality combined with Itô isometry see [17].

\[ \lim_{t \to \infty} \sup \frac{1}{t} \log (I + wR) \leq -M \quad \text{almost surely} \]

Then we have
\[ \lim_{t \to \infty} \sup \frac{1}{t} \log I \leq -M, \quad \text{so I is almost surely} \]
\[ \lim_{t \to \infty} \sup \frac{1}{t} \log R \leq -M, \quad \text{so R is almost surely.} \]

**Theorem 3.**

If \( \frac{(m + d)}{l + n} - \frac{g e^{-\nu \lambda}}{\beta} - 2(m + d) < 0 \), \( S(t) \) converge exponentially almost surely to \( \frac{l + n}{m + d} \).

**Proof:**

Applying Itô formula to the first equation in system (8), we obtain

\[ d \log \left[ S - \frac{l + n}{m + d} \right] = \left( \mu + d \right) + \frac{sI}{N} - \frac{l + n}{m + d} - \frac{S}{l + n} \frac{s (m + d) \theta - 2 \omega \left( S \right)^2}{(l + n) \frac{S}{m + d}} - \frac{1}{(S - \frac{l + n}{m + d})^2} \left( \frac{S}{l + n} \right) \frac{a SI}{S - \frac{l + n}{m + d}} db, \]
\[ d \log S - \frac{l + n}{m + d} \leq \left( \mu + d \right) - \frac{\beta a(m + d)}{l + n} - \frac{\dot{I} e^{\nu t}}{\dot{I}} \cdot \frac{SI}{S - \frac{l + n}{m + d}} - \frac{1}{2} a^2 \frac{SI}{S - \frac{l + n}{m + d}} dt - a \cdot \frac{SI}{S - \frac{l + n}{m + d}} db, \]

We suppose that
\[ F(x) = \left( \mu + d \right) - \frac{\beta a(m + d)}{l + n} - \frac{\dot{I} e^{\nu t}}{\dot{I}} \cdot \frac{SI}{S - \frac{l + n}{m + d}} - \frac{1}{2} a^2 x^2, \quad x = \frac{SI}{S - \frac{l + n}{m + d}}, \]

If the determinant of the equation is negative, then for all \( x \).
\[ F(x) \leq \frac{D}{a^2} < 0, \quad \text{with} \quad D = \frac{\beta a(m + d)}{l + n} - \frac{\dot{I} e^{\nu t}}{\dot{I}} \cdot \frac{SI}{S - \frac{l + n}{m + d}} - 2(m + d) \]

We have
\[ d \log S - \frac{l + n}{m + d} \leq \frac{D}{a^2} dt - a \cdot \frac{SI}{S - \frac{l + n}{m + d}} db, \]

With integration, we obtain
\[ \log S - \frac{l + n}{m + d} \leq \frac{D}{a^2} t - a \cdot \int_0^t S(v)I(v)db(v) - \frac{l + n}{m + d} \]

Since
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t S(v)I(v)db(v) = 0 \text{ almost surely} \]
\[ \limsup_{t \to \infty} \frac{1}{t} \log S - \frac{l + n}{m + d} \leq \frac{D}{a^2}. \]

Conclusion:
In this paper, the epidemic model has a disease free equilibrium \( E_0 \), which is locally asymptotically stable if and only if
\[ \alpha < \frac{\sigma(v + \lambda)}{\mu + d} \quad \text{and the endemic equilibrium} \quad E^*_t. \]

We proof \( I(t) \) and \( R(t) \) are exponentially almost surely stable if \( R_0 < 1 \). Finally \( S(t) \) converges exponentially almost surely to \( \frac{l + n}{m + d} \) if
\[ \frac{\beta a(m + d)}{l + n} - \frac{\dot{I} e^{\nu t}}{\dot{I}} \cdot \frac{SI}{S - \frac{l + n}{m + d}} - 2(m + d) < 0. \]
References:
10. James M. Hyman, Jia Li. The reproductive number for an HIV model with differential infectivity and staged progression.