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RESEARCH ARTICLE

CONFORMAL CHANGE OF DOUGLAS SPECIAL FINSLER SPACE WITH SECOND APPROXIMATE MATSUMATO METRIC.

Thippeswamy K R and Narasimhamurthy. S.K.
Department Of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, India.

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Abstract

In this paper, we find the necessary and sufficient conditions for a Finsler space with the metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ to be a Douglas space and also to be a Berwald space, where α is a Riemannian metric and β is a differential one-form. Further, we study the conformal change of Douglas space with the above mentioned second approximate matsumato metric.

Introduction:-

The conformal theory of Finsler spaces has been introduced by M. S. Knebelman in 1929 and later on M. Hashiguchi developed such theory which was based on Matsumoto's approach to Finsler geometry. The conformal theory of two-dimensional Finsler space has been studied by M. Matsumoto and based on the above work, B. N. Prasad and D. K. Diwedi discussed the theory of conformal change of three-dimensional Finsler space.

Nabil L. Youssef, S. H. Abed and A. Soleiman investigated intrinsically conformal changes in Finsler geometry. Also they studied conformal change of Barthel connection and its curvature tensor, the conformal changes of Cartan and Berwald connections as well as their curvature tensors. Shun-ichi Hojo, M. Matsumoto and K. Okubo discussed the theory of conformally Berwald Finsler spaces and its applications to $(\alpha,\beta) -$ metrics.

The notion of Douglas space has been introduced by M. Matsumoto and S. Bacso as a generalization of Berwald space from the view point of geodesic equations. It is remarkable that a Finsler space is a Douglas space or is of Douglas type if and only if the Douglas tensor vanishes identically.

M. Matsumoto studied on Finsler spaces with $(\alpha,\beta) -$ metric of Douglas type. Hong-Suh Park and Eun-Seo Choi explained Finsler spaces with an approximate Matsumoto metric of Douglas type. The authors S. Bacso and I. papp studied on a generalized Douglas space. Also the team of authors Benling Li, Yibing Shen and Zhongmin Shen studied on a class of Douglas metrics.

The first part is devoted to find the condition for the Finsler space with Second approximate matsumato metric to be Berwald space (Theorem 3.1) The second part of the present paper is devoted to study the condition for the Finsler space with Second approximate matsumato metric to be a Douglas type (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the metric of Douglas type (Theorem 4.1). We study conformal change of

Corresponding Author:- Thippeswamy K R.
Address:- Department Of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, India.

Douglas space with Second approximate matsumato metric $\mathrm{L} = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ and also we obtain the conditions for Finsler space with an Second approximate matsumato metric to be conformally Berwald.

Preliminaries:-

Definition 1.1: A Finsler metric on M is a function $L : T\,M \rightarrow [0,\infty)$ with the following properties:

1. L is C^{∞} on TM_0,
2. L is positively $1-$homogeneous on thefibers of tangent bundle TM, and
3. The hessian of F^2 with element $g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\partial_j\dot{L}^2$, is regular on TM_0 ,

$$\text{i.e, } \det(g_{ij} \neq 0)$$

The pair (M^n, L) is then called a Finsler space. L is called fundamental function and g_{ij} is called fundamental tensor.

Let $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ be a Cartan tensor. Consider the Finsler space $F^n = (M^n.L)$ equipped with an (α,β) $-$metric $L(\alpha,\beta)$ Let γ^i_{jk} denote the Christoffel symbols in the Riemannian space (M^n,α). Denote by; $b_{i:j}$, the covariant derivative of the vector field b^i with respect to Riemannian connection γ^i_{jk}, i.e., $b_{i:j} = \frac{\partial b_i}{\partial x^j} - b^k\,\gamma^i_{jk}$.
Consider the following notations **[8].**

$$r_{ij} = \frac{1}{2}\,(b_{i:j} + b_{j:i}),\, r^i_j = a^{ih} r_{hj},\, r_j = b_i r^i_j$$
$$s_{ij} = \frac{1}{2}\,(b_{i:j} - b_{j:i})\,,\, s^i_j = a^{ir} s_{rj},\, s_j = b_r s^r_j$$
$$b^i = a^{ih} b_h,\, b^2 b^i_i.$$

To find the condition for the Finsler space with special (α,β) $-$ metric to be Berwald space (Theorem 3.1) and finally, we apply the conformal change of Finsler space with the special (α,β) $-$ metric of Douglas type (Theorem 4.1). We study conformal change of Douglas space with Second approximate matsumato metric $\mathrm{L} = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ and also we obtain the conditions for Finsler space with an Second approximate matsumato metric to be conformally Berwald.

The Condition To Be A Berwald Space:-

In the present section, we find the condition that a finsler space F^n with a Second approximate matsumato metric

$$\mathrm{L} = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} \tag{2.1}$$

A Finsler space is called Berwald space if the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is linear. In **[3],** the function G^i of a Finsler space with an (α,β) $-$metric are given by $2G^i = \gamma^i_{00} + 2B^i$ then we have $G^i_j = \gamma^i_{oj} + B^i_j$ and $G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$ where $B^i_j = \dot{\partial}_j B^i$. Thus a Finsler space with an (α,β)-metric is a Berwald space iff $G^i_{jk} = G^i_{jk}(x)$ equivalently $B^i_{jk} = B^i_{jk}(x)$. Moreover on account of **[6]** B^i_j is determined by

$$L_\alpha B^t_{ji} y^j y_t + \alpha L_\beta \left(B^t_{ji} b_t - b_{j:i}\right) y^j = 0 \tag{2.2}$$

Where $y_k = a_{ik} y^i$. For the special (α,β) $-$metric **(2.1)** we have,

$$L_\alpha = 1 - \frac{\beta^2}{\alpha} - \frac{2\beta^3}{\alpha^3},\, L_\beta = 1 + \frac{2\beta}{\alpha} + \frac{3\beta^2}{\alpha^2},\, L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3} + \frac{6\beta^3}{\alpha^4},\, L_{\beta\beta} = \frac{2}{\alpha} + \frac{6\beta}{\alpha^2}. \tag{2.3}$$

Substituting **(2.3)** in **(2.2)** equation, we have

$$\left(\alpha^3 - \alpha\beta^2 - 2\beta^3\right) B^t_{ji} y^j\, y_t + \alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)\left(B^t_{ji}\, b_t - b_{j:i}\right) y^j = 0 \tag{2.4}$$

Assume that F^n is a Berwald space, i.e., $B^i_{jk} = B^i_{jk}(x)$. Separating **(2.4)** in rational and irrational terms of y^i

$$\text{as}\left(\alpha^3 - \alpha\beta^2 - 2\beta^3\right) B^t_{ji} y^j\, y_t + \alpha^4\left(B^t_{ji} b_t - b_{j:i}\right) y^j + 2\alpha^3\beta\left(B^t_{ji} b_t - b_{j:i}\right) y^j$$
$$+3\alpha^2\beta^2\left(B^t_{ji} b_t - b_{j:i}\right) y^j = 0 \tag{2.5}$$

which yields two equations

$$(\alpha^3 - \alpha\beta^2 - 2\beta^3) B^t_{ji} y^j y_t + \alpha^4\left(B^t_{ji} b_t - b_{j:i}\right) y^j + 2\alpha^3\beta\left(B^t_{ji} b_t - bj;i\right) y^j \tag{2.6}$$

and

$\left(B_{ji}^{t} b_t - b_{j:i}\right) y^j = 0$ (2.7)

Substituting **(2.7)** in **(2.6),** we have

$(\alpha^3 - \alpha\beta^2 - 2\beta^3) B_{ji}^{t}\ y^j y_t = 0.$ (2.8)

Case(i): If $B_{ji}^{t} y^j y_t = 0$, we have

$B_{ji}^{t} a_{th} + B_{hi}^{t}\ a_{tj} = 0$ and $B_{ji}^{t} b_t - b_{j:i} = 0$ (2.9)

Thus we obtain $B_{ji}^{t} = 0$ by Christoffel process in the first equation of **(2.9)** and from second of **(2.9),** we have $b_{i:j} = 0$.

Case(ii): If $(\alpha^3 - \alpha\beta^2 - 2\beta^3) = 0$,

⇒ α is a one form, which is a contradiction.

Conversly, if $b_{i:j} = 0$, then $B_{ij}^{t} = 0$ are uniquely determined from **(2.4).**

Hence, we conclude the following

Theorem 2.1. A Finsler space with a second approximate matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is a Berwald space iff $b_{i;j} = 0$.

The Condition To Be A Douglas Space:-

In this section, we find the condition for a Finsler space F^n with a Second approximate matsumato metric **(2.1)**, to be Douglas type.

J. Douglas introduced a curvature namely Douglas curvature, which always vanishes for a Riemannian metrics **[2].** Finsler metrics with vanishing Douglas curvature are called Douglas metric and the space is called douglas space for which $B^{ij} = B^i y^j - B^j y^i$ are homogeneous polynomials of degree 3, in short we write B^{ij} is hp(3) **[1].**

In view of **[6],** if $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$ then the function $G^i(x,y)$ of F^n with an (α, β)-metric is written in the form

$$2G^i = \gamma_{00}^{i} + 2B^i$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + c^*\left[\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i\right)\right]$$

$$C^* = \frac{\alpha\beta\left(\gamma_{00} L_\alpha - 2\alpha s_0 L_\beta\right)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_\alpha)}$$

Where $\gamma^2 = b^2\alpha^2 - \beta^2, b^i = a^{ij} b^j,\ and\ b^2 = a_{ij} b_i b_j$.

The vector $B^i(x,y)$ is called the difference vector. Hence B^{ij} is written as

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}\left(s_0^i y^j - s_0^j y^i\right) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i).$$

$\{\alpha^3(1+2b^2) + 3\beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)B^{ij} - \beta(2\alpha + 3\beta)(s_0^i y^j - s_0^j y^i)\} -$
$\alpha^2(\alpha + 3\beta)\{\gamma_{00}\alpha^3 - \alpha\beta^2 - 2\beta^3\} - 2\alpha^2\beta^2(2\alpha - 3\beta)(b^i y^j - b^j y^i)$. (3.1)

Suppose that F^n is a Douglas space, that is, B^{ij} are hp(3). Arranging the rational and irrational terms, equation **(3.1)** can be written as

$\alpha^3(1 + 2b^2) + \beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)B^{ij} - \beta(2\alpha + 3\beta)\ (s_0^i y^j - s^j y_i)\} -$
$\alpha^2(\alpha + 3\beta)\{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2\alpha^2\beta^2(2\alpha - 3\beta)\}(b^i y^j - b^j y^i)$

$+\alpha^2[2s_0\alpha^4(\alpha + 3\beta)(b^i y^j - b^j y^i) - (\alpha + 3\beta)\{\alpha^3(1 + 2b^2)$
$+\beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}(s_0^i y^j - s_0^j y^i) = 0$ (3.2)

Separating rational and irrational terms of y^iin (3.1) we have the following two equations

$$\alpha^3(1 + 2b^2) + \beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)B^{ij} - \beta(2\alpha + 3\beta)$$
$$(s_0^i y^j - s^j y_i)\} - \alpha^2(\alpha + 3\beta)\{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2\alpha^2\beta^2(2\alpha - 3\beta)\}(b^i y^j - b^j y^i) . \quad (3.3)$$

And

$$2s_0\alpha^2(\alpha + 3\beta)(b^i y^j - b^j y^i) - (\alpha + 3\beta)\{\alpha^3(1 + 2b^2)$$
$$+\beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}(s_0^i y^j - s_0^j y^i) = 0 \quad (3.4)$$

Substituting **(3.4)** in **(3.3),** we have

$$\{\alpha^3(1 + 2b^2) + \beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)B^{ij}$$

$$-\alpha^2(\alpha + 3\beta)r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3)(b^i y^j - b^j y^i) = 0. \quad (3.5)$$

only the term $4\beta^5 B^{ij}$ of **(3.5)** does not contain α^2. Hence we must have hp(6) v_6^{ij} satisfying

$$4\beta^5 B^{ij} = \alpha^2 v_6^{ij} \quad (3.6)$$

Now we study the following two cases:

Case(i): $\alpha^2 \neq 0(mod\beta)$

In this case**,** **(3.6)** is reduced to $B^{ij} = \alpha^2 v^{ij}$ are hp(1).Thus **(3.5)** gives

$$\alpha^3(1 + 2b^2) - \beta^2(-3\alpha - 2 - 6\beta)B^{ij} - r_{00}(b^i y^j - b^j y^i) = 0. \quad (3.7)$$

Transvecting this by $b_i y_j$, where $y_j = a_{jk}y^k$, we have

$$\alpha^3(1 + 2b^2)v^{ij}\, b_i y_j - b^2 r_{00} == \beta^2(r_{00} - 8v^{ij}\, b_i y_j) \quad (3.8)$$

Since $\alpha^2 \neq$ 0(modβ) there exist a function h(x) satisfying

$(1 + 2b^2)v^{ij}\, b_i y_j - b^2 r_{00} = h(x), \beta^2(r_{00} - 8v^{ij}\, b_i y_j) = h(x)\alpha^2.$

Eliminating $v^{ij}\, b_i y_j$ from the above two equations, we obtain

$$(1 + b^2)r_{00} = h(x)\{(1 + 2b^2)\alpha^2 - 8\beta^2\} \quad (3.9)$$

from **(3.9)**, we get

$$b_{i;j} = k\{(1 + 2b^2)a_{ij} - 3b_i b_j\} \quad (3.10)$$

where $k = \frac{h(x)}{(1+b^2)}$. Hence, b^i is a gradient vector.

Conversely, if **(3.10)** holds, then $s_{ij} = 0$ and we get **(3.9).** Therefore, **(3.3)** is written as follows:

$$B^{ij} = k\{\alpha^2(b^i y^j - b^j y^i)\}$$

which are hp(3), that is, F^n is a Douglas space.

Case(ii): [3]$\alpha^2 = 0$(modβ).

Consider the following lemma,

Lemma 3.1. If $\alpha^2 = 0(mod\beta)$, that is, $a_{ij}\,(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimention n is equal to 2 and b^2 vanishes. In this case we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2.$

The equation **(3.6)** is reduced to $B^{ij} = \delta w_2^{ij}$, where w_2^{ij} are hp(2).

Hence, the equation **(3.4)** leads to

$$2s_0\delta(b^i y^j - b^j y^i) - (\delta - 3\beta)(s_0^i y^j - s_0^j y^i) = 0. \quad (3.11)$$

Transvecting the above equation by $y_i b_j$, we have $s_0 = 0$. Substituting $s_0 = 0$ in the above equation, we have $s_{ij} = 0$. Therefore, **(3.7)** reduces to

$$(\delta - 3\beta)\, w_2^{ij}\, biyj - r_{00}\beta^2 = 0,$$

which is written as

$$\delta w_2^{ij}\ biyj\ =\ \beta(\beta r_{00}\ -\ 3w_2^{ij}\ b_i y_j).$$

Therefore, there exists an hp(2), $\lambda\ =\ \lambda_{ij}\ (x)y^i y^j$ such that

$$w_2^{ij}\ b_i y_j\ =\ \beta\lambda, \beta r_{00}\ +\ 3w_2^{ij}\ b_i y_j\ =\ \delta\lambda.$$

eliminating $w_2^{ij}\ b_i y_j$ from the above equations, we get

$$\beta r_{00}\ =\ 3\beta\lambda\ -\ \delta\lambda\ =\ \lambda(3\beta\ -\ \delta) \qquad (3.12)$$

which implies there exists an hp(1), $v_0\ =\ v_i(x)y^i$ such that

$$r_{00}\ =\ v_0(3\beta\ -\ \delta)\ =\ v_0\beta \qquad (3.13)$$

From r_{00} given by **(3.13)** and $s_{ij}\ =\ 0$, we get

$$b_{i:j} = \frac{1}{2}\{v_i(3b_j\ +\ d_j\ )\ +\ v_j\ (3b_i\ +\ d_i)\}. \qquad (3.14)$$

Hence b_i is gradient vector.

Conversely, if **(3.14)** holds, then $s_{ij} = 0$, which implies $r_{00}\ =\ v_0(3\beta\ +\ \delta)$**.** Therefore, **(3.3)** is written as follows:

$$B^{ij}\ =\ v_0\delta(b^i y^j - b^j y^i),$$

which are hp(3). Therefore, F^n is a Douglas space.

Thus, we have

Theorem 3.1: *A Finsler space with* second approximate matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ *is a Douglas space if and only if*

(1) $\alpha^2 \not\equiv 0(mod\ \beta), b^2 \neq 1$: $b_{i|j}$ *is written in the form (3.11).*

(2) $\alpha^2 \equiv 0(mod\ \beta)$: $n = 2$ *and* $b_{i|j}$ *is written in the form (3.14), where* $\alpha^2 = \beta\delta, \delta = d_i(x)y^i, v_o = v_i(x)y^i$.

Conclusion:-

M. Matsumoto and S. Bacso introduced the notion of Douglas space as a generalization of Berwald space from the view point of geodesic equations. Douglas metrics can be viewed as generalized Berwald metrics. The study on Douglas metrics will enhance our understanding on the geometrical meaning of non-Riemannian quantities. In this paper, we found the conditions for a Finsler space with second approximate matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ to be a Douglas space and also to be a Berwald space. Further, we found the conditions for a conformally transformed Douglas space with the above mentioned second approximate matsumato metric to be a Douglas space.

The important findings of this paper are as follows:-

1. A Finsler space with a second approximate matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is a Douglas space if and only if
   a. $\alpha^2 \not\equiv 0(mod\ \beta), b^2 \neq 1$: $b_{i|j}$ is written in the following form:

   b. $\alpha^2 \equiv 0(mod\ \beta)$: $n = 2$ and $b_{i|j}$ is written in the following form:
   $$b_{i|j} = \frac{1}{2}\{v_i(3b_j + d_j) + v_j(3b_i + d_i)\},$$
   where $\alpha^2 = \beta\delta, \delta = d_i(x)y^i, v_0 = v_i(x)y^i$
2. $\alpha^2 \not\equiv 0(mod\ \beta)$, then the Douglas space with second approximate matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is conformally transformed to a Douglas space if and only if the transformation is homothetic.

References:-

1. S. Bacso and M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berwald space, Publ. Math. Debrecen, 51 (3) (1997), 385406.
2. J. Douglas, The general geometry of paths, Annals of Math., 29 (1927-28), 143168.
3. Il-Yong Lee and Hong-Suh Park, Finsler spaces with infinite series (α, β)-metric, J. Korean Math. Soc., 41(3) (2004), 567589.
4. Il-Yong Lee and Myung-Han Lee, On Weakly-Berwald spaces of special (α, β)-metric, Bull. Korean Math. Soc., 43 (2) (2006), 425441.
5. B. Li, Y. Shen and Z. Shen, On a class of Douglas metrics in Finsler geometry, Studia Sci. Math. Hungarica, 46 (3) (2009), 355365.
6. M. Matsumoto, The Berwald connection of a Finsler space with an (α, β)-metric, Tensor, N. S., 50 (1991), 1821.
7. M. Matsumoto, Theory of Finsler spaces with (α, β)-metric, Rep. On Math. Phys., Vol. 31 (1992), 43-83.
8. Z. Shen and G. Civi Yildirim, On a Class of Projectively Flat metrics with Constant Flag Curvature, Canadian Journal of Mathematics, 60 (2008), 443-456.
9. S. K. Narasimhamurthy, Ajith and C. S. Bagewadi On conformal β-change of Douglas Space with (α, β)-metric, in press.
10. S.K. Narasimhamurthy and G.N. Latha Kumari, On a hypersurface of a special Finsler space with a metric $L = \alpha + \frac{\beta^2}{\alpha} + \beta$, *ADJM,* 9(1)(2010), 36-44.
11. S.K. Narasimhamurthy and D.M. Vasantha, Projective change between two Finsler spaces with $(\alpha,\beta)-$metric, *Kyunpook Mathematical Journal.*
12. B. N. Prasad, B. N. Gupta and D. D. Singh On Conformal transformation in Finsler spaces with an (α, β)-metric, Indian J. pure appl. Math., Vol. 18, No. 4, (1987), 290-301.
13. B. N. Prasad, Conformal change of Finsler space with (α, β)-metric of Douglas type,in press.
14. Pradeep Kumar, S.K. Narasimhamurthy, H.G. Nagaraja, S.T. Aveesh, On a special hypersurface of a Finsler space with $(\alpha,\beta)-$metric, *Tbilisi Mathematical Journal,* 2(2009), 51-60.
15. Z. Shen and S. S. Chern, Riemann-Finsler Geometry, Nankai Tracts in Mathematics, World Scientific, 6 (2004).
16. L. Zhou, A Local classiffication of a class of (α, β)-metric with constant flag curvature, Differential Geometry and its Applications, 28 (2010), 170-193.
17. Z. Shen, On Landsberg (α,β) -metrics, math.iupui.edu., Preprint (2006).
18. Benling Li, Yibing Shen and Zhongmin Shen, On a class of Douglas metrics, *Comm. Korean Math. Soc.*, 14(3)(1999), 535-544.
19. Nicoleta Aldea and Gheorghe Munteanu, On Complex and Landsberg spaces, Journal of Geometry and Physics, 62 (2012), 368-380.