ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

Triple Dirichlet Average of Trigonometrical Functions and Fractional Derivative

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Manuscript Info

Manuscript History:

Received: 10 October 2014
Final Accepted: 22 November 2014
Published Online: December 2014

Key words:

Dirichlet average, $\cos x$, $\sin x$, fractional derivative and Fractional calculus operators.

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Abstract

The aim of present paper is to establish some results of Triple Dirichlet average of Trigonometrical functions i.e. $\cos x$ and $\sin x$, and using fractional derivative.

Introduction

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

Also, Gupta and Agarwal [10,11] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional calculus [8]. Sharma and Jain [ ]

Recently, Sharma and Jain [ ] obtained double Dirichlet average of Trigonometry function $\cos x$ using fractional derivative. We can say that every analytic function can be measured as Dirichlet average, using fractional derivative. and they have also found the Triple Dirichlet Average of e^x by using fractional calculus.

Saxena, Pogany, Ram and Daiya [ ] investigated the Dirichlet Averages of Generalized Multi-index Mittag-Leffler in terms of Riemann-Liouville integrals and hypergeometric functions of several variables Functions.

Kilbas and Kattuveetti [ ] established a correlation among Dirichlet averages of the generalized Mittag-Leffler function nwith Riemann-Liouville fractional integrals and of the hyper- geometric functions of many variables.

In this paper, the Dirichlet average of Trigonometrical functions has been obtained by using fractional calculus.

1. Definitions:

We give blew some of the definitions which are necessary in the preparation of this paper.

1.1 Standard Simplex in $R^n, n \geq 1$:

The standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \ldots u_n) : u_1 \geq 0, \ldots u_n \geq 0, u_1 + u_2 + \cdots + u_n \leq 1\}$$

1.2 Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by E[1].

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \ldots u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots - u_{k-1}) b_k^{-1} du_1 \ldots du_{k-1} \qquad (2.2.1)$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b1, \ldots bk) = \frac{\Gamma(b_1) \ldots \Gamma(b_k)}{\Gamma(b_1 + \cdots + b_k)}, \qquad C_> = \{z \in z: z \neq 0, |ph\, z| < \pi/2\},$$

Open right half plane and $C_>k$ is the k^{th} Cartesian power of $C_>$

1.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_>$, let $z = (z_1, \ldots, z_k) \in \Omega^k, k \geq 2$ and let $u.z$ be a convex combination of $z_1, \ldots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b, z) = \int_E f(u.z) d\,\mu_b(u) \qquad (2.3)$$

We shall call F the Dirichlet measure of f with variables $z = (z_1, \ldots, z_k)$ and parameters $b = (b_1, \ldots b_k)$.

Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots - u_{k-1.}$$

If $k = 1$, define $F(b, z) = f(z)$.

1.4 Fractional Derivative [9, p.181]:

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi [9]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^\alpha F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z - t)^{-\alpha-1} dt \qquad (2.4)$$

Where $Re(\alpha) < 0$ and F(x) is the form of $x^p f(x)$, where f(x) is analytic at $x = 0$.

2.5 Average of cosh x (from [16]):

let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1}; $k \geq 2$. For every $z \in C^k$

$$S(b, z) = \int_E \cos(u.z)\, d\mu_b\ (u) \qquad (2.5)$$

If $k = 1, S = (b, z) = \cos(u.z)$.

2.6 Triple averages of functions of one variable (from [1, 2]): let z be species with complex elements z_{ijk}. Let $u = (u_1, \ldots u_l)$, $v = (v_1, \ldots v_m)$ and $w = (w_1, \ldots w_n)$ be an ordered l-tuple, m-tuple and n-tuple of real non-negative weights $\sum u_i = 1$, $\sum v_j = 1$, and $\sum w_k = 1$ respectively.

We define

$$u.z.v.w = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} u_i z_{ijk} v_j w_k \qquad (2.6)$$

If z_{ijk} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11}, \dots z_{kx})$, denote by $H(z)$.

Let $\mu = (\mu_1, \dots \mu_k)$ be an ordered l –tuple of complex numbers with positive real part (Re (μ) > 0) and similarly for $(\alpha = \alpha_{1,} \dots \alpha_m)$ and $\beta = (\beta_{1,} \dots \beta_n)$. Then we define $dm_\mu(u), dm_\alpha(v)$ and $dm_\beta(w)$ as (2.2.1).

Let f be the holomorphic on a domain D in the complex plane. If $Re(\mu) > 0, Re(\alpha), Re(\beta) > 0 \text{and} H(z) \subset D$, we define

$$F(\mu, z, \alpha, \beta) = \iint \int f(u, z, v, w)\, dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \tag{2.7}$$

Corresponding to the particular function coshx , $z^t$and e^z, we define,

$$S(\mu, z, \alpha, \beta) = \iiint \cos(u, z, v, w)\, dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \tag{2.8}$$

$$S(\mu, z, \alpha, \beta) = \iiint \sin(u, z, v, w)\, dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \tag{2.9}$$

$$R_t(\mu, z, \alpha, \beta) = \iiint (u, z, v, w)^t dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \tag{2.10}$$

$$S(\mu, z, \alpha, \beta) = \iiint (e)^{u.z.v.w}\, dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \tag{2.11}$$

2. Main Results and Proof:

Theorem: Following equivalence relation for Triple Dirichlet Average is established for (l = m = n = 2) of $\cosh(u, z, v, w)$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu} (x-y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \cos x\, (x-y)^{\mu-1} \tag{3.1}$$

Proof:

Let us consider the triple average for (l = m = n = 2 of $\cos(u.z.v.w)$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \int_0^1 \int_0^1 \cos(u.z.v.w) dm_{\mu,\mu'}(u) dm_{\alpha,\alpha'}(v) dm_{\beta,\beta'}(w)$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} \int_0^1 \int_0^1 [u.z.v.w]^{2n} dm_{\mu,\mu'}(u) dm_{\alpha,\alpha'}(v) dm_{\beta,\beta'}(w) \tag{3.2}$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\alpha) > 0, Re(\alpha') > 0, Re(\beta) > 0, Re(\beta') > 0$ and

$$u.z.v.w = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} u_i z_{ijk} v_j w_k = \sum_{i=1}^{2} \sum_{j=1}^{2} [u_i v_j (z_{ij1} w_1 + z_{ij2} w_2)]$$

$$u.z.v.w = \sum_{i=1}^{2} [u_i (v_1 z_{i11} w_1 + v_1 z_{i12} w_2 + v_2 z_{i21} w_1 + v_2 z_{i22} w_2)]$$

$$u.z.v.w = [u_1 v_1 z_{111} w_1 + u_1 v_1 z_{112} w_2 + u_1 v_2 z_{121} w_1 + u_1 v_2 z_{122} w_2 + u_2 v_1 z_{211} w_1 + u_2 v_1 z_{212} w_2 + u_2 v_2 z_{221} w_1 + u_2 v_2 z_{222} w_2]$$

let in first species $z_{111} = a, z_{112} = b, z_{121} = c, z_{122} = d$ and second species $z_{211} = e, z_{212} = f, z_{221} = g, z_{222} = h$

and $\begin{cases} u_1 = u, & u_2 = 1 - u \\ v_1 = v, & v_2 = 1 - v \\ w_1 = w, & w_2 = 1 - w \end{cases}$

such that

$$u.z.v.w = [uvw(a - b - c + d - e + f + g - h) + uv(b - d - f + h) + vw(e - f - g + h) + wu(c - d - g + h) + u(d - h) + v(f - h) + w(g - h) + h]$$

$$dm_{\mu,\mu'}(u)=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}u^{\mu-1}(1-u)^{\mu'-1}du$$
$$dm_{\alpha,\alpha'}(v)=\frac{\Gamma(\alpha+\alpha')}{\Gamma\alpha\Gamma\alpha'}v^{\alpha-1}(1-v)^{\alpha'-1}dv$$
$$dm_{\beta,\beta'}(w)=\frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'}w^{\beta-1}(1-w)^{\beta'-1}dw$$

Putting these values in (3.2), we have,

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'}$$
$$\times\sum_{v=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^1\int_0^1\int_0^1[uvw(a-b-c+d-e+f+g-h)+uv(b-d-f+h)$$
$$+vw(e-f-g+h)+wu(c-d-g+h)+u(d-h)+v(f-h)+w(g-h)+h]^{2n}$$
$$\times u^{\mu-1}(1-u)^{\mu'-1}du\,v^{\alpha-1}(1-v)^{\alpha'-1}dv\,w^{\beta-1}(1-w)^{\beta'-1}dw \quad (3.3)$$

In order to obtained the fractional derivative equivalent to the above integral.

Case-1:

If $a=x,e=y,b=c=d=f=g=h=0$ then we have

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'}$$
$$\times\sum_{v=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^1\int_0^1\int_0^1[uvw(x-y)+vwy]^{2n}\,u^{\mu-1}(1-u)^{\mu'-1}\,v^{\alpha-1}(1-v)^{\alpha'-1}\,w^{\beta-1}(1-w)^{\beta'-1}dudvdw$$

Using the definition of beta function and due to suitable adjustment, Deora and Banergy [ ] we arrive at

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}$$
$$\times\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^1[ux+(1-u)y]^{2n}\,u^{\mu-1}(1-u)^{\mu'-1}du$$
$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}S(\mu,\mu';x,y) \quad (3.4)$$

By using the definition of fractional derivative we get,

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma\mu}(x-y)^{1-\mu-\mu'}D_{x-y}^{-\mu'}cosx\,(x-y)^{\mu-1} \quad (3.5)$$

This is complete proof of (3.1).

Similarly we can show equivalence of triple Dirichlet average of $sinh\,x$ $(l=m=n=2)$ with the fractional derivative i.e

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma\mu}(x-y)^{1-\mu-\mu'}D_{x-y}^{-\mu'}sinx\,(x-y)^{\mu-1} \quad (3.6)$$

Case 2:

If we put $a=b=c=d=x;\,e=f=g=h=y$ in equation (3.3) then Triple Dirichlet average of $cosh\,x$ is same as single Dirichlet average of $cos\,x$ Sharma and Jain [14], i.e.

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'}$$
$$\times\sum_{v=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^1\int_0^1\int_0^1[u(x-y)+y]^{2n}\,u^{\mu-1}(1-u)^{\mu'-1}du\,v^{\alpha-1}(1-v)^{\alpha'-1}dv\,w^{\beta-1}(1-w)^{\beta'-1}dw$$

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\times\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^1[u(x-y)+y]^{2n}\,u^{\mu-1}(1-u)^{\mu'-1}du$$

Putting $u(x-y)=t$, we obtain

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma}\times\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^{x-y}[y+t]^{2n}\left(\frac{t}{x-y}\right)^{\mu-1}\left(1-\frac{t}{x-y}\right)^{\mu'-1}\frac{dt}{(x-y)}$$

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}(x-y)^{1-\mu-\mu'}\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}\int_0^{x-y}[y+t]^{2n}\,(t)^{\mu-1}(x-y-t)^{\mu'-1}dt$$

On changing the order of integration and summation, we have

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}(x-y)^{1-\mu-\mu'}\int_0^{x-y}cos(y+t)\,(t)^{\mu-1}(x-y-t)^{\mu'-1}dt \qquad (3.7)$$

Using definition of fractional derivative (2.4), we get

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu}(x-y)^{1-\mu-\mu'}D_{x-y}^{-\mu'}cosx\,(x-y)^{\mu-1}$$

Hence we obtained Triple Dirichlet average of $cosh\,x$ is same as single Dirichlet average of $cos\,x$.

Similarly we can show equivalence of triple Dirichlet average of $sinh\,x$ $(l=m=n=2)$ with the fractional derivative i.e.

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma\mu}(x-y)^{1-\mu-\mu'}D_{x-y}^{-\mu'}sinx\,(x-y)^{\mu-1} \qquad (3.8)$$

3. Particular Cases:

(i) If $\mu'=v-\mu$ and $y=0$ in (3.1)

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\left(\frac{\Gamma v}{\Gamma\mu}(x)^{1-v}D_x^{\mu-v}cosx\,(x)^{\mu-1}\right)$$

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{1}{2}\left[\frac{\Gamma v}{\Gamma\mu}(x)^{1-v}D_x^{\mu-v}e^{ix}\,(x)^{\mu-1}+\frac{\Gamma v}{\Gamma\mu}(x)^{1-v}D_x^{\mu-v}e^{-ix}\,(x)^{\mu-1}\right]$$

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{1}{2}\left[{}_1F_1(\mu,v;ix)+{}_1F_1(\mu,v;-ix)\right] \qquad (4.1)$$

(ii) If $\mu=-n$, $\mu'=1+\gamma+n$ and $y=0$ in (3.1) we have,

$$S(-n,1+\gamma+n;z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{1}{2}\left[\frac{\Gamma(1+\gamma)}{\Gamma(-n)}x^{-\gamma}D_x^{-n-\gamma-1}e^{ix}\,x^{-n-1}+\frac{\Gamma(1+\gamma)}{\Gamma(-n)}x^{-\gamma}D_x^{-n-\gamma-1}e^{-ix}\,x^{-n-1}\right] \qquad (4.2)$$

$$S(-n,1+\gamma+n;z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{1}{2}\left[{}_1F_1(-n,1+\gamma;ix)+{}_1F_1(-n,1+\gamma;-ix)\right] \qquad (4.3)$$

$$S(-n,1+\gamma+n;z;\alpha,\alpha',\beta,\beta')=\frac{1}{2}\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\left[\frac{L_n^{\gamma}(ix)}{L_n^{\gamma}(0)}+\frac{L_n^{\gamma}(-ix)}{L_n^{\gamma}(0)}\right] \qquad (4.4)$$

Where L_n^{γ} is the Laguerre polynomial of degree n.

5. Acknowledgement:

Authors are grateful to referee for his valuable comment and improvement upon the paper.

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