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RESEARCH ARTICLE

Z- PRIMESUBMODULES.

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Abstract

LetR be a commutative ring with identity and Mbe a unitary R-module, a submodul$N$ of an R-module M is called prime if for each $x \in M, r \in R$ such that $rx \in N$ implies thateither $x \in N$or $r \in [N: M]$. In this paper we say that $N$is a Z-prime submodule of an R-module Mif for each$x \in M, f \in M^* = Hom(M, R)$, such that $f(x).x \in N$implies that either $x \in N$or$f(x) \in [N: M]$, where $[N: M] = \{r: r \in R and rM \subseteq N\}$. We study also a$Z$-prime module in which (0) is a Z-prime submodulein$M$. We give many properties of Z-prime submodule and Z-prime module.

Introduction:-

Aproper submodule N of module M over ring R is said to be prime (or P-prime) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $r \in [N:M]$ or $x \in N$ where $[N:M]=\{r \in R : rM \subseteq N\}$, [1].And M is called prime module if the zero submodule of M is a prime submodule of M, [2].In the last year many studies, researches are published about prime submodule by many people who care with the subject of commutative algebra and some of them are J..Dauns ,R.L. .Mcsland, C.P.LU, P.F.Smith, M..E..Moore. M is called faithful module if $[0: M] = 0$, [3].Where $[0: M] = \{r \in R: rM = 0\}$ and $[0: N]$is define as $[0: N] = \{r \in R: rN = 0\}$. There are several generalization of the notion of a prime submodules as like S-prime, [4]. In section one we define Z-prime submodule of M and it is clear every prime submodule is$Z$-prime submodule but the converse is not true in general . However we give condition under which a proper submodule of Z-prime is a prime submodule, also we study some properties and example of Z-prime submodule. In section two we define the Z-prime module and we show the relations between aZ-prime module and a faithful module, also we see if R is integral domain then R as R-module is Z-prime module, but the converse is not true in general. In this paper we study a Z-prime submodule of M and a Z-prime module as in the following

Z-Prime Submodules:-

Recall that a proper submodule$N$ of an R-module M is called prime if $r.x \in N$where$r \in R, x \in M$ implies that either $x \in N$or$r \in [N: M]$, where $[N: M] = \{r: r \in R and rM \subseteq N\}$, [1]. In this section we introduce the following:

Definition:-

A proper submodule$N$ of an R-module M is called Z-prime if $f(x).x \in N$such that $f \in M^* = Hom(M, R)$, $x \in M$ implies that either $x \in N$or$f(x) \in [N: M]$.

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Remarks and examples:-

Every prime submodule of an R-module is Z-primesubmoduleof M. But the converse is not truein general as in the following example:

Let$M = Z_8$ as$Z$-module, $N = \{\bar{0}, \bar{4}\}$ a submodule of Mbut$\bar{2} \notin N and\ 2Z_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \nsubseteq \{\bar{0}, \bar{4}\}$, then $2 \notin [N: M]$.Thus N is not prime.

To show that $N = \{\bar{0},\ \bar{4}\}$ is Z-prime submoduleof M. Since$Hom(Z_8, Z) = 0$, then $f = 0$, $0.x \in N$, $\forall x \in Z_8$, since $0Z_8 \subseteq \{\bar{0}, \bar{4}\}$, then $0 \in [N: Z_8]$.Hence N is Z-prime but not prime submodule of M

Consider the Z-module, $M = Z \oplus Z$. The submodule $N = 2Z \oplus 0$ is not a Z-prime submoduleof M, since if we define $f: Z \oplus Z \to Z$by $f(n, m) = 2n$,then $(18,0) = f(3,0)(3,0) = 6(3,0) \in N, but(3,0) \notin N and\ 6 \notin [N: M] = (0)$.

Every maximal submodule of an R-module M is Z-prime submoduleof M.
Since every maximal submodule is prime.

We can find a faithful R-moule M but N is not a prime submodule of M, as the following example show:
Let $M = Z$ as Z-module and $N = 4Z$ a submoduleof M. $2.2 = 4 \in N$, but $2 \notin N$and $2 \notin [4Z: Z]$, since $2Z \nsubseteq 4Z$. Thus N is not prime

If M is cyclic R-module and N is submodule of, then N may not be a Z-prime submodule of, see example in (1.2)(1).

Let I be an ideal of a ring R. $I$is called a Z-prime ideal if for every$a \in R$, $f \in R^* = Hom(R, R)$ such that $f(a). a \in I$implies that either $a \in I$ or $f(a) \in I$

In case the R-module M is cyclic, faithful, then the prime and Z-prime submodule of M are equivalent.
Recall that an R-module M is called faithful if $ann(M) = 0$, [3]

Proposition:-

Let M be a cyclic, faithful R-module and Nbe a submoduleof$M$, if N is aZ-prime submoduleof M, then N is a prime submodule of M.

Proof:-

Let $rx \in N$, $where r \in R$, $x \in M$and suppose$x \notin N$, we want prove $r \in [N: M]$. Suppose$M = < a >$, then $x = ta$; $t \in R$. Define $f: M \to R$ by$f(x) = f(ra) = r for each x \in M$,since M is faithful , then f is well defined , this implies that $f(x). x \in N$, $x \notin N$, but N is Z-prime submodule, then $r = f(x) \in [N: M]$.Thus$r \in [N: M]$.

Corollary:-

Let N be aproper submodule of a cyclic faithful R-module, thefollowing are equivalent:-
a)$N$is a prime submodule of$M$.
b)$N$is a Z-prime submodule of M.

proof:-

a⇒b) By*Proposition* (1.3)
b⇒a) By*Proposition* (1.2)

Corollary:-

Let R be a PID. IfN is aZ-prime submodule of a cyclic, faithful$R$-module, then$[N: M]$is aZ-prime ideal of$R$.

Proof:-

Let N is Z- Primesubmodule of an R-module M,since R is cyclic faithful ring, then N is prime submodule, thus $[N: M]$ is prime ideal.since$R$ is cyclic faithful ring, therefore by proposition(1.3) $[N: M]$ is a Z-primary ideal of R.

The converse of corollary (1.5) is not true in general for example:
Let$M = Z \oplus Z$ as Z-module and consider the submodule$N = 4Z \oplus (0)$ of the $Z \oplus Z$, then $[N: M] = [4Z \oplus (0): Z \oplus Z] = (0)$ which is a Z-prime ideal of Z, but$N$ not a Z-prime submodule of M, since if we take$f: Z \oplus Z \to Z$, define by $f(n, m) = n$, $(4,0) = f(2,0)(2,0) = 2(2,0) \in N$, but neither$2 \in [4Z \oplus (0): Z \oplus Z] = (0)$ nor $(2,0) \in N$.

Proposition:-
Let M be an R-module and P be a prime ideal of$R$.If $[N: K] \subseteq P$, for each submodule K of M containing$N$ properly and $P \subseteq [N: M]$, then $N$is aZ-primesubmodule of M

Proof:-
Let $f(x).x \in N$,where $f \in M^* = Hom(M, R), x \in M$,suppose $x \notin N$, let $K = N + (x)$, so $K$is submoduleproperly containing$N$ but$f(x)K = f(x)N + (f(x).x) \subseteq N$,hence $f(x) \in [N: K] \subseteq P \subseteq [N: M]$. Thus N is a Z-prime submoduleof M.

Proposition:-
Let M be an R-module, N be a proper submodule of$M$, and t K be submodule of M,if$f(K)\ K \subseteq N$, such that either $N \supseteq K or [N: M] \supseteq f(K)$ for all $f \in Hom(M, R)$, then N is Z-prime submodule of M

Proof:-
Let$f(x).x \in N$, where $f \in M^*$and $x \in M$,then$f(< x >) < x > \subseteq N$,and thus by our assumption either $< x > \subseteq N$, which implies that $x \in N$, or$f(< x >) \subseteq [N: M]$. Which implies that$f(x) \in [N: M]$, hence N is aZ-prime submodule of M.

Proposition:-
IfN is aZ-prime submodule of M and K submodule of M such that $K \nsubseteq N$, then $K \cap N$ is aZ-prime submodule of K.

Proof:-
Let$f(x).x \in N \cap K$; $f \in K^* = \text{Hom}(K, R)$, $x \in K$, suppose $x \notin K \cap N$, then $x \notin N, so there exists$ a function $h: M \to R$ such that $h \circ i = f$, where $i: K \to R$is the inclusion map.Now, $(h \circ i)(x).x \in N \cap K$, then$h\ (x).x \in N$,$x \notin N$, but N is Z-prime submodule of M,hence$h\ (x) \in [N: M]$, so that $h(x)M \subseteq N$, hence $h \circ i(x)M \subseteq N$. Thus $f(x)K \subseteq N \cap K$, such that $f(x) \in [N \cap K: K]$

Proposition:-
Let$N$ be aproper submodule of$R$-module M such that $[N: M] \nsupseteq [K: M]$ for eachsubmodule$K$of$M$containing$N$ properly. If $[N: M]$ is aZ-prime ideal of$R$, then$N$ is aZ-prime submodule of M

Proof:-
We want prove N is Z-prime submoduleof M. Let$f \in M^* = Hom(M, R), x \in M$, such that $f(x).x \in N$, suppose $x \notin N$, $N + (x) = L$is containing$N$ properly such that $[N: M] \nsupseteq [L: M]$, there exist $s \in [L: M]$ and $s \notin [N: M]$, hence $sM \subseteq L$and $sM \nsubseteq N$but$f(x)sM \subseteq N$, then $f(x) .s \in [N: M]$,since $[N: M]$ is Z-prime ideal and $s \notin [N: M]$, then $f(x) \in [N: M]$. Thus $N$is Z-prime submodule of M.

Recall that an R-module$M$is called a multiplication R-moduleif for eachsubmodule$N$ of M there exists an ideal I of R such that $N = IM$, we can take $I = [N: M]$,[5].

Remark:-
If$M \neq 0$ is a multiplication R-module, then $[N: M] \nsupseteq [K: M]$ for each submodule$K$ of M containing$N$ properly where N is a proper submoduleof M.

Proof:-
Suppose there existsa submodule$K$ containing$N$ properly such that $[N:M] \supseteq [K:M]$, then $[N:M]M \supseteq [K:M]M$, but M is multiplication$R$-moduletherefore,$N \supseteq K$ and thisisa contradiction. Hence$[N:M] \supseteq [K:M]$ for eachsubmodule$K$ of M containing$N$properly.

Corollary:-
Let$N$ be a proper submodule of a multiplication R-module M, then N is Z-prime submodule if and only if $[N:M]$ is Z-prime ideal of R.

Proof:-
By remark (1.10) we have $[K:M] \nsubseteq [N:M]$ for each submodule K of M such that $K \nsupseteq N$ and by corollary (1.5) we get a result .

Remark:- Every family ofsubmoduleN_i ; $i \in I$ of R-module Mbe:
$\bigcap_{i \in I}[N_i : M] = [\bigcap_{i \in I} N_i : M]$. [4]

Proposition:-
Let$\{N_i\}_{i \in I}$ be a family of Z-prime submoduleof M,then$\bigcap_{i \in I} N_i$ is Z-prime submodule of M.

Proof:-
Let $f \in M^* = Hom(M,R)$, $x \in M$such that $f(x).x \in \bigcap_{i \in I} N_i$, suppose $x \notin \cap N_i$, then $f(x).x \in N_i$,for each $i \in I$, since N_i is$Z$-prime submodules $\forall i \in I$ of $M$and $x \notin N_i$, hence $f(x) \in [N_i : M]$ $\forall i \in I$, thus $f(x) \bigcap_{i \in I} \in [N_i : M] = [\bigcap_{i \in I} N: M]$, thus $\bigcap_{i \in I} N_i$ is a Z-prime submodule of M.

Proposition:-
Let M_1 and M_2 be two R-modules and $M = M_1 \oplus M_2$, if $N = N_1 \oplus N_2$ is aZ-prime submodule of M, then N_1 and N_2 are Z-prime submodules of $M_1$and M_2respectively.

Proof:-
To prove N_1 is Z-prime R-submodule of M_1, let $f \in M_1^*$, $x \in M_1$such that $f(x).x \in N_1$, then $(f \circ j)(x,0)(x,0) \in N_1 \oplus N_2$, where $j: M_1 \oplus M_2 \rightarrow M_1$, but $N_1 \oplus N_2$ is Z-prime submodule,so either $(x,0) \in N_1 \oplus N_2$or $f(x) \in [N_1 \oplus N_2 : M_1 \oplus M_2]$. Thuseither $x \in N_1$or $f(x) \in [N_1 : M_1] \cap [N_2 : M_2]$, hence either $x \in N_1$or $f(x) \in [N_1 : M_1]$, therefore N_1 is Z-prime submodule of M_1, by a similar proof N_2 is a Z-prime submodule of M_2.
The converse of the proposition is not true always. Consider the Z-module M=$Z_8 \oplus Z_8$ and let $N = <\bar{4}> \oplus <\bar{2}>$. It is clear that $<\bar{4}>$is a Z-primesubmodulesof $M$by remark (1.2) (1)and$<\bar{2}>$ is prime, then is Z-prime. But $N = <\bar{4}> \oplus <\bar{2}>$is not Z-prime submodule of $Z \oplus Z$ since if we define $f: Z \oplus Z \rightarrow Z$by $f(n,m) = n$,then$f(2,3).(2,3) = 2.(2,3) = (4,6) \in N$, but $(2,3) \notin N$and $f(2,3) = 2 \notin [<\bar{4}> \oplus <\bar{2}> : Z \oplus Z]$.

Proposition:-
Let $f: M \rightarrow$M' be an epimorphism. If $N$is a Z-prime submodule of Msuch that $kerf \subseteq N$, then $f(N)$is Z-prime submodule of M'.

Proof:-
Suppose that $f(N) = M'$ but f is an epimorphism, thus $f(N) = f(M)$ and hence $M = N + kerf$ but $kerf \subseteq N$. This implies that $M = N$, which is acontradiction
Now, let$h(x).x \in f(N)$, $x \in M'$, h $\in (M')^*$, and $x \notin f(N)$, we want to prove $h(x) \in [f(N):M']$, Since f is anepimorphism and $x \in M'$, then there exists$w \in M$ such that $f(w) = x$,$w \notin N$,$h(x).x = h(x)f(w) \in f(N)$ $=f(h(x).w) \in f(N)$ and since $kerf \subseteq N$ we get$h(x).w \in N$and N is Z-primesubmodule of M in M, $w \notin N$, then $h(x) \in [N:M]$, thus $h(x)M \subseteq N$. Now, $f(h(x)M) \subseteq f(N)$, h(x) f(M) $\subseteq$ f(N), then $h(x) \in [f(N):M']$.

Z-Prime Modules:-

Recallthat An R-module M is said to be a prime module if (0) is a prime submoduleof M,[2].

Definition:-

An R-module M is said to be a Z-prime module if (0) is a Z-prime submoduleof M.
A ring R is called Z-prime ring if (0) is a Z-prime ideal of R as R-module.

Remarks and examples:-

1. Every simple R-module is Z-prime. But the convers is not true in general. For example: The Z-module Z is a Z-prime module but not simple
2. It clear that every prime R-module is Z-prime module, but the converse is not true in general as the following example show: Z_8 as Z-module is Z-prime trivially but not primemodule, since $0 = \bar{8} = 2.\bar{4} \in (0)$ but $\bar{4} \notin (0)$ and $2 \notin [(0): Z_8]$
3. It is clear that$M = Z \oplus Z$ *is a* Z-prime Z- module, since $Z \oplus Z$ is prime module.
4. Every non-zero submodule of a Z-prime R-module is Z-prime.
5. A ring R is called a prime ring if for any $a, b \in R$, $a\,R\,b = \{0\}$, implies that either $a = 0$ or $b = 0$.Any domain is a prime ring.
6. Every prime ring is Z-prime ring, but the converse is not true, since if we take $R = Z_6$, then R is a Z-prime ring but not a prime ring $2.3 = \{0\}$ but either $a \neq 0$ or $b \neq 0$.

Recall that a submodule N of an R-module M is said to be pure if $IM \cap N = IN$ for all ideal $I \in R$,[5]

Proposition:-

Let M be a Z-prime module over principal ideal ring R. Then every proper puresubmodule of $M$is a Z-prime.

Proof::-

Len Na pure submoduleof an R-module M, $f \in M^* = Hom(M,R), x \in M$such that $f(x).x \in N$ and suppose $x \notin N$, thus $f(x).x \in < f(x) > M \cap N$, but N is a puresubmodule of M,implies that $f(x).x \in < f(x) > N$,which means that $f(x).x = f(x).x'$, for some $x' \in N$. Hence $f(x).(x - x') = 0$ and $x - x' \neq 0$, since M is a Z-primemodule therefore, $f(x) \in ann(M)$, which means that $f(x) \in [0: M]$ but $[0: M] \subseteq [N: M]$, hence $f(x) \in [N: M]$, thus N is Z-prime submodule of M.

The condition M is a Z-prime R-module cannot be dropped from above Proposition as the following example shows: Let $M = Z \oplus Z_{16}$as Z-module.Then M is not Z-prime. The submodule $N = Z \oplus (0)$ of M is pure, but N not Z-prime submodule of Msince if we take$f: Z \oplus Z_{16} \to Z$ such that $f(n, \bar{m}) = n$. $f(4, \bar{4}).(4, \bar{4}) = (16, \bar{0}) \in N$, but $4 \notin [Z \oplus (0): Z \oplus Z_{16}] = 16Z$and $(4, \bar{4}) \notin N$.

Recall that an R-module M is called divisible if and only if $rM = M$ for each non- zero$r \in R$, [7].

Corollary:-

Let R be a (PID), M is aZ-prime R- module M. If N is a divisible submoduleof M,then N is aZ-primesubmodule of M.

Proof:-

It is enough to show N is pure in M, since N is divisible submodule of M,then $rN = N$ for every $r \in R$ and so $rM \cap N = rM \cap rN = rN$. Thus N is pure, therefore N is a Z-prime submodule of M by proposition (2.3).

Proposition:-

Let M_1 and M_2 be two$R$-modulesand let $M = M_1 \oplus M_2$, if M is Z-prime$R$-module M,then M_1 and M_2 are Z-prime module.

Proof:-
Since M=$M_1 \oplus M_2$ is Z-prime R-module,then (0,0) is Z-prime submoduleof M, but $(0,0) = (0) \oplus (0)$ so by proposition(1.13),(0) is a Z-prime submodule of M_1 and M_2 .ThereforeM_1 and M_2 are Z-prime modules R-module by definition of Z-prime R- module.

Proposition:
Let M be a multiplication R-module, then M is Z-primeif and only if $ann(M)$ is a Z-prime ideal of R.

Proof:-
By definition of Z-prime module and corollary (1.6)

Proposition:-
Let R be an integral domain,if M is a faithful multiplication R-module, then M is a Z-prime R- module$M$.

Proof:-
Since M is faithful R-module, then $ann(M) = (0)$ but R is an integral domain,therefore $ann(M) = (0)$ is a prime idealof R, hence it is Z-prime ideal but M is a multiplication R-module,so M is Z-prime R- module by proposition(2.7).

Proposition:-
Let M be a multiplication faithful R-module, if R is Z-prime ring, then M is Z-prime R-module$M$.

Proof:-
Since R is a Z-prime ring,then (0) is Z-prime ideal of R. But$M$is a faithful R-module, hence$annM = (0)$, therefore $ann(M)$ is Z-prime ideal on the other hand M is multiplication R-module, therefore by proposition(2.6) M is a Z-prime module R-module M.

Corollary:-
Let R be an integral domain and M be faithful cyclic R-module, then M is Z-prime R-module M.

Corollary:-
Let R be an integral domain and M be divisible multiplication R-module M , then M is a Z-prime R-module .

Proposition:-
Let Nbe a proper submodule of an R-module M,if M/N is aZ-prime R- module,then $N$is$Z$-prime submodule of M.

Proof:-
Let $f(x).x \in N$; $f \in M^* = Hom(M,R)$, $x \in M$, supposex $\notin$ N, then $f(x).x + N = f(x)(x + N) = \bar{O}_{M/N} \in R$, $\bar{O}_{M/N}$is aZ-primesubmodule so either$rx + N = \bar{O}_{M/N}$ or$f(x) \in ann(M/N) = [N:M]$. Hence$x \in N$, which is a contradiction or$f(x) \in [N:M]$. Thus N is Z-prime submodule of M

Proposition:-
Let$N$ be a simple submodule of an R-module M, if M/N is a Z-prime, then M is Z-prime$module$.

Proof:-
Let$f(x).x = 0$; $f \in M^* = Hom(M,R)$,$x \in M$. Now, $f(x).x = 0 \in N$but M/N is a Z-prime$R$-module, so N is a Z-prime submodule of M by proposition(2.11), thus either $x \in N$or $f(x) \in [N:M]$ and so either $(x) \subseteq N$ or $f(x).M \subseteq N$, but N is a simple submodule of N, then either $x = 0$ or $f(x) \in ann(M)$ which means that (0) is a Z-prime submodule of M.Thus M is a Z-prime module.

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