Advance DSP – Geortzel Algorithm Implementation

***Abstract*— The Goertzel algorithm is one of the turns in digital signal processing that efficiently detects specific frequency components of a signal. In contrast to methods such as the Fast Fourier Transform, which examines all the frequencies, the Goertzel algorithm is applied to the selection of single frequencies or a few of them for analysis; hence, in situations where resources are limited, this method becomes very useful. This abstract considers the operational principles of the algorithm, which basically contains two major phases: processing and evaluation. During the processing phase, the algorithm works as a kind of digital filter, ringing on the target frequency; at the same time, the evaluation phase calculates the magnitude of this frequency component. Because of computational efficiency and simplicity, the Goertzel algorithm is applied perfectly in real-time systems, especially in systems with limited processing power. An especially notable application of the Goertzel algorithm in Dual-Tone Multi-Frequency decoding is that it will precisely detect the pairs of frequencies a telephone keypad generates and enables applications such as automated call routing. This abstract evidence the relevance and applicability of the Goertzel algorithm in DSP, with specific applications where the detection of certain, pre-defined targeted frequencies has to be made with very minimal computational overhead**.

Keywords—Goertzel Algorithm, Digital Signal Processing (DSP), Frequency Analysis, Discrete Fourier Transform (DFT), Infinite Impulse Response (IIR) Filter, Real-Time Signal Processing, DTMF Decoding, Frequency Detection, Single Frequency Bin, Tone Detection, Embedded Systems, Signal Processing Algorithms, Targeted Frequency Analysis

I. Introduction

The Goertzel algorithm is, therefore, a very core technique in the realm of digital signal processing, essentially related to the analysis and detection of some frequency components of a signal. Indeed, DSP is the backbone of modern technology in that it enables the manipulation and interpretation of digital signals in very diversified applications, be they in telecommunications, audio processing, or medical imaging. One of the cardinal problems in DSP is how, efficiently, to detect certain frequencies within a signal, especially when real-time performance is required and computational resources are limited. In the late 1950s, Gerald Goertzel suggested an algorithm that provided a computationally efficient solution for the isolation and examination of specific frequencies in a digital signal.

Much of the DSP application is centred on frequency analysis because the important information may be encoded in the frequency content of the signals. In telecommunication applications, for example, various channels or streams of data can be modulated on various frequencies, while in audio processing frequency analysis of a signal may become very vital in noise reduction or equalization. Traditional frequency analysis tools, such as the Fast Fourier Transform, are incredibly powerful means of gaining an overview of the frequency spectrum of a signal. However, they are sometimes rather computationally expensive when only a very few frequencies need to be known. That is where the Goertzel algorithm excels. The Goertzel algorithm is of immense use where computational efficiency is a big factor, since it is optimized for individual frequency or small sets of frequencies detection. In contrast to the FFT, the computation of the whole spectrum is done with some redundancy.

The Goertzel algorithm comprises two major phases: the processing phase and the evaluation phase. In the processing phase, it applies a difference equation to the input signal, much like a digital filter. This filter, however, is designed with coefficients that make it resonate at a target frequency, effectively amplifying the frequency component of interest while attenuating others. The difference equation used in this phase is such that it accumulates energy at the target frequency over time; the final value reflects the magnitude of the frequency component in question within the signal.

The evaluation phase of the Goertzel algorithm is the second stage, in which the algorithm uses the final states of the filter to compute the magnitude of the target frequency component. This involves final states of the filter derived for magnitude and effectively quantifying the strength of the target frequency within the signal to allow for precise detection and analysis.

One of the major advantages of the Goertzel algorithm is its efficiency. The algorithm performs a constant number of operations for each frequency to be estimated; hence, in cases where only a small set of frequencies is under analysis, it is much more efficient than the FFT. This makes the Goertzel Algorithm very useful in real-time applications and on systems with limited processing power or low memory, such as embedded systems. This algorithm is simple, and its simplicity makes it possible to be implemented on very basic hardware platforms; this ranges from microcontrollers to even digital signal processors.

An extremely practical application of the Goertzel algorithm is in Dual-Tone Multi-Frequency decoding. In fact, when any key on a dual-tone phone keypad is pressed, it essentially generates two different frequencies: one from the low-frequency bank and one from the high-frequency bank. Since the Goertzel algorithm is excellent at detecting such frequencies, it can easily detect which key has been pressed. This capability is very critical in telecommunication systems, including in applications such as automated call routing and IVR systems. The robustness and accuracy of the algorithm also make it quite effective in noisy environments, further greatly enhancing its utility in real-world applications.

Put simply, the Goertzel algorithm is an important tool in the area of digital signal processing, providing a very focused and efficient approach to analysing frequencies. It is an irreplaceable technique for applications as far apart as telecommunications and audio processing, due to the capability of this algorithm to isolate and analyse specific frequencies with minimum computational overhead in doing so. Understanding and implementing the Goertzel algorithm not only enhances the capacity of performing a selective frequency detection but also deepens the broader understanding of DSP concepts and their practical application in the different technological domains.

II. Equations

A. Difference Equation

The Goertzel algorithm is based on a difference equation at the core, which virtually equals to a digital resonator, for frequency detection. The basic difference equation is:

$$s[n]=x[n]+2\cos(\omega)\cdot s[n-1]-s[n-2]$$

In this equation, s[n] represents the current output of the filter at sample n, while x[n] is the input sample at the same point. The terms

s[n−1] and s[n−2] are the outputs of the filter from the previous two samples. The term ω denotes the normalized angular frequency of the target frequency, defined as $\omega=2\pi f/fs$, where f is the frequency of interest and fs is the sampling rate. By iterating this difference equation across all input samples, the algorithm effectively isolates the energy associated with the target frequency, enabling precise detection within the signal.

B. *Final State Calculation*

Once all samples have been processed through the difference equation, the Goertzel algorithm calculates the final states to determine the frequency component's real and imaginary parts. The real part is computed using:

$$Sr = s[N-1] - \cos(\omega) * s[N-2]$$

Here, s[N−1] and s[N−2] are the final outputs of the filter, representing the most recent samples. The term cos(ω) adjusts for the phase of the target frequency. The imaginary part is given by:

$$Si = \sin(\omega) \cdot s[N-2]$$

where sin(ω) accounts for the phase shift in the orthogonal direction. These components, Sr and Si, are then used to compute the magnitude of the target frequency:

$$|X(f)|=\sqrt{s_r^2 + S_i^2}$$

This magnitude provides a measure of the strength of the frequency f within the signal, allowing for accurate frequency detection.

C. *Alternate Difference Equation*

An alternate formulation of the Goertzel algorithm utilizes a variant of the difference equation:

$$Qn = x[n]+2\cos(2\pi k/N) * Q(n-1) - Q(n-2)$$

In this version, Qn is the output of the filter at sample n and $2\pi k/N$ represents the normalized angular frequency for discrete frequency bins. Here, k is the bin index corresponding to the frequency of interest, and N is the total number of samples in the analysis window. This formulation provides an alternative method for computing the frequency response, often used in discrete Fourier transform applications and signal analysis.

D. *Magnitude Calculation with Alternate Form*

Using the alternate difference equation, the squared magnitude of the frequency component k can be computed as:

$$|yk(N)| = Q2(n) + Q2(n-1) – 2\cos(2\pi k/N)\, Qn * Qn-1$$

In this equation, Qn and Qn-1 are the filter outputs at the final points in the analysis window. The term $2\cos(2\pi k/N)$ reflects the cosine coefficient associated with the frequency bin index k. This expression combines the squared outputs and cross-terms to determine the magnitude squared of the frequency component, offering an efficient way to compute frequency content.

E. *Simplified Final State Calculation*

A simplified form of the squared magnitude calculation is given by:

$$|Yk(N)|2 = Q2(n) + Q2(n-1) – Coeff * Qn * Qn-1$$

where Coeff is the coefficient calculated as $2\cos(2\pi k/N)$. This formula simplifies the magnitude calculation by incorporating the cosine term directly into the equation. The term Coeff adjusts for the phase shift of the target frequency, and the resulting magnitude squared reflects the strength of the frequency component in the signal.

F. *Summary*

These combined equations enable the Goertzel algorithm to be very effective in detecting frequencies within a digital signal. The difference equation serves to isolate a frequency of interest, while the final state calculations and magnitude equations follow through on the measurements and quantification of that frequency's presence. This set of alternate formulations with simplified calculations is useful for flexibility and efficiency in many signal processing situations.

III. TASKS

This will entail the study and implementation in C of the Goertzel algorithm for single frequency detection. This forms the basis of this project: understanding the basic working of the algorithm and translating it into a C program. The Goertzel algorithm will be efficient for the identification of specific frequencies in a signal. In applications like DTMF tone detection, this might turn out to be very useful. It will show how this algorithm filters out and processes a desired frequency, which becomes a stepping-stone toward more advanced frequency detection tasks.

The second task is an expansion in that it requires completing a project able to detect all the 15 DTMF digits. This involves the following steps: Copy the provided data bin file into the CCS project environment. You will need to work through the first three TODO sections of the code files provided to apply the Goertzel algorithm to decode the tone data from the file. Finally, you should fill in the last TODO section of util file so that this program outputs an audio file that contains the decoded tone and is similar to the reference data_audio.wav. The third assignment is to optimize the performance of the Goertzel algorithm. Concretely, it means that you have to modify the C code to use intrinsics and compiler optimizations and then measure computational cycles with and without these improvements. It is expected that this optimization would improve the efficiency of the algorithm so that, even with the additional complexity in it to detect multiple frequencies, it would perform well.

A. *Task 1*

The Goertzel algorithm, which is frequently used in applications like DTMF decoding in telecommunication systems, is an effective computational technique for identifying frequency components in a digital signal.

This task describes the application of this technique to identify, from provided data, a single pre-defined frequency.

The below figure represents the snippet of code written for accomplishing the task.

```
// to be completed
    input = (short) sample;

    double w = (2.0 * PI * freq1) / SAMPLING_RATE;
    coef_1 = 2.0 * cos(w);

    prod1 = coef_1 * delay_1;
    prod2 = delay_2;
    delay = input + prod1 - prod2;
    delay_2 = delay_1;
    delay_1 = delay;

    N++;

    if (N >= NO_OF_SAMPLES) {
        prod1 = delay_1 * delay_1;
        prod2 = delay_2 * delay_2;
        prod3 = delay_1 * coef_1 * delay_2;
        Goertzel_Value = sqrt(prod1 + prod2 - prod3);

        delay_1 = 0;
        delay_2 = 0;
        N = 0;

        gtz_out[0] = Goertzel_Value;
        System_printf("\n The GTZ for frequency = %d Hz , is = %d\n",freq1,gtz_out[0]);
    }

     gtz_out[0] = Goertzel_Value;

}
```

Fig. 1. Code Snippet for detecting one frequency

The implementation of the code is explained as follows :

w calculates the angular frequency in radians per sample, which is used to detect the target frequency freq. **coef_1** is calculated using the cosine of **w**, which is part of the Goertzel algorithm's recursive formula. **prod1** and **prod2** are intermediate variables used in the recursive formula. **delay**, **delay_1**, and **delay_2** are part of a recursive filter setup to compute the necessary terms over time. This sequence of assignments updates the delays which simulate the filter's memory.

The block inside the if statement executes once all the samples (**NO_OF_SAMPLES**) have been processed. It calculates the final

Goertzel value, which represents the magnitude of the target frequency component within the signal. **prod1**, **prod2**, and **prod3** compute the terms required to calculate the magnitude of the detected frequency using the Goertzel-specific algorithm. Goertzel value is computed using the square root of the sum and differences of these products, which yields the magnitude of the frequency component at freq. The delays and sample counter **N** are reset for the next batch of processing.

In this task, a sample with single tone at 697 Hz is generated and then detected.
Fig 2 shows the magnitude of the frequency 697 Hz.

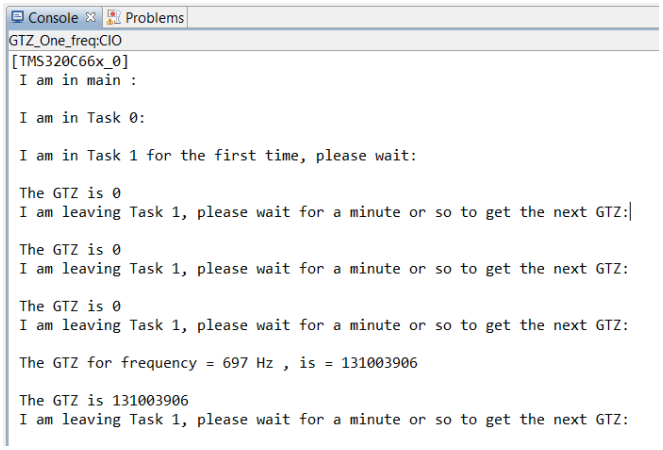

Fig. 2. Goertzel value for 697 Hz

In order to make sure that the code works properly it was run to detect other requencies. Fig 3 shows the magnitude for the frequency 941 Hz.

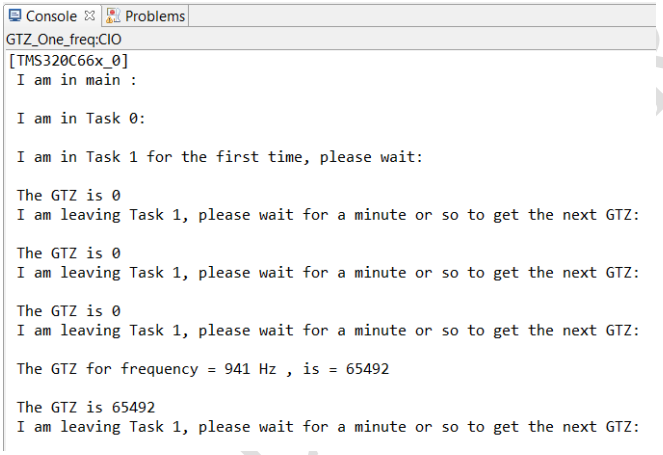

Fig. 3. Goertzel values for 941 Hz

It can be observed that the magnitude for 697 Hz is more than 941 Hz, which confirms the reliability of the code.

B. Task 2

This task explains the implementation of the Goertzel algorithm to detect Dual-tone multi-frequency (DTMF) signals. The algorithm decodes the provided 8-digit tone from a binary data file and generates an equivalent audio file.

a) This part consists of reading the data from the given binary file, as shown in the below code snippet of Fig 4.

```
47    /* Read binary data file */
48    FILE* fp = fopen("../data.bin", "rb");
49    if(fp==0) {
50        System_printf("Error: Data file not found\n");
51        System_flush();
52        return 1;
53    }
54    fread(data, 2, NO_OF_SAMPLES, fp);
55    buffer = (short*)malloc(2*8*10000);
```

Fig. 4. Code Snippet for reading binary data

The file **data.bin** is opened in read-binary mode. If the file doesn't exist or can't be opened, an error message is printed, and the program exits with an error code. The program reads **NO_OF_SAMPLES** of 2-byte elements from the file into the data array. Hence the data read is of the format 'signed int16'. Memory is allocated dynamically for buffer. The size allocated here is sufficient to hold a large number of samples i.e., 160,000 short integers.

b) This part implements the feedback and feedforward loop of the Goertzel algorithm to detect all the eight frequencies and based on the Goertzel value, the key pressed is found out.

TODO 1 implements the feedback loop of the Goertzel algorithm, shown in the Fig 5.

```
97    //Record start time
98    start = Timestamp_get32();
99    static int Goertzel_Value = 0;
100   short input = (short) (sample);
101   /* TODO 1. Complete the feedback loop of the GTZ algorithm*/
102   /* ======================== */
103   static float s_prev[8] = {0};
104   static float s_prev2[8] = {0};
105   int j,k;
106   float omega, cosine, coeff, q0;
107   for (j = 0; j < 8; j++) {
108       omega = (2*PI*freq[j])/SAMPLING_RATE;
109       cosine = cos(omega);
110       coeff = 2.0 * cosine;
111       q0 = coeff * s_prev[j] - s_prev2[j] + input;
112       s_prev2[j] = s_prev[j];
113       s_prev[j] = q0;
114   }
115   /* ======================== */
116   N++;
117   //Record stop time
118   stop = Timestamp_get32();
119   //Record elapsed time
120   tdiff = stop-start;
```

Fig. 5. Code for TODO 1

s_prev and **s_prev2** are arrays that store state values for each of the 8 frequencies being monitored. For each of the 8 frequencies, the code calculates the normalized frequency **omega**, its cosine, and the Goertzel coefficient. Using the Goertzel algorithm formula, the code updates the state variables based on the current input and the previous states. This loop effectively filters the input signal and prepares the state variables for detecting the presence of specific frequencies associated with DTMF tones.

TODO 2 completes the feedforward loop of the Goertzel algorithm, shown in the Fig 6.

```
122     if (N == 206) {
123         //Record start time
124         start = Timestamp_get32();
125         /* TODO 2. Complete the feedforward loop of the GTZ algorithm*/
126         /* ========================= */
127         for (k = 0; k < 8; k++) {
128             float prod1 = s_prev[k]*s_prev[k];
129             float prod2 = s_prev2[k]*s_prev2[k];
130             float res = prod1 + prod2 - coeff*s_prev[k]*s_prev2[k];
131             gtz_out[k] = sqrt(res);
132             s_prev[k] = 0;
133             s_prev2[k] = 0;
134             }
135         /* gtz_out[..] = ... */
136         /* ========================= */
137         flag = 1;
138         N = 0;
139         //Record stop time
140         stop = Timestamp_get32();
141         //Record elapsed time
142         tdiff_final = stop-start;
143     }
```

Fig. 6. Code for TODO 2

The algorithm runs the feedforward loop after processing a specific number of samples (206 in this case). For each of the 8 frequencies, the state variables are processed to get the final magnitude. The magnitudes are stored in the **gtz_out** array.

TODO 3 provides implementation of the Goertzel algorithm for detecting DTMF (Dual-tone multi-frequency) tones on the Goertzel values output, shown in the Fig 7.

```
56      for(n=0;n<8;n++) {
57          while (!flag) Task_sleep(210);
58          /* TODO 3. Complete code to detect the 8 digits based on the GTZ output */
59          /* ========================= */
60          int max_row_val = 0;
61          int max_col_val = 0;
62          for(i = 0; i < 4; i++) {
63              if(gtz_out[i] > max_row_val) {
64                  max_row_val = gtz_out[i];
65                  row = i;}
66          }
67          for(i = 4; i < 8; i++) {
68              if(gtz_out[i] > max_col_val) {
69                  max_col_val = gtz_out[i];
70                  col = i-4;}
71          }
72          result[n] = pad[row][col];
73          /* result[n] = ... */
74          /* ========================= */
75          printf("Key pressed = %c\n", result[n]);
76          flag = 0;
77      }
78      printf("\nDetection finished\n\n");
```

Fig. 7. Code for TODO 3

The code loops through the first four frequencies to find the row frequency with the highest magnitude and then through the next four frequencies to find the highest column frequency. The detected row and column indices help to look up the corresponding digit and store it in the **result** array. For each sample the detection loops goes and the corresponding **result** array gets populated.

c) This part implements the code to generate DTMF tones based on the keys detected, and finally generating an audio file.

The implementation part is shown in the Fig 8.

```
88  void task2_dtmfGenerate(char* keys){
89      int length = strlen(keys);
90      int total_samples = length * NUM_SAMPLES;
91      int16_t *samples = (int16_t *)malloc(total_samples * sizeof(int16_t));
92      int i;
93      for (i = 0; i < length; i++) {
94          generateDTMFTone(samples + i * NUM_SAMPLES, keys[i], NUM_SAMPLES);
95      }
96      FILE *wav_file = fopen("decoded_audio.wav", "wb");
97      if (!wav_file) {
98          perror("Unable to open file");
99          free(samples);
100     }
101     writeWAVHeader(wav_file, total_samples);
102     fwrite(samples, sizeof(int16_t), total_samples, wav_file);
103     fclose(wav_file);
104     free(samples);
105     printf("DTMF sequence saved to audio file\n\n");
106 }
```

Fig. 8. Code for generating audio file

The algorithm starts with determining the number of keys and allocating memory for the total samples required for the DTMF sequence. Here there were in total 8 keys. For each key the duration of the tone specified was 0.5 seconds. Hence with a sampling rate of
8 KHz, each key requires 4000 samples. The function **generateDTMFTone** loops through the keys detected to generate the DTMF tone and store it in the appropriate position in the samples array. Post generation, the samples are written to the audio file.

Fig 9 shows the detected keys present in the given binary data file. And the corresponding audio file is generated too.

```
Console  Problems
GTZ_all_freq_2024:CIO
[TMS320C66x_0]
 System Start

Key pressed = 1
Key pressed = 2
Key pressed = 3
Key pressed = 4
Key pressed = 5
Key pressed = 6
Key pressed = 7
Key pressed = 8

Detection finished

DTMF sequence saved to audio file

No of cycles for Feedback loop = 3996
No of cycles for Feedforward loop = 1603

Finished
```

Fig. 9. Result for task 2

- Use either SI (MKS) or CGS as primary units. (SI units are encouraged.) English units may be used as secondary units (in parentheses). An exception would be the use of English units as identifiers in trade, such as “3.5-inch disk drive”.

C. *Task 3*

This task quantifies the performance of the Goertzel algorithm by counting the number of clock cycles required to loop through the algorithm. The cycles for feedback and feedforward loops are tallied separately.
Fig 5 and Fig 6 shows the code for calculating the time elapsed through feedback and feedforward loops respectively.

Fig 9 shows the number of cycles took for feedback and feedforward loops before optimization which are 3996 and 1603 respectively.

The cycle count of the Goertzel algorithm can be reduced using compiler switches and intrinsics.

Compiler switches are commands or settings that are used during the compilation process to modify the behaviour and processing of the compiler. They are often referred to as compiler options or compiler flags. The optimization level, the kinds of warnings that are generated, the debug information that is included in the output, and many other elements of the compilation process may all be impacted by these settings.
Intrinsics are a kind of function that many compilers offer that let programmers access processor-specific instructions straight out of the box without requiring them to create assembly code. By allowing software to leverage the sophisticated features of contemporary processors straight from code, intrinsics fill the gap between high- level languages and low-level assembly.

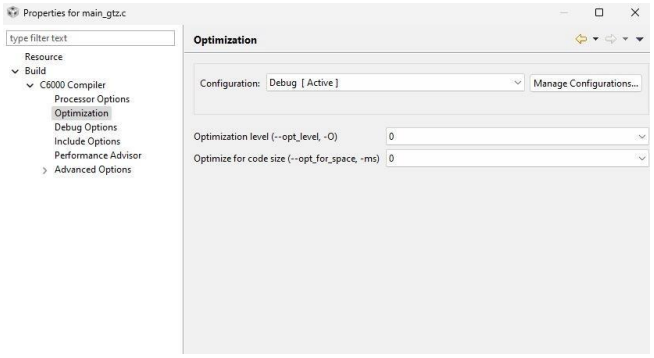

Fig. 10. Values before optimizing

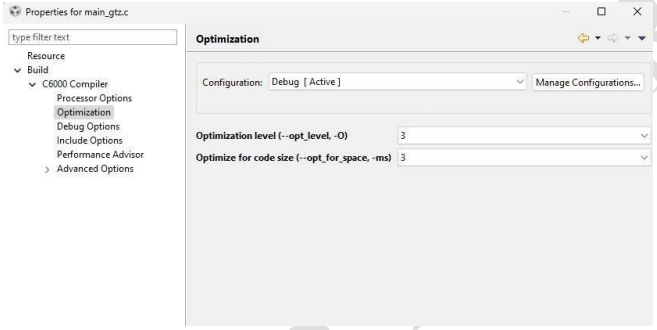

Fig. 11. Values optimizing to

The optimizations used here are **Optimization Level (--opt level, -O)** and **Optimize for code size (--opt_for_space,-ms)**. '**-O'** controls the general optimization level applied by the compiler when generating code. Here level 3 is employed. This level improves the performance of the resulting executable code at the cost of longer compile time. It enables more aggressive optimizations which includes loop unrolling, inlining of functions, and vectorization. '**- ms'** directs the compiler to reduce the size of the generated code, which is particularly useful for memory-constrained environments. Here level 3 is used. At this level, the compiler aggressively changes the way code is generated to minimize the footprint, which includes omitting inline expansions, using smaller libraries and techniques that reduce size at the cost of execution speed.

Fig 10 shows the number of cycles obtained after optimization which are 3600 and 1148 respectively.

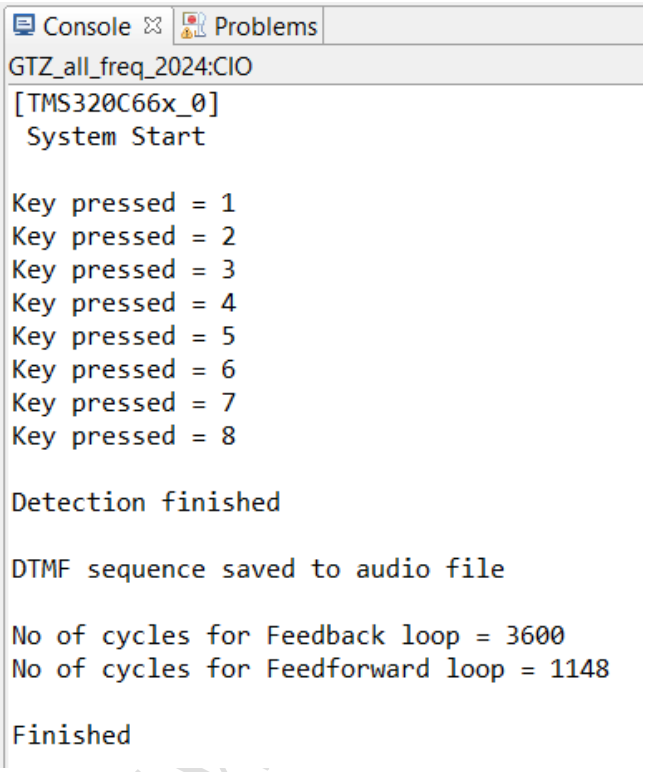

Fig. 12. Number of cycles after optimization

The optimization of the Goertzel algorithm through compiler flags and intrinsics decently enhanced its execution efficiency. These results highlight the importance of targeted optimizations in the development of high-performance DSP applications.

IV. Conclusion

This report shall therefore extensively cover all aspects of the implementation and optimization of the Goertzel algorithm in C for frequency detection tasks. The first task involves studying, then implementing, the Goertzel algorithm to detect a single frequency that has been successful in showing foundational steps needed for isolating and analyzing certain frequency components in a digital signal. This implementation was checked with the use of code snippets and frequency magnitude visualizations, which proved the accuracy and reliability of the algorithm. In turn, this created the basis for more complex tasks of multiple frequency detection.

In second task was the extension of the Goertzel algorithm to detect all 15 DTMF digits used in telecommunication systems. It read and decoded an 8-digit tone from a binary data file, generated a corresponding audio file that would compare with a reference audio, showed in detail the code snippets of the feedback and feedforward loops of the algorithm, and managed to correctly decode the tones. Finally, optimization of the Goertzel Algorithm is explained toward the end of the report, where performance was improved by activating compiler switches and intrinsics. It measured the cycle count before and after optimization. The report presented an effective decrease in processing time and pointed out the necessity of such optimizations in digital signal processing applications. All tasks performed accurately, hence ensuring that the Goertzel algorithm is effective for both single and multi-frequency detection cases.

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