Shape and Topological Optimization of a Nonlinear Elliptical Problem

Abstract

In this work, let us deal with existence and derivation results in shape optimization. It should be noted that a shape optimization problem does not generally have a solution with only its initial data. To get around the non-existence of solution, we impose geometric order restrictions (i.e. volume type) and we work with the open class checking the ε-cone property to obtain existence. On the other hand, we determine the shape derivative using the Lagrange method. And then we establish the topological derivative using the minmax method.

keywords: thermoelasticity problem, shape derivative, shape gradient, minmax.

1 Introduction

In this paper, we are interested with shape optimization problems using the functional

$$J(\Omega) = a\int_{\Omega} |\nabla u_{\Omega} - \nabla v_0|^2 dx + b \int_{\Omega} |u_{\Omega} - v_1|^2 dx. \tag{1.1}$$

where a and b two real numbers, v_0 (*respectively* v_1) are the given functions of $H_{loc}^{1}(\mathbb{R}^N)$ (*respectively* $L^2_{loc}(\mathbb{R}^N)$) and u_{Ω} is the solution of the following Neumann problem :

$$\begin{cases} -\nabla u_{\Omega} + u^q = f \ in\ \Omega \\ \frac{\partial u_{\Omega}}{\partial n} = 0 \ on\ \partial\Omega \end{cases} \tag{1.2}$$

.

The objective of this paper is fixed around three main axes, that is to say the existence of optimal shape solution, the shape derivative using vector fields and the topological derivative using the minmax method. These types of problems have been studied by many authors who can be cited [2, 3, 9, 10, 7, 8, 9, 11, 12, 15]

We will give existence results by adding constraints, either on the functional to be minimized or on the set of admissible domains. But we can also increase volume constraints. A minimization problem does not always admit a solution with only its initial data. Thus, it will be a question of giving existence results assuming that the edge is uniformly regular.

Let us denote by O_{ad} the set of admissible open sets. We assume that this set satisfies the following properties:

$$O_{ad} \subset O_{\varepsilon}$$

(the set of open checking the property of the $\varepsilon - cone$). It is also closed for one of the three types of convergence, namely convergence in the sense of Hausdorff, in the sense of characteristic functions or in the sense of compacts.

The main in this paper is to determine the shape derivative and the topological derivative of the functional J(O_ε)= J(O_ε, u_ε), where the perturbed domain Ω_ε *of* Ω is defined by $\Omega_\varepsilon = T_\varepsilon(\Omega)$ *or* $\Omega_\varepsilon = \Omega \backslash E_\varepsilon$ *on the derivative to be calculated.*

The paper is organized as follows: In the first section we give the introduction. In the second section, we establish the existence of optimal form Section 3 is reserved for the form derivative of the functional using the Lagrange method. In this part we first give an example of application of the derivative in the sense of Hadamard. Then we apply them to the energy functionals. In the section 4 we give the topological derivative using minmax method. And in the section 5, we give the conclusion of the work.

2 Existence of a solution by the -cone property

We consider a functional of the form:

$$J1(\Omega) = \int_\Omega F(x, u_\Omega, \nabla u_\Omega) dx \qquad (3.1)$$

where

$$F : B \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$$

is a continuous function, measurable in (x,r,p) and verifying the hypothesis

$$|F(x, r, p)| \leq c(1 + r^2 + |p|^2) \ \forall x \in B, \forall r \in \mathbb{R}, \forall p \in \mathbb{R}^N . \qquad (3.2)$$

With $u = u_\Omega$ of the following Neumann problem:

$$\begin{cases} -\Delta u + u^q = f \ in \ \Omega \\ \dfrac{\partial u}{\partial n} = 0 \quad on \ \partial\Omega \end{cases} \qquad (3.4)$$

with $\Omega \subset B$ (where B is an open $\mathbb{R}^N$) and $f \in L^2(B)$.

We further consider the following functional:

$$J2(\Omega) = \int_\Omega F(x, v_\Omega(x), 0) dx + \alpha \int_\Omega |\nabla v_\Omega(x)|^2 dx \qquad (3.5)$$

with $\alpha \geq 0$ and v_Ω solution of (3.3) or(3.4).

We then ask :

$$J(\omega) = \int_\omega F(x, u_\omega(x), \nabla u_\omega) dx. \qquad (3.6)$$

We notice that J is well defined, because F is, by hypothesis, a function measurable in (r,p) p.p. Thus, we show that $J(\omega) < +\infty$. Indeed, according to (3.2) we have:

$$|J(\omega)| = \left|\int_{\omega} F(x, u_{\omega}(x), \nabla u_{\omega}(x))dx\right| \leq \int_{\omega} c(1 + u_{\Omega}(x)^2 + |\nabla u_{\Omega}(x)|^2)dx,$$

$$\leq c\int_{\omega} (1 + u_{\Omega}(x)^2 + |\nabla u_{\Omega}(x)|^2)dx.$$

Furthermore, we can increase

$|J(\omega)| \leq c(\|u_{\omega}\|_{H^1} + \|\omega\|) < +\infty.$

So $|J(\omega)| < +\infty$, hence $J(\omega)$ is well defined. We therefore recall that problems (3.3) and (3.4) are well posed in the sense of Hadamard. Indeed, we have specified the conditions at the edges of Dirichlet and Neumann types.

In all that follows, we set $\varepsilon > 0$ and we consider the set O_{ε} defined by:

$$O_{\varepsilon} = \{\Omega \text{ open}, \Omega \subset D, \Omega \text{ has the property of } \varepsilon\text{-cone}\}.$$

We therefore consider the following shape optimization problem:

$$\min\{J(\Omega) : \Omega \in O_{\varepsilon}\},$$

where J designates the functional of type J_1 or J_2. In all that follows, we seek to determine optimal shape existence results for shape optimization problems. But before giving optimal form existence results, we need the following results:

Theorem 2.1 *: Let Ω_n be an open sequence in the class O_{ε}. Then there exists an open $\Omega \in \mathcal{O}_{\epsilon}$ and an sub-sequence Ω_{n_k} which converges towards Ω both in the sense of Hausdorff, in the sense of the characteristic functions and in the sense of compact. In addition, $\overline{\Omega_{n_k}}$ and $\partial\Omega_{n_k}$ converge in the Hausdorff sense respectively to $\overline{\Omega}$ and $\partial\Omega$.*

LEMMA 2.2 *: Let K be a compact and B a bounded open of R^N. Let Ω_n be a sequence of open with $\overline{\Omega_n} \subset K \subset B$, verifying the ownership of the -cone.*
Then there is an open Ω verifying the ownership of the -cone and an extracted sequence Ω_{nk} such as

$$\mathrm{u}\Omega_{n_k} \xrightarrow{H} \Omega,\ \chi\Omega_{n_k} \xrightarrow{L^1 p.p} \chi\Omega$$

$$\overline{\Omega_{n_k}} \xrightarrow{H} \overline{\Omega},\quad \partial\Omega_{n_k} \xrightarrow{H} \partial\Omega$$

It is a result which will allow us to characterize the existence of solution.

Preuve. Pour la preuve, voir [1].

Consider the following homogeneous Neumann equation:

$$\begin{cases} -\Delta u_\Omega + {u_\Omega}^q = f \ in\ \Omega \\ \dfrac{\partial u_\Omega}{\partial n} = 0 \qquad on\ \partial\Omega \end{cases} \tag{3.7}$$

So, by doing the variational formulation and integrating, we have according to Green's formula:

$$v \in H^1(\Omega), \int_\Omega \nabla u_\Omega . \nabla v dx + \int_\Omega {u_\Omega}^q dx = \int_\Omega f v dx,$$

with $f \in L^2(B)$.
In what follows, we focus on the fundamental result of the game.

Theorem 2.3 *Let $O_{ad} \subset O_\varepsilon$ be a non-empty set of open sets satisfying a closure property for convergence in the sense of Hausdorff, F a function which satisfies (3.2) and J_1 (respectively J_2) defined by (3.1) (respectively by (3.5)). Then, there exists $\Omega \in O_{ad}$ which minimizes J_1 (respectively J_2).*

Proof. Let us show that J_1 is bounded.
We have

$$|J_1(\Omega_n)| = \left| \int_{\Omega_n} F\left(x, u_{\Omega_n}(x), \nabla u_{\Omega_n}(x)\right) dx \right| \leq c\left(\|u_{\Omega_n}\|_{H^1} + |\Omega_n|\right) < +\infty,$$

which shows that $J_1(\omega_n)$ is increased. Moreover,

$$|J_1(\Omega_n)| = \left| \int_{\Omega_n} F\left(x, u_{\Omega_n}(x), \nabla u_{u_{\omega_n}}(x)\right) dx \right|$$

and $J_1(\Omega_n) > -\infty$ because $u_{\Omega_n} \in H^1$. So $J_1(\Omega_n)$ is reduced. Thus, $J_1(\Omega_n)$ is bounded. Let's ask

$$\mathrm{m} = \inf_{\Omega \in O_\epsilon \ or \ O_{ad}} J_1(\Omega) \tag{3.8}$$

Then, according to the properties of the lower bound, there exists a minimizing sequence (ω_n) of O_{ad} such that

$J_1(\Omega) \to \mathrm{m} = \inf_{\Omega \in O_\epsilon \ or \ O_{ad}} J_1(\Omega)$.

Let $\Omega_n \in O_{ad}$. According to Theorem 2.1, there exists an open $\Omega \in O_\varepsilon$ and an extracted sequence (Ω_{n_k}) which converges to Ω in the Hausdorff sense. Like $\Omega_n \in O_{ad} \subset O_\varepsilon$, the sequence (Ω_n) verifies the property of the ε-cone. According to lemma (2.2), we can extract from the sequence (Ω_n) a subsequence $(u_{\Omega_{n_k}})$ which verifies the following convergences:

$$\Omega_{n_k} \xrightarrow{H} \Omega, \ \chi\Omega_{n_k} \to \chi_\Omega \text{ in } L^1 \text{ (p.p.)},$$

$$u_{\Omega_{n_k}} \xrightarrow{H} \Omega, \qquad \partial u_{\Omega_{n_k}} \xrightarrow{H} \partial\Omega.$$

with Ω verifying the ε-cone property.
It will now be a matter of showing that:

$$lim\, J1(\Omega n) = J(\omega) = \inf_{\Omega \in O_\epsilon \ or \ O_{ad}} J_1(\Omega) \quad .$$

■

Let us make the variational formulation of the Poisson problem with Neumann condition at the boundary:

$$\begin{cases} -\Delta u_\Omega + {u_\Omega}^q = f \ in\ \Omega \\ \frac{\partial u_\Omega}{\partial n} = 0 \qquad on\ \partial\Omega \end{cases}$$

By multiplying the above equation by a test function $\phi \in H^1(\Omega)$ and integrating, we have :

Or, $\int_{\partial\Omega} \frac{\partial u_\Omega}{\partial n} \varphi d\sigma = 0$ then:

$$\int_\Omega \nabla_{u_\Omega} . \nabla\varphi dx + \int_\Omega {u_\Omega}^q \varphi dx = \int_\Omega f\varphi dx$$

Thus, according to the Lax-Milgram theorem, we can show the existence of a unique solution to this problem.

So, in Ω_n, we have the following variational formulation:

$$\int_{\Omega_n} \nabla_{u_\Omega} . \nabla\varphi dx + \int_{\Omega_n} {u_\Omega}^q \varphi dx = \int_{\Omega_n} f\varphi dx.$$

Since $O_{ad} \subset O_\varepsilon$, we can define an extension in Ω_n by: there exists an operator

$$P_{\Omega n} : H^1(\Omega_n) \rightarrow H^1(B),$$

with B a bounded open of R^N, such that

$$P\Omega_n(u_{\Omega_n}) = \begin{cases} u_{\Omega_n} & \text{if } x \in \Omega n \\ 0 & \text{otherwise,} \end{cases}$$

So either

$$\widetilde{u}_n = \begin{cases} u_n & \text{if } x \in \Omega_n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\varphi} = \begin{cases} \varphi \ if\ x \in \Omega n \\ 0 \ otherwise \end{cases}$$

So, in $u_{\Omega_{n_k}}$ we have the following variational formulation:

$$\int_{\Omega_{n_k}} \nabla_{u_\Omega} . \nabla\varphi dx + \int_{\Omega_{n_k}} {u_{\Omega_{n_k}}}^q \varphi dx = \int_{\Omega_{n_k}} f\varphi dx. \qquad \text{(a)}$$

Taking $\varphi = u_{\Omega_{n_k}}$, we obtain:

$$\int_B |\nabla u\Omega_{nk}|^2 dx + \int_B u|^2 \Omega_{nk} dx + \int_B f u_{\Omega_{nk}} dx.$$

Thus, we have:

$$\left\| u_{\Omega_{nk}} \right\|^2 H_{1(B)} \leq C \|f\| L^2(\Omega) \left\| u_{\Omega_{nk}} \right\| L^2(\Omega)$$

from which it follows that

$$\left\|u_{\Omega_{n_k}}\right\| \; H^{1}(B) \leq C\|f\|L^{2}(\Omega)$$

Therefore, the sequence $(u_{\Omega_{n_k}})$ is bounded in $H^1(B)$.

Since $H^1(B)$ is a reflexive Hilbert space, there exists $u^* \in H^1(B)$ such that

$$u_{\Omega_{n_k}} \rightarrow u^* \quad \text{weakly in } H^1(B),$$

$$u_{\Omega_{n_k}} \rightarrow u^* \quad \text{in } L^2(B) \text{ (strongly)}.$$

Let us now show that:

$$\int_\Omega \nabla u^* \cdot \nabla\varphi dx + \int_\Omega (u^*)^q \varphi dx = \int_\Omega f\varphi dx, \qquad \forall\varphi\epsilon\, H^1(\Omega).$$

For this, given that $\varphi \in D(\Omega)$, there exists a certain rank from which $\varphi \in D(\Omega_n)$. Thus, by multiplying equality (a) by $\chi\Omega_{n_k}$, we have:

$$\int_B \chi_{\Omega_{n_k}} \nabla u_\Omega . \nabla\varphi dx + \int_B \chi_{\Omega_{n_k}} u_{\Omega_{n_k}}{}^q \varphi dx = \int_B \Omega_{n_k} f\varphi dx, \quad \forall\varphi\epsilon H^1(B).$$

Since $\chi\Omega_{n_k} \rightarrow \chi_\Omega$ in $L^1(B)$ (p.p.),

and using the weak convergence in $H^1(B)$ of $u_{\Omega_{n_k}}$, passing to the limit when $k \rightarrow +\infty$, we ob

$$\chi_{\Omega_{n_k}} \frac{\partial\varphi}{\partial_{x_i}} \rightarrow \chi_\Omega \frac{\partial\varphi}{\partial_{x_i}} \quad in\ L^2(B),$$

$$\frac{\partial\Omega_{n_k}}{\partial_{x_i}} \rightarrow \frac{\partial u^*}{\partial_{x_i}} \quad in\ L^2(B)$$

$$\chi u_{\Omega_{n_k}} \nabla u_{\Omega_{n_k}} \rightarrow \chi_\Omega \nabla u^* \qquad \text{in } L^2(B),$$

$$\chi\Omega_{n_k} \mathrm{u}\Omega_{n_k} \rightarrow \chi_\Omega u^* \qquad \text{in } L^1(B).$$

Thus, we have:

$$\int_B \chi u_{\Omega_{n_k}} \nabla u_{\Omega_{n_k}} . \nabla\varphi dx \rightarrow \int_B \chi_\Omega \nabla u^* . \nabla\varphi dx = \int_\Omega \nabla u^* \nabla\varphi dx.$$

Finally, we obtain:

$$\int_\Omega \nabla u_\Omega{}^* . \nabla\varphi dx + \int_\Omega (\nabla u_\Omega^*)^q \varphi dx = \int_\Omega f\varphi dx$$

Let us show that $u_\Omega^* = u_\Omega$

Using Green's formula in the variational formulation (b), we have:

$$-\int_{\Omega} \nabla u_{\Omega}{}^{*} \varphi dx + \int_{\Omega} (u_{\Omega}^{*})^{q} \varphi dx = \int_{\Omega} f \varphi dx, \qquad \forall \varphi \epsilon H^{1}(\Omega), \quad .$$

Thus, we obtain:

$$-\nabla u_{\Omega}{}^{*} + (u_{\Omega}^{*})^{q} = f \text{ with } \frac{\partial u_{\Omega}^{*}}{\partial n} = 0.$$

We also need to show that the sequence $u_{\Omega_{n_k}} \rightarrow u_{\Omega}$ in $H^1(\Omega)$. Taking $\phi = u_{\Omega_{nk}}$ in (a) and $\phi = u_{\Omega}$ in (b), we have:

$$lim \int_{\Omega_{nk}} \left(|\nabla u_{\Omega_{nk}}|^2 + \Omega_{nk} \right) dx = lim \int_{B} \chi u_{\Omega_{nk}} f u_{\Omega_{nk}} dx.$$

However, we also have:

$$\int_{B} \chi u_{\Omega_{nk}} f u_{\Omega_{nk}} dx. \rightarrow \int_{B} \chi \Omega_{nk} f u_{\Omega}^{*} dx = \int_{\Omega} |\nabla u_{\Omega}^{*}|^2 + (u_{\Omega}^{*})^{2} dx.$$

Since $\chi u_{\Omega_{n_k}} u_{\Omega_{n_k}}$ converges strongly in $L^2(B)$ to $\chi_{\Omega} \nabla u_{\Omega}$, we have:

$$\int_{\Omega_{nk}} |\nabla u_{\Omega_{nk}} - \nabla u_{\Omega}|^{2} dx = \int_{\Omega_{nk}} |\nabla u_{\Omega_{nk}}|^{2} dx - 2\int_{\Omega_{nk}} \nabla u_{\Omega_{nk}} \cdot \nabla u_{\Omega} dx + \int_{\Omega_{nk}} |\nabla u_{\Omega}|^{2} dx$$

By taking the limit, the second term on the right becomes zero, and therefore:

$$\text{Lim} \int_{\Omega_{nk}} |\nabla u_{\Omega_{nk}} - \nabla u_{\Omega}|^{2} = 0$$

In the same way, we show that:

$$\text{Lim} \int_{\Omega_{nk}} |u_{\Omega_{nk}} - u_{\Omega}| dx = 0$$

Thus, we obtain:

$$\nabla u_{\Omega_{nk}} \overset{L^2}{\rightarrow} \nabla u_{\Omega}, \; u_{\Omega_{nk}} \overset{L^2}{\rightarrow} u_{\Omega}$$

Since F is a continuous function, we have:

$$J(\Omega_{n\,k}) = \int_{\Omega_{nk}} F(x, u_{\Omega_{nk}}, \nabla \text{u}\Omega_{nk}) dx \rightarrow \int_{\Omega} F(x, u_{\Omega}) dx.$$

3 Shape derivative :

The objective of this section to calculate the shape derive of the functional 3.1. Before going further we first prove the following results which as useful for the main result. The idea is to use the celebrated method of Hadamard for the shape functional that we considered. This method was introduced by Hadamard in [14] and many other authors [2]. In there papers, the notions of shape derivative is given.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class C^2. For $t \geq 0$, let $\Omega_t = \phi_t(\Omega)$, where for all t, ϕ_t associated for V is diffeomorphism of R^2. These properties holds:

$$\phi_0 = V, |\det(\nabla\phi\ t)| = j(t,x), \frac{d\emptyset_t}{dt} = -V, |\det(\nabla\emptyset_t^{-1}| = j(-t,x).$$

Let $\Omega_t = (Id + V)(\Omega)$ be a demain of class C^2. For $t \geq 0$, very small, and $V \in C^1 \cap W^{1,\infty}(\mathbb{R}^2)$. Let us consider also, the function J in Ω_t. We have the following definition:

Definition 3.1 One function $J(\Omega)$ of the domain is said to be shape differentiable at Ω if the mapping $t \rightarrow J(\Omega_t)$ from R into R is Frechet differentiable at $t = 0$. The corresponding Frechet derivative (or differential) is denoted by $DJ(\Omega,V)$ and the following expansion holds:

$$J(\Omega_t) = J(\Omega) + tDJ(\Omega,V) + o(t).$$

In the following, consider also then functional defined in Ω_t, by

$$J(\Omega_t) = a\int_{\Omega_t} |\nabla u_\Omega - \nabla v0|^2 dx + b\int_{\Omega_t} |u_\Omega - v_1|^2\, dx \tag{3.1}$$

where u_t be the solution to the following problem

where a and b two real numbers, v_0 (*respectively* v_1) are the given functions of $H_{loc}^1(\mathbb{R}^N)$ (*respectively* $L^2_{loc}(\mathbb{R}^N)$) and u_Ω is the solution of the following Dirichlet problem :

$$\begin{cases} -\nabla u_{\Omega_t} + u_{\Omega_t}^q = f \ in\ \Omega_t \\ u_{\Omega_t} = 0\ on\ \partial\Omega_t \end{cases} \tag{3.2}$$

We look, in this section for the shape derivative of the functional $J(\Omega)$. The key point in the calculation of the shape derivative $DJ(\Omega,V)$ is in general, the definition of an appropriate derivation for the mapping $\Omega \rightarrow u_\Omega$. This mapping has a Lagrangian derivative $\dot{u}_\Omega$ and an Eulerian derivative u'_Ω linking with the Laplacian derivative by $u'_\Omega = \dot{u}_\Omega - \nabla u_\Omega \cdot V$. For the definition of the Laplacian and eulerian derivative, we refer to [1], [2]. The following result is devoted to the shape derivative of the functional.

Theorem 3.1 *Lets Ω a class domain $C^1(\mathbb{R}^N)$ and V a class vector field C^1.*
Let.$F \epsilon C^1\left((0,\epsilon) C^0(\overline{\Omega_t})\right) \cap C^0\left((0,\in), C^1(\overline{\Omega_t})\right)$. *The function defined by*

$$J_1(\in) = \int_{\Omega_t} F(\in,x)dx \tag{3.3}$$

is differentiable and its derivative is given by :

$$DJ_1(\Omega t, V) = \int_{\Omega_\in} \frac{\partial}{\partial \in} F(\in, x) + divF(\in, x)V(x))dx \tag{3.4}$$

$$DJ_1(\Omega t, V) = \int_{\Omega_\in} \frac{\partial}{\partial \in} F(\in, x) + \int_{\partial_{\Omega_\in}} F(\in, \sigma)V.nd\sigma. \tag{3.5}$$

Proof. See [1] ■

Theorem 3.2 *Let Ω be a domain of class* C^2 *over* $\mathbb{R}^N$ *and V a field of vectors of class* C^2. *Let G be a function belonging to the space*

$$C^1((0,\varepsilon),C^0(\Omega_t)) \cap C^0((0,\varepsilon),C^1(\Omega_t)).$$

The function defined by:

$$J_2(t) = \int_{\partial_{\Omega_t}} G(t, \sigma)d\sigma, \tag{4.4}$$

is differentiable and its derivative is given by:

$$dJ_2(\Omega_t, V) = \int_{\partial\Omega_t} \frac{\partial}{\partial_t} G(t, \sigma)d\sigma + \int_{\partial\Omega_t} \left[H(\sigma)G(t, \sigma) + \frac{\partial G(t, \sigma)}{\partial n}\right] d\sigma, \tag{4.5}$$

where $H(\sigma)$ *is the mean curvature on the edge* σ *and* $\frac{\partial G(t,\sigma)}{\partial n}$ *is the usual normal derivative.*

Proof. For the proof of these two theorems, see [1]. ■

3.1 Exemple

Let Ω be a domain of class C^1.

The perimeter and the volume being differentiable, we have:

$$\frac{d|\Omega_t|}{dt}|_{t=0} = \frac{d}{dt}| \int_{\Omega_t} dx = \int_\Omega divVdx = \int_{\partial\Omega} V.nd\sigma.$$

The normal derivative is positive if it points outward and negative otherwise.

Likewise,

$$\frac{d}{dt}|_{t=0} \int_{\Omega_t} fdx = \int_\Omega div(fV)dx = \int_{\partial\Omega} fV\ .nd\sigma$$

and for the perimeter:

$$\frac{d}{dt}P(\Omega_t) = \int_{\partial\Omega_t} d\sigma = \int_{\partial\Omega_t} (n_t\ .n_t)d\sigma = \int_{\partial\Omega_t} N_t\ .n_t)d\sigma.$$

Integration by parts in the other direction gives:

$$\frac{d}{dt}P(\Omega_t) = \int_{\partial\Omega_t} d\sigma = \int_{\partial\Omega_t} (n_t\ .n_t)d\sigma = \int_{\partial\Omega_t} N_t\ .n_t)d\sigma.$$

We consider N_t as a trace, and we have:

$$\frac{d}{dt}P(\Omega_t) = \int_{\partial\Omega_t} div\, N_t.$$

where N_t is an extension of n_t to R^N (unitary norm 1). So :

$$\frac{d}{dt}P(\Omega_t) = \int_{\partial\Omega_t} div\, N_t\,|_{t=0} = \int_\Omega \frac{\partial}{\partial t}(div\, N_t) + div(V.divN_0)dx.$$

By application of the divergence:

$$\int_\Omega div\left(\frac{\partial}{\partial t}N_t\right) + div(V\ .divN_0)dx = \int_{\partial\Omega} \frac{\partial}{\partial t}N_t\ .nd\sigma + \int_{\partial\Omega}(V.n)divN_0 d\sigma$$

However, we know that:

$$\frac{\partial}{\partial t}N_t\ .n = 0$$

Hence: $\frac{d}{dt}\int_{\Omega_t} f|_{t=0} =$

$\int_{\partial\Omega} fV.nd\sigma.$

And

$$\frac{d}{dt}P(\Omega_t)|_{t=0} = \int_{\partial\Omega} fV.nd\sigma.$$

where H is the mean curvature.

We deduce that the derivative of the perimeter is equal to the mean curvature.

3.2 Shape derivative via Lagrange

In this part, we apply the results of the previous paragraph to the following functional :

$$J(\Omega) = a\int_\Omega |\nabla u_\Omega - \nabla v_0|^2 dx + b\int_\Omega |u_\Omega - v_1|^2\, dx \qquad (3.6)$$

where a and b two real numbers, v_0 (*respectively* v_1) are the given functions of $H_{loc}^1\ \mathbb{R}^N$) (*respectively* $L^2_{loc}(\mathbb{R}^N)$) and u_Ω is the solution of the following Dirichlet problem :

$$\begin{cases} -\nabla u_\Omega + u^q = f \ \ in\ \Omega \\ u_\Omega = 0 \quad on \quad \partial\Omega. \end{cases} \tag{3.7}$$

In Ω_t the functional J is written :

$$J(\Omega_t) = a\int_{\Omega_t} |\nabla u_\Omega - \nabla v_0|^2\, dx + b\int_{\Omega_t} |u_{\Omega - v_1}|^2 dx.$$

Using the derivation formula (4.2) gives us (assuming enough regularity $\Omega \in C^1, f \in L^2{}_{loc}$

$$dJ(\Omega,V) = 2a\int_\Omega (\nabla u_\Omega - \nabla v_0).\nabla u' dx + 2b\int_\Omega (\nabla u_\Omega - v_1)u' dx$$
$$+ a\int_{\partial\Omega} |\nabla u_\Omega - \nabla v_0||v_1|^2 n d\sigma + b\int_{\partial\Omega} |v_1|^2 V.n d\sigma$$

for any vector field V with u^0 the form derivative of u_t.

In everything that follows, we look for the equation verified by u^0.

Let us make the variational formulation of the Dirichlet problem in Ω_t.

Let$v \in H_0^1(\Omega_t)$, by multiplying the first equation of the previous problem by v and integrating over Ω we obtain :

$$\forall v \in H_0^1(\Omega_t), \quad \int_{\Omega_t} \nabla u_t.\nabla v + u^q v dx = \int_{\Omega_t} f v dx \tag{3.8}$$

For t small enough, we can differentiate equality(1.10) with ($v = \phi$) fixed. By applying formula (4.2) we have :

$$\int_\Omega (\nabla u'.\nabla\varphi + quu^{q-1}\varphi)dx + \int_{\partial\Omega} (\nabla u.\nabla\varphi + u^q\varphi)V.nd\sigma = \int_{\partial\Omega} f\varphi V.nd\sigma \tag{3.9}$$

Now if ϕ is zero on the edge (on a neighborhood of the edge), the integrals of the limit disappear and we have :

$$\int_\Omega (\nabla u'.\nabla\varphi + qu'u^{q-1}\varphi)dx = 0 \tag{3.10}$$

So we have

$$\int_\Omega (-\nabla u_\Omega + qu'u^{q-1}\varphi)dx = 0.$$

And so we get:

$$-\nabla u + qu'u^{q-1} = 0 \ in\ \Omega$$

In the sense of distributions, now to recover the boundary condition, let's remember the equality:

$$u'(\Omega, \mathrm{V}) = \dot{u}(\Omega, \mathrm{V}) - \mathrm{V}.\nabla u.$$

The function $u_t o(Id + tV)$ defines on the fixed domain Ω disappears on the edge of Ω or all t. We then deduce that :

$$\frac{d}{dt}(u_t)o(Id + tV)|_{t=0} = \dot{u}(\Omega, \mathrm{V}) = 0 \text{ on } \partial\Omega..$$

In other words,$u_t o(Id + tV) \in H_0^1(\Omega)$ for all t, therefore, according to the equality :

$$u'(\Omega, \mathrm{V}) = \dot{u}(\Omega) - \mathrm{V}\nabla u$$

$u^{'}$ satisfied

$u^{'} = \nabla u.V = \frac{\partial u}{\partial n} V.n \ on \ \partial\Omega.$

The last equality comes from the fact that the gradient of u is normal to the edge. We therefore have the following result.

Theorem 3.3 *Lets* Ω *be a domain of class* $C^1(\mathbb{R}^N)$ *and J be the functional defined by*

$$J(\Omega) = \text{a} \int_\Omega |\nabla u_\Omega - \nabla v_0|^2 dx + \int_\Omega |u_\Omega - v_1|^2 dx,$$

where a and b are positive real numbers.
The functional J is differentiable and we have

$$dJ(\Omega; V\ ) = 2\text{a} \int_\Omega (\nabla u - \nabla v_0).\nabla u^{'} dx + 2 \int_\Omega (u - v_1) u^{'} dx$$

$$+ a \int_{\partial\Omega} |\nabla u - \nabla v_0|^2 V.n d\sigma + b \int_{\partial\Omega} |v_1|^2 V.n d\sigma$$

where u^0*, the form derivative satisfies*

$$\begin{cases} -\Delta u + q u^{'} u^{q-1} \ \ in \ \Omega \\ u^{'} = \nabla u.V = -\dfrac{\partial u}{\partial n} V.n \ on \ \partial\Omega \end{cases}$$

(3.11).

Proof. The proof of this theorem follows directly from the previous application. ■

4 Topological derivative via minmax method

In this subsection, we describe how to calculate the topological derivative using the min-max approach, see e.g. [5], [6], [10], [7]. To begin with, we will look at the following definitions and notations.

Definition 4.1 A Lagrangian function is a function of the form

$$(t, x, y)\ 7 \rightarrow \ L(t, x, y) \ : \ [0, \tau] \times X \times Y \rightarrow \mathbb{R}\, \tau > 0$$

where X is a vector espace, Y a non empty subset of vector space and the function y 7→ $L(t, x, y)$ is affine.
Associate with the parameter *t* the parametrized minimax

$$t \longmapsto \mathrm{g(t)} = \inf_{x \in X} \sup_{y \in Y} \mathrm{L(t,x,y)}: [0,\tau] \rightarrow \mathbb{R} \ \ and\ d\mathrm{g}(0) = \lim_{t \to 0^+} \frac{\mathrm{g(t)} - \mathrm{g(0)}}{t}$$

When the limits exist, we will use the following notations

$$d_t L(0,x,y) = \lim_{t \to 0^+} \frac{\mathrm{L(t,x,y)} - L(0,x,y)}{t}$$

$$\varphi \in X, d_x \mathrm{L(t,x,y;\varphi)} = \lim_{\theta \to 0^+} \frac{\mathrm{L(t,x,+\theta\varphi,y)} - \mathrm{L(t,x,y)}}{\theta}$$

$$\emptyset \in Y\ d_y \mathrm{L(t,x,y;\emptyset)} = \lim_{\emptyset \to 0^+} \frac{\mathrm{L(t,x,+\theta\emptyset,y)} - \mathrm{L(t,x,y)}}{\theta}.$$

Since $\mathrm{L(t,x,y)}$ is affine en y, for all $(t,x) \in [0,\tau] \times X$,

$$\forall y, \psi \ \in\ Y\ dyL(t,x,y;\psi) = \ L(t,x,\psi) - \ L(t,x,0) = dyL(t,x,0;\psi), \qquad (4.1)$$

The state equation at $t \geq 0$

Find $x^t \in X$ such that for all $\psi \ \in\ Y, d_y \mathrm{L}(\mathrm{t}, x^t, 0; \psi)$ (4,2) .

The set of states x^t at $t \geq 0$ is denoted

$$E(t) = \{x^t \ \in X, \forall \psi \in Y, d_y \mathrm{L}(\mathrm{t}, x^t, 0; \psi) \qquad (3.4).$$

The adjoint equation at $t \geq 0$ is

Find $p^t \in Y$ such that for all $\phi \in X$, $d_x L(t,x^t,p^t;\phi) = 0$. (4.4)

The set of solutions p^t at $t \geq 0$ is denoted (4.5)

$$Y(t,x^t) = \{p^t, \in Y\ \forall \varphi \in X, d_x L(t,x^t,p^t,\varphi) = 0\}$$

Finally the set of minimisers for the minimax is given by

$$X(t) = \{x^t \ \in X, \mathrm{g(t)} = \inf_{x \in X} \sup_{y \in Y} \mathrm{L(t,x,y)} = \sup_{y \in Y} \mathrm{L}(t,x^t,y)\} \qquad (4.6)$$

LEMMA 4.1 **(Constrained infimum and minimax)** *We have the following assertions*

(i) $\inf_{x \in X} \sup_{y \in Y} \mathrm{L(t,x,y)} = \inf_{x \in E(t)} (t,x,y)$

(ii) *The minimax* $g(t) = +\infty$ *if and only if* $E(t) = \emptyset$. *And in this case we have* $X(t) = X$.

(iii) *If* $E(t)$ 6= $\emptyset$, *then*

(iv) $X(t) = \{x^t \ \in E(t): L(t,x^t,0) = \inf_{x \in E(t)} (t,x,0)\} \complement E(t)$ *and* $g(t) < +\infty$.

Proof. See [5], [8], [6]. ■

To end this subsection, we give definitions and theorems on d-dimensional Minkowski content and d-rectifiability.

Definition 4.2 Let E be a subset of a metric space X. $E \subset X$ is d-rectifiable if it is the image of a compact subset K of $\mathbb{R}^d$ by a continuous lipschitzian function $f : \mathbb{R}^d \rightarrow X$.

Let E be a closed compact set of $\mathbb{R}^N$ and $r \geq 0$, the distance function d_E and the r-dilatation E_r of E are defined as follows:

$d_{E(x)} = \inf_{x \in E} |x - x_0|, \;\; E_r = \{x \in \mathbb{R}^N : d_{E(x)} \leq r\,.$

Definition 4.3 Given d, $0 \leq d \leq N$ the upper and lower d-dimensional Minkowski contents of a set E are defined by an r-dilatation of this set as follows

$$M^{*d}(E) = \lim_{r \to 0^+} sup \frac{m_N(Er)}{\alpha N - d^{r^{N-d}}}; \quad M_*^d(E) = \lim_{r \to 0^+} inf \frac{m_N(Er)}{\alpha N - d^{r^{N-d}}}$$

where m_N is the Lebesgue measure in R^N and α_{N-d} is the volume of the ball of radius 1 in $\mathbb{R}^{N-d}$.

Both concepts can be found in [5], [6].

4.1 Some preliminary results

We need the following assumption for everything that follows:
Hypothesis (H0) Let X
be a vector space.

(i) : For all $t \in [0,\tau]$, $x^0 \in X(0), x^t \in X(t)$ $^0 \in X(0)$, and $y \in Y$, the function $\theta \rightarrow 7$ $L(t, x^0 + \theta(x^t - x^0), y) : [0,1] \rightarrow R$ is absolutely continuous. This implies that for almost all θ the derivative exists and is equal to $d_x L(t, x^0 + \theta(x^t - x^0), y;\, x^t - x^0)$ and it is the integral of its derivative. In particular

(ii) $L(t, x^s, y) = L(t, x^0, y) + \int_0^1 d_s L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0) d\theta.$

ii) : For all $t \in [0,\tau]$, $x^0 \in X(0)$, $x^t \in X(s)$ and $y \in Y$, $\varphi \in X$ and for almost all $\theta \in [0,1]$, $d_s L(t, x^0 + \theta(x^t - x^0), y; \emptyset)$ exist et the functions $\theta \rightarrow 7$ $d_s L(t, x^0 + \theta(x^t + x^0), y; \emptyset)$ belong to $L^1[0,1]$

Definition 4.4 Given $x^0 \in X(0)$ and$x^t \in X(t)$, the averaged adjoint equation is:

$$y^t \in Y \; \forall X, \int_0^1 d_s L(t, x^0 + \theta(x^t + x^0), y; \emptyset) d\theta$$

Find

and the set of solutions is noted $Y(t, x^{0,} x^t)$.
$Y(0, x^0, x^0)$ clearly reduces to the set of standard adjoint states $Y(0, x^0)$ at $t = 0$.

Theorem 4.2 *Consider the Lagrangian functional*

$$(t,x,y) \mapsto L(t,x,y) : [0,\tau] \times X \times Y \to \mathbb{R},\ \tau > 0$$

where X and Y are vector spaces and the function $y \mapsto L(t,x,y)$ *is affine. Assume that (H0) and the following hypotheses are satisfied*

H1a *for all* $t \in [0,\tau]$, $g(t)$ *is finite,* $X(t) = \{x^t\}$ *and* $Y(0,x^0) = \{p^0\}$ *are singletons,*

H2a $d_t L(0,x^0, y^0)$ *exists,*

H3a *The following limit exists*

$R(x^0, y^0) = \lim_{t\to 0^+} \int_0^1 d_s L\left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t}\right) d\theta.$

Then, $dg(0)$ *exists and* $dg(0) = d_t L(0,x^0, y^0) + R(x^0, p^0)$.

Proof. See [4, 5]. ■

COROLLARY 4.3 *Consider the Lagrangian functional*

$$(t,x,y) \mapsto L(t,x,y): [0,\tau] \times X \times Y \to \mathbb{R}, \tau > 0$$

where X and Y are vector spaces and the function $y \mapsto L(t,x,y)$ *is affine. Assume that (H0) and the following assumptions are satisfied:*

(H1a) for all $t \in [0,\tau]$, $X(s)$ not equal $\emptyset$, $g(t)$ *is finite, and for each* $x \in X(0)$, $Y(0,x)$ not equal $\emptyset$,

(H2a) for all $x \in X(0)$ *and* $p \in Y(0, x)$ $d_t L(0,x,p)$ *exists,*

(H3a) there exist $x^0 \in X(0)$ *and* $p^0 \in Y(0, x^0)$ *such that the following limit exists*

$$R(x^0, p^0) = \lim_{t\to 0^+} \int_0^1 d_s L\left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t}\right) d\theta.$$

Then, $dg(0)$ *exists and there exist* $x^0 \in X(0)$ *and* $p^0 \in Y(0,x^0)$ *such that* $dg(0) = d_t L(0,x^0,p^0) + R(x^0,p^0)$.

In what follows, we focus on the main result of this part. And for information on the tools used the reader can consult [5].

4.2 Topological derivative

Let us consider the functionnal defined in Ω_t by

$$F(\Omega t) = a \int_{\Omega_t} |\nabla u_\Omega - \nabla v_0|^2 dx + b \int_{\Omega_t} |u_\Omega - v_1|^2 dx. \tag{4.7}$$

where u_{Ω_t} be the solution to the Neumann Problem

$$\begin{cases} -\Delta\Omega_t + u_{\Omega_t}^q = f \ in \ \Omega_t \\ u_{\Omega_t} = 0 \ on \ \partial\Omega_t \end{cases} \tag{4.8}$$

where $q > 1$ is an integer.

Let us consider as shape functional F define by

$$F(\Omega) = \mathrm{a}\int_{\Omega} |\nabla u_{\Omega} - \nabla v_0|^2 dx + b\int_{\Omega} |u_{\Omega} - v_1|^2 dx. \tag{4.9}$$

And$u_{\Omega} \in H_0^1(\Omega)$ is solution to the variational problem

$$\forall v \in H_0^1(\Omega), \qquad \int_{\Omega} \nabla u_{\Omega}.\Delta v + u_{\Omega}^q.v dx = \int_{\Omega} f v dx. \tag{4.10}$$

We aim to compute the topological derivative of the functional $F(\Omega_t)$

$$dF = \lim_{t\to 0^+} \frac{F(\Omega_t) - F(\Omega)}{\alpha_N - d r^{N-d}}.$$

Thus, the Lagrangian dependent on t will be written in the form :

$$L(t, \emptyset, \Phi) = a\int_{\Omega_t} |\nabla\Phi - \nabla v_0|^2 dx + b\int_{\Omega} |\Phi - \nabla v_1|^2$$

$$+\int_{\Omega} \nabla\emptyset.\nabla\Phi + \emptyset^q \nabla\Phi dx - \int_{\Omega} f\Phi dx$$

From this, we can now evaluate the derivative of the Lagrangian, dependent on t, with respect to $\emptyset$.

$$d_{\emptyset}L(t, \emptyset, \Phi, \emptyset') = 2a\int_{\Omega} (\nabla\emptyset - \nabla v_0).\nabla\emptyset' dx + 2b\int_{\Omega} (\Phi - v_1)\emptyset' dx + \int_{\Omega} \nabla\emptyset'\nabla\Phi + q\emptyset'\emptyset^{q-1}\Phi dx$$

The initial adjoint state p_{Ω_0} is a solution of $d_{\emptyset}L(0, u_{\Omega_0}, p_{\Omega_0}, \emptyset') = 0$ for all $\emptyset'$ for $t = 0$. Thus the variational formulation of the adjoint equation of state is given by

$$2\mathrm{a}\int_{\Omega} (\nabla u_{\Omega_0} - \nabla v_0)\nabla\emptyset' dx + 2b\int_{\Omega} (u_{\Omega_0} - v_1)\emptyset' dx + \int_{\Omega} \nabla\emptyset' \nabla p_{\Omega_0} + q\emptyset' u_{\Omega_0}^{q-1} p_{\Omega_0} dx = 0$$

.

And we have

$$\int_{\Omega} [2\mathrm{a}(\nabla u_{\Omega_0} - \nabla v_0)\nabla\emptyset' dx + 2b(u_{\Omega_0} - v_1)\emptyset' dx + \nabla\emptyset'\nabla p_{\Omega_0} + q\emptyset' u_{\Omega_0}^{q-1} p_{\Omega_0}] dx = 0 \tag{4.11}$$

Next, we derive the Lagrangian with respect to Φ.

$$d_{\emptyset}L(t, \emptyset, \Phi, \emptyset') = \int_{\Omega} \nabla\emptyset.\nabla\Phi + \emptyset^q \Phi' dx - \int_{\Omega} f\Phi' dx$$

The initial state u_{Ω_0} is a solution of $d_{\emptyset}L(0, u_{\Omega_0}, 0, \Phi'_{\Omega_0}) = 0 \ \forall\ \Phi'_{\Omega_0} \in H_0^1(\Omega)$ and in this case, we have:

$$\int_{\Omega} \nabla u_{\Omega_0}.\Delta\Phi' + u_{\Omega_0}^q \Phi' dx. - \int_{\Omega} f\Phi' dx = 0$$

.

$$\int_{\Omega}[\nabla u_{\Omega_0}.\Delta\Phi' + u_{\Omega_0}^{q}\Phi' dx. - f\Phi']dx = 0 \tag{4.12}$$

And we have

$$L(t,\emptyset,\Phi) - L(0,\emptyset,\Phi) = \int_{\Omega_t} \mathrm{f}(x)\,\Phi(x)dx - \int_{\Omega_t} f(x)\,\Phi(x)$$

$$L(t,\emptyset,\Phi) - L(0,\emptyset,\Phi) = \int_{\Omega_t} \mathrm{f}(x)\,\Phi(x)dx - \int_{\mathrm{w}_t} f(x)\,\Phi(x) + \int_{\Omega_t} \mathrm{f}(x)\,\Phi(x)dx$$

$$L(t,\emptyset,\Phi) - L(0,\emptyset,\Phi) = -\int_{\mathrm{w}_t} f(x)\,\Phi(x)$$

$$d_sL(t,\emptyset,\Phi) = \lim_{s\to 0}\frac{1}{|B(x_0,s)|}\left[\int_{B(x_0,s)} f(x)\Phi(x)\right]$$

$$d_sL(0,\emptyset,\Phi) = f(x_0)\Phi(x_0).$$

We will now define $R(t)$ by.

$$R(t) = \int_0^1 d_{\emptyset}L\left(t, u_{\Omega_t} + \psi(u_{\Omega_t} - u_{\Omega_0}, p_{\Omega_0}, \left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)\right)d\psi.$$

By substituting $\emptyset' = \frac{u_{\Omega_t} - u_{\Omega_0}}{t}$ and $\psi = \frac{u_{\Omega_t} - u_{\Omega_0}}{2}$ into the adjoint equation for p_{Ω_0}, we obtain:

$$R(t) = 2a\int_{\Omega}\left[\nabla\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right) - \nabla v_0\right].\nabla\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)dx$$

$$+2b\int_{\Omega}\left[\nabla\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right) - v_1\right].\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)dx$$

$$+\int_{\Omega}\nabla\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)\nabla p_{\Omega_0} + \mathrm{q}\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right)^{q-1} p_{\Omega_0}dx$$

$$= 2a\int_{\Omega}\left[\nabla\left(\frac{u_{\Omega_t}}{2}\right) - \nabla\left(\frac{u_{\Omega_t}}{t}\right) + \nabla\left(\frac{u_{\Omega_0}}{2}\right) + \nabla\left(\frac{u_{\Omega_0}}{2}\right) - \nabla v_0\right].\nabla\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)dx$$

$$+2b\int_{\Omega}\left[\frac{u_{\Omega_t}}{2} + \frac{u_{\Omega_0}}{2} - \frac{u_{\Omega_0}}{2} - \frac{u_{\Omega_0}}{2} - v_1\right].\nabla\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)dx$$

$$+\int_{\Omega}\left[\nabla\left(\frac{u_{\Omega_t}}{2}\right) - \nabla\left(\frac{u_{\Omega_0}}{t}\right)\right]\nabla p_{\Omega_0} + q\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right)^{q-1} p_{\Omega_0}dx$$

$$+\int_{\Omega} q\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)[u_{\Omega_0}^{q-1} - u_{\Omega_0}^{q-1}]p_{\Omega_0}dx$$

$$= 2a\int_{\Omega}\nabla\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{2}\right).\nabla\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)dx + 2b\int_{\Omega}\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{2}\right)\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)dx$$

$$+\int_{\Omega} q\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)\left[\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right)^{q-1} - u^{q-1}\right]p_{\Omega_0}dx$$

$$R(t) = \frac{a}{t}\int_{\Omega} |\nabla u_{\Omega_t} - u_{\Omega_0}|^2 dx + \frac{b}{t}\int_{\Omega} |\nabla u_{\Omega_t} - u_{\Omega_0}|^2 dx$$

$$+\frac{q}{t}\int_{\Omega} (u_{\Omega_t} - u_{\Omega_0}) dx \left[\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2} \right)^{q-1} - u^{q-1} \right] p_{\Omega_0} dx$$

Theorem 4.4 *Let* $0 \le d < N$, *E verify* ***Hypothesis1*** *and* $t = \alpha_{N-d} r^{N-d}$. *The topological derivative exists if the function* $\mathcal{R}(\epsilon)$ *has a finite limit. Therfore, the topological derivative of the function is given by the expression:*

$$dJ = \lim_{t \to 0} sup \frac{J(\Omega_t) - J(\Omega)}{\alpha N - d^{r^{N-d}}}$$

$$dJ = R(x_0, p_{\Omega_0}) - f(x_0) p_{\Omega_0}(x_0).$$

where $p_{\Omega_0}, u_{\Omega_0}$ *are solutions of systems*

$$\int_{\Omega} [2a(\nabla u_{\Omega_0} - \nabla v_0). \nabla \emptyset' + 2b(u_{\Omega_0} - v_1)\emptyset' dx + \nabla \emptyset' \nabla p_{\Omega_0} + q \emptyset' u_{\Omega_0}^{q-1} p_{\Omega_0}] dx = 0 .$$

5 Conclusion

In this paper we start by establishing an existence result of optimal form. Then we proved the shape drift using the Lagrange method. The last part of the document was devoted to the topological derivative of the functional. we plan to look at the numerical problem of these already established derivatives.

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