Design of a stepped disk for continuous kinetic energy storage: an educational toy

Abstract

In this study, the design of a stepped disk for continuous kinetic energy storage is considered.

The system includes a stepped disk made of a large disk for kinetic energy storage and a small disk to wrap and unwrap a string; and a relatively small mass attached to the string for potential energy to drive the stepped disk. The disk rotates around a fixed point and the mass drops down in a straight-line motion.

While the mass drops down, the string is unwrapped, the stepped disk rotates and transfers the potential energy into kinetic energy. When the mass reaches its lowest point, the string wraps around the small disk and the mass gains potential energy. By neglecting friction, the stepped disk rotates back and forth continuously.

The importance of the stepped disk system is twofold: it is used as an educational toy thus enabling research activities and demonstrating the process of kinetic energy storage.

1. introduction

Gravitational energy and flywheel technology represent two promising avenues for renewable energy generation and storage. Gravitational energy, often overlooked, can be harnessed through innovative systems that capitalize on the potential energy derived from gravitational forces. Shyu Shyu (2011) introduced a Vertical Type Potential Energy Generator (VTPEG), demonstrating that gravitational energy can be effectively converted into usable energy, thus positioning it as a viable renewable resource. This concept is further supported by Shyu (Shyu, 2010), who posits that universal gravitation itself can be viewed as an ultimate renewable energy source, emphasizing the potential for large-scale applications. The integration of gravitational energy systems could significantly contribute to sustainable energy solutions, particularly in regions where conventional energy sources are limited.

Flywheel energy storage systems (FESS) also play a crucial role in the renewable energy landscape. These systems store kinetic energy in a rotating mass, allowing for rapid energy release when needed. Li et al. (2013) highlighted advancements in flywheel technology, particularly the use of composite materials that enhance the mechanical properties and efficiency of energy storage. The application of flywheels in conjunction with renewable energy sources, such as wind power, has been explored by Shao et al. (Shao et al., 2014), who demonstrated that integrating flywheels with lead-acid batteries can optimize energy storage and reduce losses. Furthermore, Wen et al. (2013) discussed the design of multi-ring carbon fiber composite flywheels, which exhibit high energy density and longevity, making them suitable for various applications, including aerospace and automotive industries.

The synergy between gravitational energy and flywheel technology can lead to innovative energy solutions. For instance, the potential for using flywheels to store energy generated from gravitational systems is an area ripe for exploration. Erd et al. (2023) provided insights into the power flow simulation of flywheel systems, particularly in tramway applications, where energy savings can be achieved through regenerative braking. This highlights the practical implications of combining Additionally, Ratniyomchai et al. (2014) emphasized the importance of energy storage devices in electrified railways, showcasing the effectiveness of flywheels in managing regenerative braking energy.

In summary, both gravitational energy and flywheel technology present significant opportunities for renewable energy generation and storage. The integration of these systems could lead to enhanced energy efficiency and sustainability. As research continues to evolve in these fields, the potential for innovative applications and improvements in energy systems remains promising.

In this article, a stepped disk system is proposed to demonstrate kinetic energy storage. The system includes a stepped disk as a simplified version of a flywheel. The stepped disk continuously stores the potential energy supplied by a moving mass in a gravitational field as kinetic energy.

The article is arranged as follows: system description is given in section 2; mathematical model is described in section 3; solutions of the mathematical model are given in section 4; numerical example is given in section 5; and finally summary and conclusion are given in section 6.

2 System description

in this section the system is described including its components and its function.

2.1 components

The stepped disk system is designed to run continuously in a prosses to store and release kinetic energy. The successful operation of the system is granted by proper design and assembly of its components.

There are three main elements in the system including a stepped disk; a relatively small mass and a string and a wood bar to serve as a disk holder.

The stepped disk has a mass M with radius R, and a smaller disk with radius r (see figure 1).

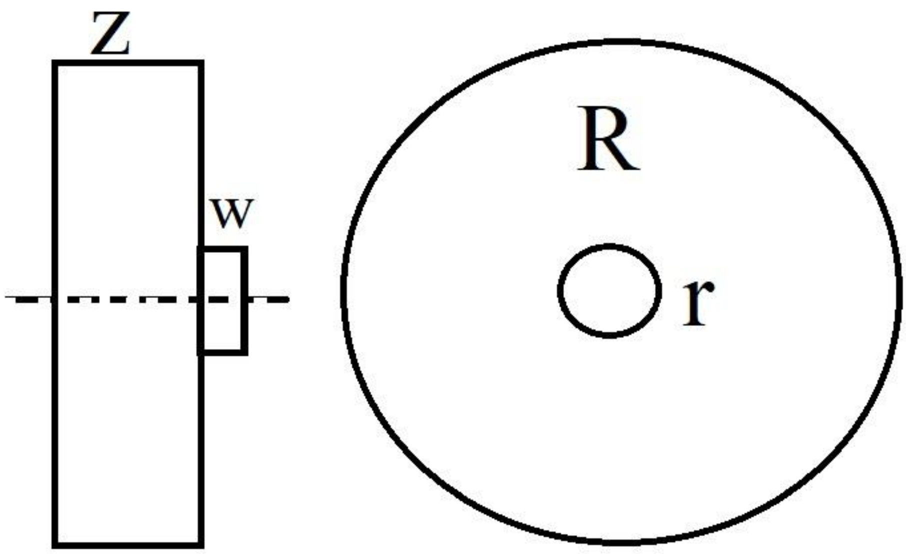

Figure 1: Schematics of the stepped disk. The large disk with a radius *R* and thickness *z*. the small disk with a radius *r* and thickness *w*.

The second component is a relatively small mass *m* which is connected to the small disk with a string (the third component), (see figure 2).

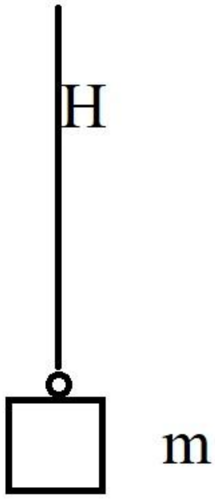

Figure 2: Schematics of the small mass attached to a string with length *H*.

The last component is a wood bar to serve as a disk holder.

2.2 Assembly and operation

The system is assembled together in a few simple steps as follows: the stepped disk is fixed on top of a table by means of a wooden bar. The height of the table and the wooden bar should be greater than the length of the string to insure smooth operation of the system. Then the string is pinned or screwed to the smaller disk, and finally the mass is attached to the free side of the string, (see figure 3).

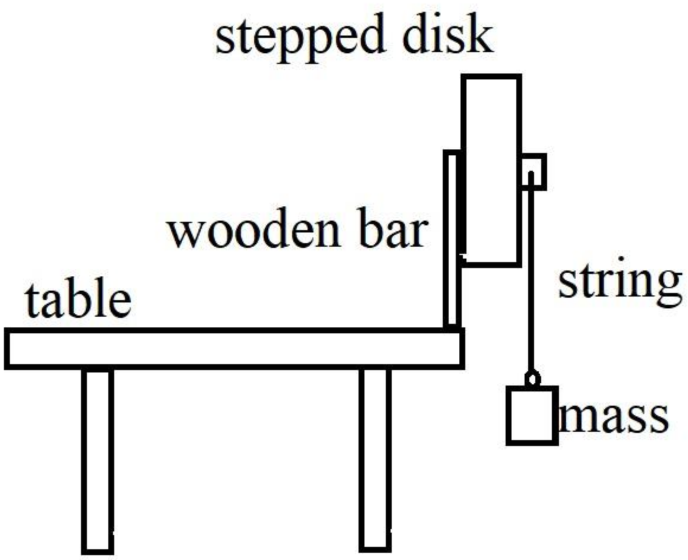

Figure 3: Schematics of the stepping disk system.

The string is wrapped around the smaller disk number of rounds n such that $2nr=H$. The maximal displacement of the mass is $H+r$ meters.

When the system us released from rest, the mass m drops down $H+r$ meters and afterwards it restores the same height in the case of negligible friction and thus moves up and down forever.

3. Mathematical model

In section 2, the stepped disk system for continuous kinetic energy storage was described. The motion of the disk is rotational and the motion of the mass is linear, thus the potential energy of the mass is converted into rotational kinetic energy and vice versa.

3.1 Separate models

The system is split by imagination into two separate components. The free body diagrams are shown in figure 4. The internal force in the spring now acts as an external force.

Newton's second law of motion is written for the disk and for the mass by introducing the tension force in the string as an external force, (see figure 4).

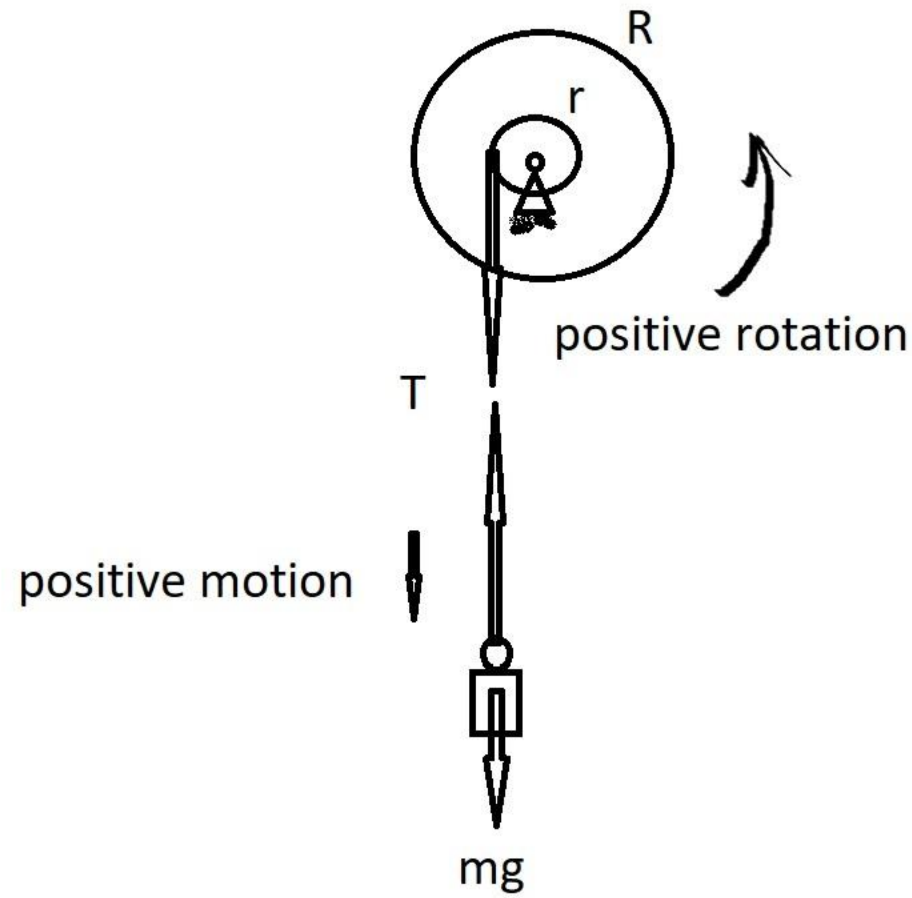

Figure 4: free body diagrams for separated components.

Newton's second law for a rotational motion is given by:

$$\sum M_o = I_o \, \alpha \qquad (1)$$

Where M_o is the external moment and I_o is the moment of inertia of the disk around its axis of rotation, and α is the appropriate angular acceleration.

After writing the external moment explicitly, equation (1) is rewritten and is given by:

$$Tr = I_o \propto \qquad (2)$$

Where T is the tension in the string and r is the radius of the small disk.

Similarly, Newton's second law is written for the linear motion of the mass and is given by:

$$\sum F = ma \qquad (3)$$

Where F is the external force, m the mass of the driving component and a is the appropriate acceleration.

After writing the balance of the external forces explicitly, equation (3) is rewritten accordingly and is given by:

$$mg - T = ma \qquad (4)$$

The relation between the linear acceleration and the angular acceleration is given by:

$$a = \propto r \qquad (5)$$

An expression for the angular acceleration is derived from equations

(2), (4) and (5) after mathematical manipulations and is given by:

$$\propto = \frac{mgr}{I_o + mr^2} \qquad (6)$$

3.2 Whole system model

Equation (6) could be derived based on the whole system model without splitting the systems into two free body diagrams. Applying Newton's second law for the whole system (see figure 5), equation (1) is rewritten and is given by:

$$mgr = (I_o + mr^2) \propto \qquad (7)$$

Clearly, equation (6) could be derived easily from equation (7).

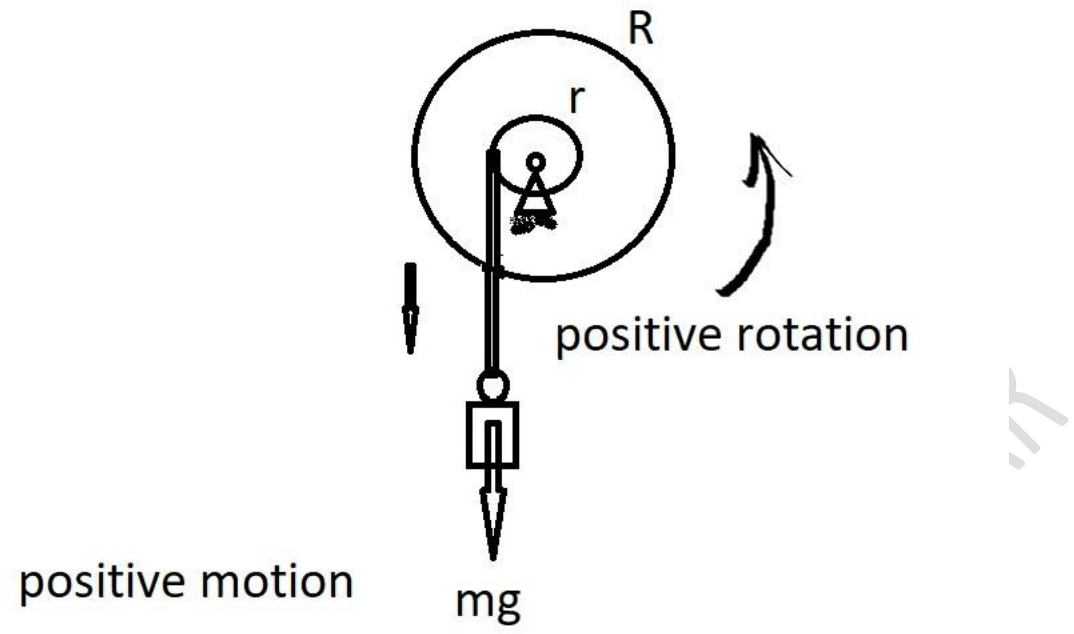

Figure 5: free body diagram for the whole system.

The moment of inertia I of the whole system while neglecting the smaller disk contribution about the axis of the disk is given by:

$$I = \frac{1}{2}MR^2 + mr^2 \quad (8)$$

It is important to note that for the range of the vertical motion from r to H the acceleration is constant, but for the range between from zero to r the acceleration is not constant and should be considered appropriately.

3.3 Non-constant acceleration

when the mass drops down H meters the driving gravitational force is constant but its effect on the moment is changed with respect to the cos(θ), where θ is the angle of rotation measured from the x-axis in the counterclockwise direction, (see figure 6).

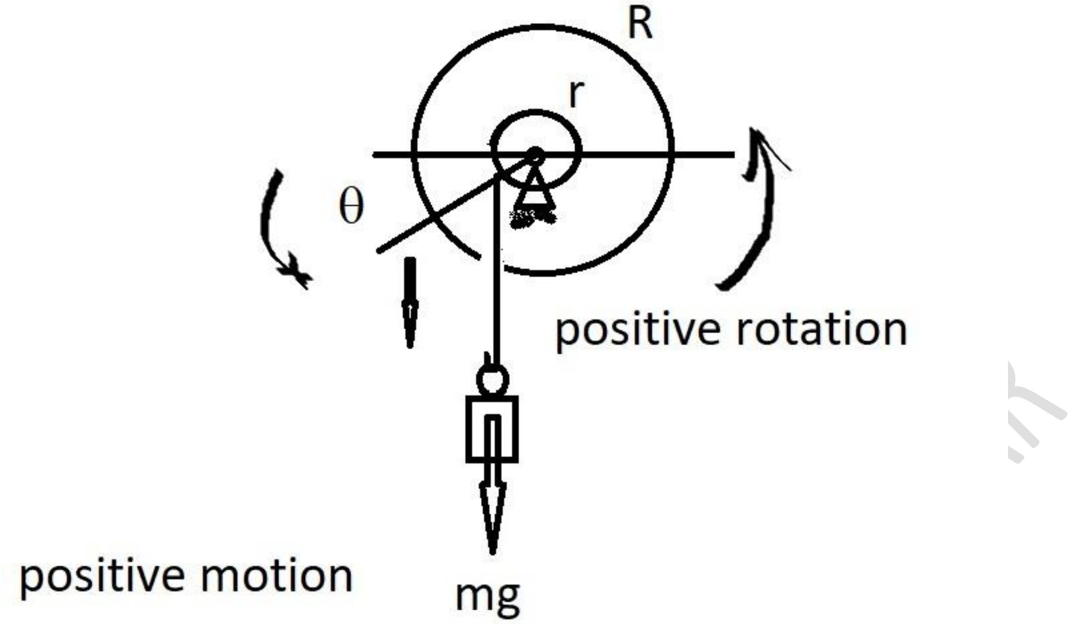

Figure 6: non-constant acceleration model

Equation (6) is modified to account for the dependence of the angular acceleration with respect to angle θ and is given by:

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \alpha\cos(\theta) \qquad (9)$$

Towards finding a solution equation (9) is rearranged in terms of the angular velocity $\dot{\theta}$ and is given by:

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \alpha\cos(\theta) \qquad (10)$$

Equation (10) is integrated to express the angular velocity as a function of the angle and is given by:

$$\dot{\theta} = \sqrt{\dot{\theta}_i^2 + 2\alpha\sin(\theta)}$$

(11)

Where $\dot{\theta}_i$ is the initial angular velocity.

Now, by applying the method of separation of variables to equation (11) , a relation between the time and the angle could be easily derived and it is given by:

$$t = \int_0^{\theta} \frac{d\theta}{\sqrt{\dot{\theta}_i^2 + 2\alpha \sin(\theta)}} \qquad (12)$$

4 Solution of the equation of motion

The motion is descibed by two regions: height between r and H and height between zero and r.

4.1 Height between r and H

In this region the motion is described by a constant acceleration motion. The acceleration is given by equations (5) and (6).

The angular velocity ω is given by:

$$\omega = \alpha t \qquad (13)$$

And the linear velocity v is given by:

$$v = \omega r \qquad (14)$$

And finally, the vertical position is given by:

$$y = H + r - \frac{1}{2} a t^2 \qquad (15)$$

The elapsed time t_{rH} from r to H is extracted from equation (15) by substituting y=r, and is given by:

$$t_{rH} = \sqrt{\frac{2H}{a}} \tag{16}$$

4.2 Height range from zero to r

Equation (12) could be solved exactely or by approximations.

4.2.1 Exact solution

This equation coul be solved exactely by using a calculator; wolfram alpha or by using trapezoidal integration rule from angle zero to $\frac{\pi}{2}$ with the aid of microsoft excel.

4.2.2 Zero order approximation

The sine term is neglected, due to small contribution, and the elapsed time from zero to r is given by:

$$t_{0r} = \frac{\theta}{\dot{\theta}_i} \tag{17}$$

4.2.3 First order approximation

The separated diffrential equation is given by:

$$dt = \frac{d\theta}{\sqrt{\dot{\theta}_i^2 + 2\alpha \sin(\theta)}} \tag{18}$$

The square root is approximated up to the first term of Talor's expamsion and the approximation of equation (18) is given by:

$$dt = \frac{d\theta}{\dot{\theta}_i (1 + \varepsilon \sin(\theta))} \tag{19}$$

Where ε is given by:

$$\varepsilon = \frac{\alpha}{\dot{\theta}_i^2} \tag{20}$$

Equation (19) is solved analytically by using the substitution $u = \tan\left(\frac{\theta}{2}\right)$

And the time t as a function of angle θ is given by (see appendex1 for more details.):

$$t = \frac{2}{\dot{\theta}_i(1-\varepsilon^2)}\left(arctan\left(\frac{u+\varepsilon}{\sqrt{(1-\varepsilon^2)}}\right) - arctan\left(\frac{\varepsilon}{\sqrt{(1-\varepsilon^2)}}\right)\right) \tag{21}$$

Finally, the vertical position in the range zero to r is given by:

$$y = r - rsin(\theta) \tag{22}$$

5. Numerical example

For demonstration, the following values are used as appropriate for lab considerations:

Large disk

Radius R=0.35 m

Radius r=0.03 m

Mass M=3.5kg

Moment of inertia (neglecting small contributions of the small disk and the relatively small mass) =0.0536kgm^2,

Small mass

Mass m=0.1kg

String

Length H=1m

Based on these values the following are calculated:

Angular acceleration

$$\alpha - \frac{mgr}{I} = \frac{0.1\ x\ 9.81\ x0.03}{0.0536} = 0.55\frac{rad}{s}$$

The fall time

$$t_{rH} = \sqrt{\frac{2H}{\alpha r}} = \sqrt{\frac{2x1}{0.55x0.03}} = 11s$$

Fall time in the range 0-r

The initial angular velosity is needed

$$\dot{\theta}_i = \alpha t = 0{,}55x11 = 6.04rad/s$$

Calculator calculation

$$t_{0r} = 0.258\ s$$

Zero order approximation

$$t_{0r} = \frac{\pi}{2\dot{\theta}_i} = 0.26s$$

First order approximtion (equation 21)

$$t_{0r} = 0.2592s$$

The time in the range 0-r is calculated by approximating the integral by a trapezoidal rule and compared with the zero orde approximation. The results are shown in figure 7, (see figure 7).

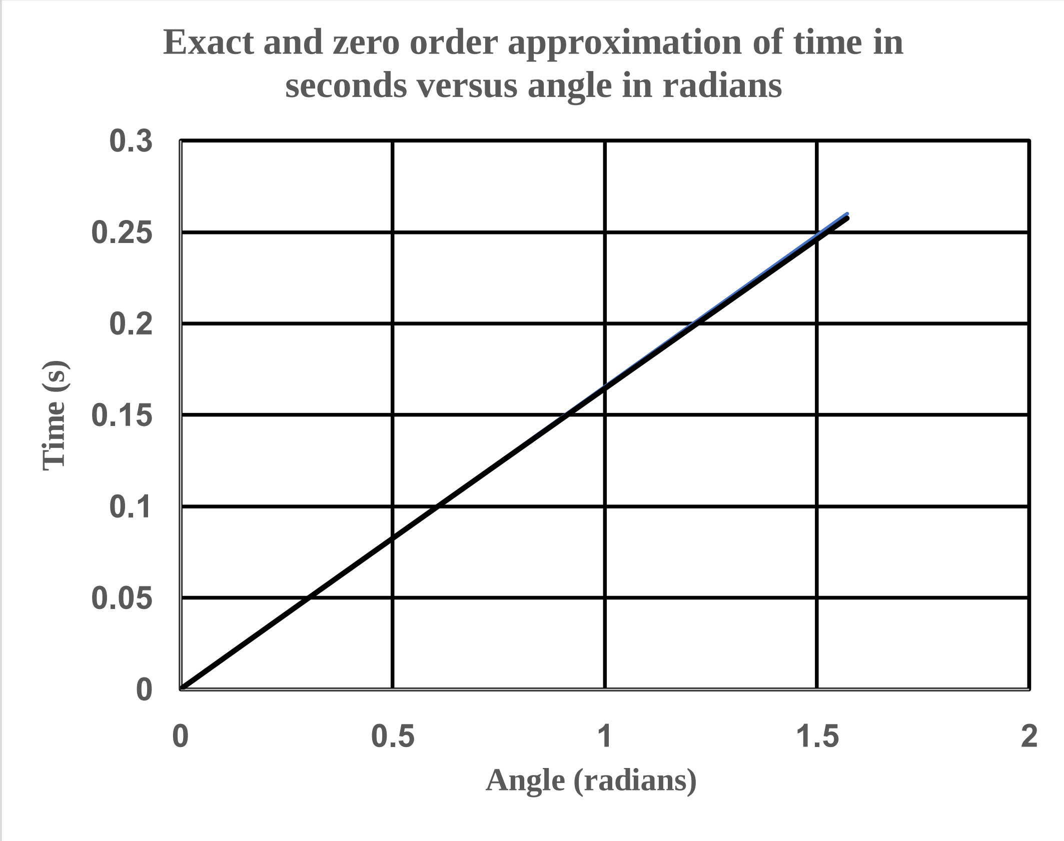

Figure 7: exact and zero order approximation of time versus angle in radians in the range 0-r.

In order to check the accuracy of the approximation, the relative difference between the exact and zero order approximation of time is shown in figure 8, (see figure 8). It is shown in figure 8 that the relative error is less than the 1% for the whole range of the considered angle of rotation.

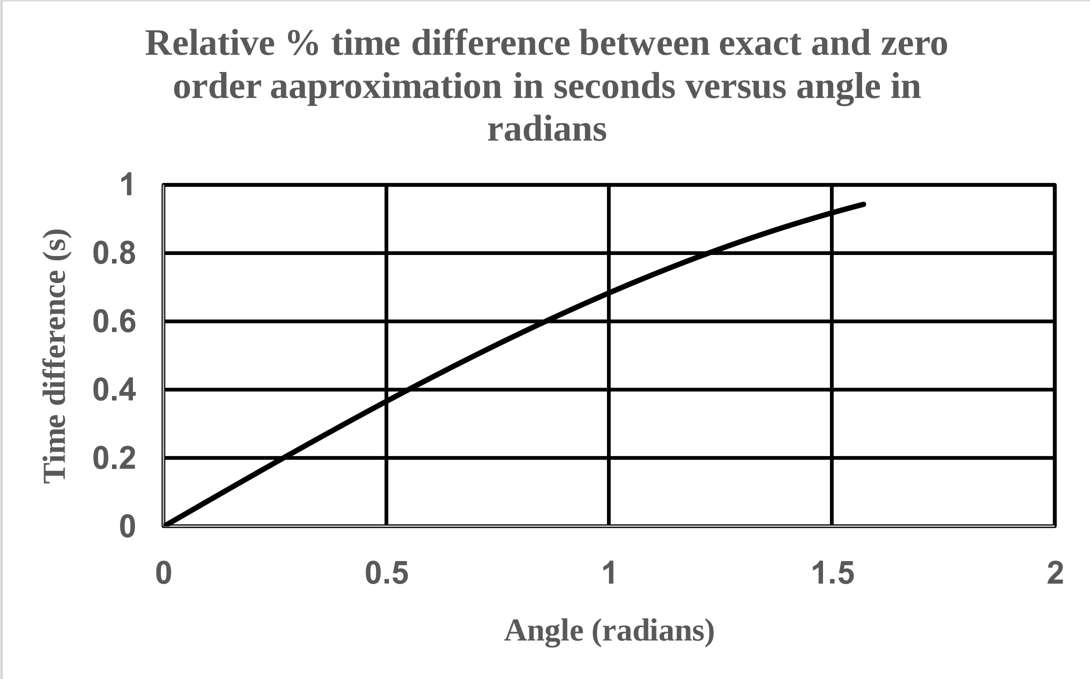

Figure 8: Relative % time difference between exact and zero order approximation versus angle in radians for the range 0-r.

The fallig time (s) in the range r-H is plotted is plotted versus the mass m (kg) in figure 9, see figure 9). It is shown that the time is inversly related to mass in non-linear fashion.

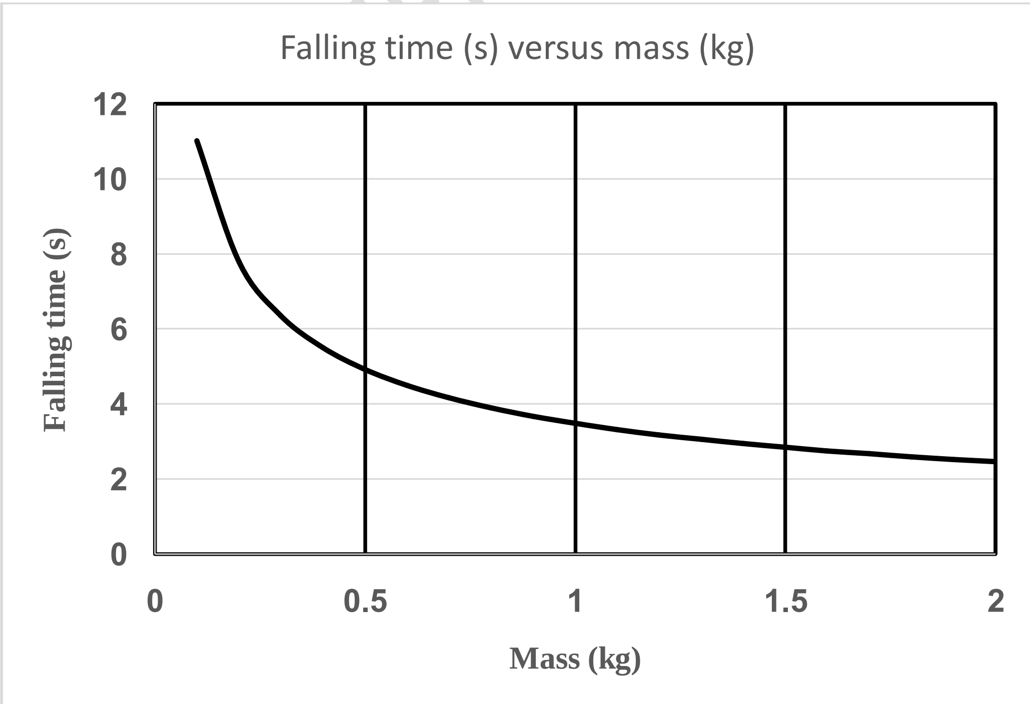

Figure 9: falling time (s) in the range r-H versus mass (kg).

Finally the vertical position versus time is shown fo the range 0-H. Figure 10 shows the drop down position, (see figure 10). The motion upwards is symetric to the motion downwards and could be visualised as its mirror image.

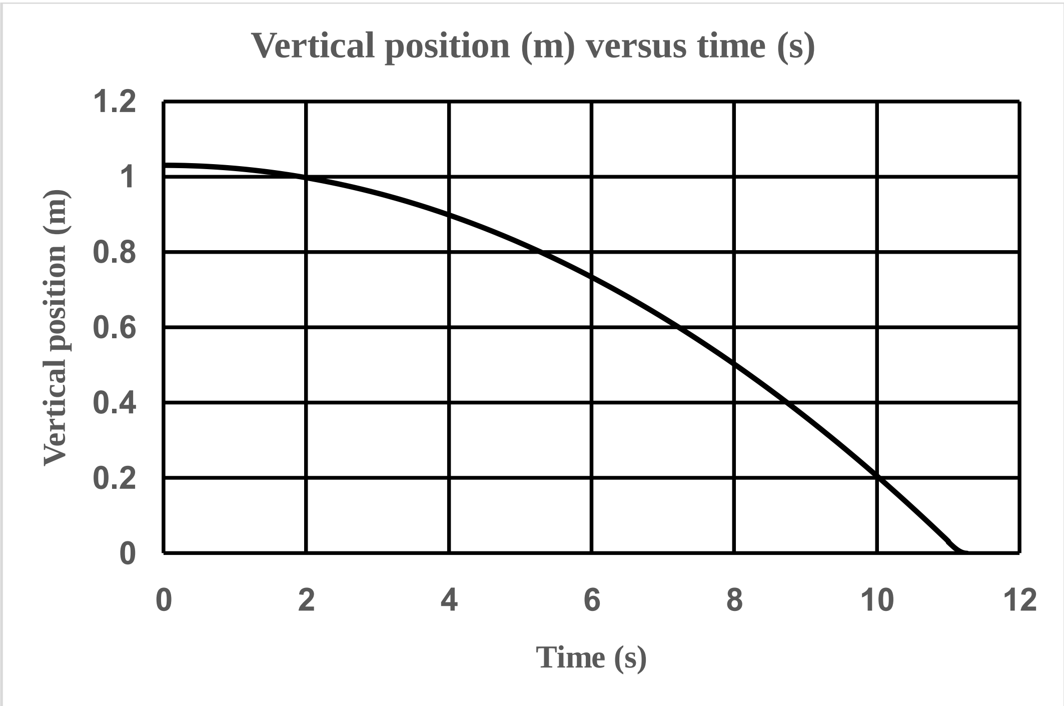

Figure 10: Vertical position (m) versus time (s) for the range 0-H.

6. Summary and conclusions

In this article, the stepped system for countiouous kinetic energy storage is introduced and described. The system includes a stepped disk, a relatively small mass and a connecting string. The mass drops in a gravitational field and its potential energy trasfered and stored in the disk. The mass gains potential energy after reaching its minimal position. In the case of neglicting friction, the system runs forever.

The importance of the system is two folds: it serves as an educational toy for research activities and it demonstrates kinetic energy storage as a renewble source of energy.

The system was described mathematically and its motion was derived analytically and approximatly, especially for the region 0-r where the acceleration is not constant but rather depends on the rotational angle. Due to its small contribution, the depencence could be described by zero order approximation.

As a final note it might be possuble to further develop the system to include a generator to convert the kinetic energy into electricity, and thus it would be possible to demonstrate the applicablility of the system for producing electrical power.

Appendix 1

Derivation of equation (21)

The substitution $u = \tan\left(\frac{\theta}{2}\right)$ leads to the following relations:

$$\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+u^2}}$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{u}{\sqrt{1+u^2}}$$

$$d\theta = \frac{2du}{1+u^2}$$

$$dt = \frac{2du}{\dot{\theta}_i(1+u^2)\left(1+\varepsilon 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)}$$

$$dt = \frac{2du}{\dot{\theta}_i(1+u^2)\left(1+2\varepsilon\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}\right)}$$

After simplification the expresion is given by:

$$dt = \frac{2du}{\dot{\theta}_i(1+u^2+2\varepsilon \mathrm{u})}$$

By completing the square we get the following expression:

$$dt = \frac{2du}{\dot{\theta}_i((u+\varepsilon)^2+1-\varepsilon^2)}$$

Finally this could be arranged as an arctangens differential and is given by:

$$dt = \frac{2du}{\dot{\theta}_i(1-\varepsilon^2)\left(1+\left(\frac{u+\varepsilon}{\sqrt{1-\varepsilon^2}}\right)^2\right)}$$

After integration the previous expression from angle zero to 90 degrees in radians, equation (21) is derived.

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