Least Squares Estimators of Drift Parameter for Discretely Observed Fractional Vasicek-type Model

Abstract: We study the drift parameter estimation problem for a fractional Vasicek-type model $X := \{X_t, t \geqslant 0\}$, that is defined as $dX_t = \theta(\mu + X_t)dt + dB_t^H$, $t \geqslant 0$ with unknown parameters $\theta > 0$ and $\mu \in \mathbb{R}$, where $\{B_t^H, t \geqslant 0\}$ is a fractional Brownian motion of Hurst index $H \in]0, 1[$. Let $\widehat{\theta_t}$ and $\widehat{\mu_t}$ be the least squares-type estimators of θ and μ, respectively, based on continuous observation of X. In this paper we assume that the process $\{X_t, t \geqslant 0\}$ is observed at discrete time instants t_i=iΔ_n, i=1,...,n. We analyze discrete versions $\widetilde{\theta_n}$ and $\widetilde{\mu_n}$ for $\widehat{\theta_t}$ and $\widehat{\mu_t}$ respectively. We show that the sequence $\sqrt{n\Delta_n}\left(\widetilde{\theta_n} - \theta\right)$ *is tight and* $\sqrt{n\Delta_n}(\widetilde{\mu_n} - \mu)$ *is not tight. Moreover, we prove the stronge consistency of* $\widetilde{\theta_n}$ *.*

Key words: Fractional Brownian motion; Vasicek-type model; Young integral; Parameter estimation; Discrete observations; Tightness.

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1. Introduction

Let $B^H := \{B_t^H, t \geqslant 0\}$ be a fractional Brownian motion (fBm) of Hurst index $H \in]0,1[$, that is, a centered Gaussian process starting from zero with covariance

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

Note that when $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a standard Brownian motion.

Consider the fractional Vasicek-type of the first kind $X := \{X_t, t \geqslant 0\}$, defined as the unique (pathwise) solution to

$$\begin{cases} dX_t = \theta(\mu + X_t)dt + dB_t^H, & t > 0, \\ X_0 = 0, \end{cases} \qquad (1.1)$$

where $\mu \in \mathbb{R}$ and $\theta > 0$ are considered as unknown parameters.

Let $\widehat{\theta_T}$ and $\widehat{\mu_T}$ be the least squares-type estimators of and μ, respectively, based on continuous observation of X. As we known, least squares estimators method are motivated by the argument of minimize a quadratic function μ a and θ, respectively,

$$(\mu, \theta) \mapsto \int_0^T |\dot{X}_t - \theta(\mu + X_t)|^2 \, dt$$

where $\dot{X}_t$ denotes the differentiation of X_t with respect to t. Taking the partial derivative for μ a and θ, separately. Then solving the equations, we can obtain the east squares estimators of μ a and θ, denoted by $\widehat{\theta_T}$ and $\widehat{\mu_T}$ respectively,

$$\widehat{\theta_T} = \frac{\frac{1}{2}TX_T^2 - X_T \int_0^T X_s ds}{T\int_0^T X_S^2 ds - \left(\int_0^T X_s ds\right)^2} \tag{1.2}$$

$$\widehat{\mu_T} = \frac{\int_0^T X_S^2 ds - \frac{1}{2}X_T \int_0^T X_S ds}{\frac{1}{2}TX_T - \int_0^T X_s ds} \tag{1.3}$$

In recent years, the study of various problems related to the model (1.1) has attracted interest. In finance modeling μ can be interpreted as the long-run equilibrium value of X whereas θ represents the speed of reversion. For a motivation in mathematical finance and further references, we refer the reader to [2,3, 4, 5]. When B^H is replaced by a standard Brownian motion, the model (1.1) with $\mu = 0$ was originally proposed by Ornstein and Uhlenbeck and then it was generalized by Vasicek, see [14]. In the ergodic case, the statistical inference for several fractional Ornstein-Uhlenbeck (fOU) models has been recently developed in the papers [8], [11] and [15]. The case of non-ergodic fOU process can be found in [1], [6], [7], [9] and [10].

Let us describe what is known about the asymptotic behaviors of the estimators (1.2) and (1.3), studied in [9]:

- for every $H \in (0,1)$, we have almost surely, as $T \to \infty$,

$$\left(\widehat{\theta_T}, \widehat{\mu_T}\right) \to (\theta, \mu) \tag{1.4}$$

- assume that $H \in (0,1)$, and $N_1 \sim \mathcal{N}(0,1)$, $N_2 \sim \mathcal{N}(0,1)$, and B^H are independent, then as $T \to \infty$,

$$\left(e^{\theta T}\left(\widehat{\theta_T} - \theta\right), T^{1-H}\left(\widehat{\mu_T} - \mu\right)\right) \xrightarrow{Law} \left(\frac{2\theta\sigma_{B^H} N_2}{\mu + \zeta_{B^H,\infty}}, \frac{1}{\theta}N_1\right), \tag{1.5}$$

$\sigma_{B^H}^2 = \frac{H\Gamma(2H)}{\theta^{2H}}$, and $\zeta_{B^H,\infty} \sim \mathcal{N}\left(0, \sigma_{B^H}^2\right)$ is independent of N_1 and N_2.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for (1.1) based on discrete observations. Then, in the present paper, we will assume that the process *X* given in (1.1) is observed equidistantly in time with the step size Δ_n: t_i=iΔ_n, i=1,...,n and $T_n = n\Delta_n$ denotes the length of the "observation window".

Here, based on discrete-time observations of *X* defined in (1.1), we will analyse the following discrete versions $\widetilde{\theta_n}$ and $\widetilde{\mu_n}$ for $\widehat{\theta_t}$ and $\widehat{\mu_t}$ respectively, defined as

$$\widetilde{\theta_n} = \frac{\frac{1}{2}X_{T_n}^2 - \frac{X_{T_n}}{n}\sum_{i=1}^n X_{t_{i-1}}}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2 - \frac{\Delta_n}{n}\left(\sum_{i=1}^n X_{t_{i-1}}\right)^2} \tag{1.6}$$

$$\widetilde{\mu_n} = \frac{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2 - \frac{1}{2}X_{t_n}\Delta_n \sum_{i=1}^n X_{t_{i-1}}}{\frac{1}{2}T_n X_{T_n} - \Delta_n \sum_{i=1}^n X_{t_{i-1}}} \tag{1.7}$$

Our paper is organized as follows. In Section 2, we give the basic knowledge about Young integral and some preliminary results, which will be very useful to our main proof. In Section 3, based on discrete observations of X defined in (1.1), we study the rate consistency of the estimators $\widetilde{\theta_n}$ and $\widetilde{\mu_n}$.

2. Preliminaries

In this section, we briefly recall some basic elements of Young integral (see [16]), which are helpful for some of the arguments we use.

For any $\alpha \in [0,1]$, , we denote by $\mathcal{H}^\alpha([0,1])$ the set of Holder continuous functions, that is, the set of functions $f: [0,T] \to \mathbb{R}$ such that

$$|f|_\alpha = \underset{0 \le s < t \le T}{Sup} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty .$$

We also set $|f|_\infty := Sup_{t\in[0,T]}|f(t)|$ and equip $\mathcal{H}^\alpha(|[0,T]|)$ with the norm

$$\|f\|_\alpha := |f|_\alpha + |f|_\infty \ .$$

Let $f \in \mathcal{H}^\alpha([0,T])$, and consider the operator $T_f : C^1([0,T]) \to C^0([0,T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \qquad t \in [0,T].$$

It can be shown (see, [13]) that, for any $\beta \in]1-\alpha, 1[$, there exists a constant $C_{\alpha,\beta,T} > 0$ depending only on α, β and T such that, for any $g \in \mathcal{H}^{\alpha}([0,T])$,

$$\left\| \int_0^{\cdot} f(u)g'(u)du \right\|_{\beta} \leq C_{\alpha,\beta,T} \|f\|_{\alpha} \|g\|_{\beta}.$$

We deduce that, for any $\alpha \in]0,1[$ any $f \in \mathcal{H}^{\alpha}([0,T])$ and any $\beta \in]1-\alpha, 1[$ the linear operator $T_f : C^1([0,T]) \subset \mathcal{H}^{\beta}([0,T]) \to \mathcal{H}^{\beta}([0,T])$, defined as $T_f(g) = \int_0^{\cdot} f(u)g'(u)du$ is continuous with respect to the norm $\|.\|_{\beta}$.

By density, it extends (in an unique way) to an operator defined on $\mathcal{H}^{\beta}$. As consequence, if $f \in \mathcal{H}^{\alpha}(|[0,T]|)$, if $g \in \mathcal{H}^{\beta}(|[0,T]|)$ and if $\alpha + \beta > 1$ then the (so-called) Young integral $\int_0^{\cdot} f(u)dg(u)$ is (well) defined as being $T_f(g)$.

The Young integral obeys the following formula. Let $f \in \mathcal{H}^{\alpha}([0,T])$ with $\alpha \in]0,1[$ and $g \in \mathcal{H}^{\beta}([0,T])$ with $\beta \in]0,1[$ such that $\alpha + \beta > 1$. Then $\int_0^{\cdot} f_u dg_u$ and $\int_0^{\cdot} f_u dg_u$ are well-defined as Young integrals. Moreover, for all $t \in [0,T]$,

$$f_t g_t = f_0 g_0 + \int_0^t g_u \, df_u + \int_0^t f_u dg_u \ . \qquad (2.1)$$

In order to study the strong consistency, we will need the following direct consequence of the BorelCantelli Lemma (see Kloeden and Neuenirch (2007)), which allows us to turn convergence rates in the p-th mean into pathwise convergence rates .

Lemma 2.1. ([12]) Let $\beta > 0$ and let $(Z_n)_{n \in \mathbb{N}}$ *be a sequence of random variables. If for every* $p \geq 1$ *there exists a constant* $c_p > 0$ *such that for all* $n \in \mathbb{N}$,

$$(E|Z_n|^p)^{1/p} \leq C_p . n^{-\beta} \quad ,$$

then for all $\varepsilon > 0$ *there exists a random variable* η_{ε} *such that*

$$|Z_n| \leq \eta_{\varepsilon} . n^{-\beta+\varepsilon} \text{ almost surely}$$

for all $n \in \mathbb{N}$. Moreover, $E|\eta_{\varepsilon}|^p < \infty$ for all $p \geq 1$.

Next, let us note that the unique solution to (1.1) can be written as

$$X_t = \mu\left(e^{\theta t} - 1\right) + e^{\theta t} \int_0^t e^{-\theta s} \, dB_s^H \ , \ t \geq 0 \, . \qquad (2.2)$$

We will also need the following processes, for every $t \geq 0$

$$\zeta_t := \int_0^t e^{-\theta t} dB_s^H \ ; \quad \Sigma_t := \int_0^t X_s ds Z_t := \int_0^t e^{-\theta t} B_s^H ds \qquad (2.3)$$

Using (2.2), we can write

$$X_t = u\left(e^{-\theta t} - 1\right) + e^{-\theta t}\zeta_t \ . \qquad (2.4)$$

Furthermore, by (1.1),

$$X_t = \mu\theta t + B_t^H \ . \qquad (2.5)$$

Moreover, applying the formula (2.1), we have

$$\zeta_t = e^{-\theta t} B_t^H + \theta \int_0^t e^{-\theta t} B_s^H ds = e^{-\theta t} B_t^H + \theta Z_t \ . \qquad (2.6)$$

From (2.4) we can also write

$$X_t = e^{\theta t} Z_t, \quad \text{With } Z_t = \mu\left(1 - e^{-\theta t}\right) + \zeta_t \qquad t \geq 0. \qquad (2.7)$$

Lemma 2.2. ([6]). *Assume that the process B^H has Hölder continuous path of order* $\gamma \in \left]0,1\right[$. *Let ζ be given by (2.3). Then for all* $\varepsilon \in \left]0,\gamma\right[$ *the process ζ admits a modification with $(\gamma - \varepsilon)$-Hölder continuous paths.*

Moreover

$$Z_t \to Z_\infty := \int_0^\infty e^{-\theta t} B_s^H ds, \qquad \zeta_t \to \zeta_\infty := \theta Z_\infty \qquad (2.8)$$

almost surely and in $L^2(\Omega)$ as $T \to \infty$.

Lemma3.2. ([9]). *Assume that H* $\in$(0, 1)*. Then, almost surely, as*

$$e^{-\theta T} X_T \to \mu + \zeta_\infty \qquad (2.9)$$

$$e^{-\theta T} \int_0^T X_s \, ds \to \frac{1}{\theta}(\mu + \zeta_\infty) \qquad (2.10)$$

$$\frac{e^{-\theta T}}{T} \int_0^T s X_s \, ds \to \frac{1}{\theta}(\mu + \zeta_\infty) \qquad (2.11)$$

$$\frac{e^{-\theta T}}{T^\delta} \int_0^T |X_s| \, ds ds \to 0 \quad \text{for any } \delta > 0 \qquad (2.12)$$

$$e^{-2\theta T} \int_0^T X_s^2 \, ds \to \frac{1}{2\theta}(\mu + \zeta_\infty)^2 \qquad (2.13)$$

where is defined in Lemma 2.2.

From now on, the generic constant is always denoted by $C(.)$ which depends on certain parameters in the parentheses.

3.Main results

Lemme 3.1. Let $(\mathrm{S_n}, \mathrm{n} \geq 1)$ and $(\mathrm{R_n}, \mathrm{n} \geq 2)$ be a random sequences defined by

$$\mathrm{S_n} := \Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2 \quad ; \quad \mathrm{S_n} := \Delta_n \sum_{i=1}^{n-1} e^{-2\theta(T_n - t_i)}(Z_{t_i}^2 - Z_{t_{i-1}}^2). \quad (3.1)$$

Then for every $\mathrm{n} \geq 2$,

$$S_n e^{-2\theta T_n} = \frac{\Delta_n}{e^{2\Delta_n} - 1}(Z_{t_{n-1}}^2 - R_n)\ . \quad (3.2)$$

In addition if $\Delta_n \to 0$ and $\mathrm{n}\Delta_\mathrm{n}^{1+\alpha} \to \infty$ for some $\alpha > 0$,

$$\mathrm{R_n} \to 0 \text{ almost surely as } \mathrm{n} \to \infty. \quad (3.3)$$

In particular,

$$S_n e^{-2\theta T_n} \to \frac{(\mu + \zeta_\infty)^2}{2\theta} \text{ almost surely as } \mathrm{n} \to \infty. \quad (3.4)$$

Proof. Using (2.7), we can write for every $n \geq 2$,

$$S_n e^{-2\theta T_n} = \Delta_n \sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n} e^{-2\theta\Delta_n} Z_{t_{i-1}}^2$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n} \left(1 - \frac{1}{e^{2\theta\Delta_n}}\right) Z_{t_{i-1}}^2.$$

This imply that

$$S_n e^{-2\theta T_n} = \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} \left(e^{-2\theta(n-i)\Delta_n} - e^{-2\theta(n+1-i)\Delta_n}\right) Z_{t_{i-1}}^2$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\left[Z_{t_{n-1}}^2 - \sum_{i=1}^{n} \left(Z_{t_{i-1}}^2 - Z_{t_{i-2}}^2\right) e^{-2\theta(n+1-i)\Delta_n}\right]$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\left[Z_{t_{n-1}}^2 - R_n\right] \quad ,$$

which implies (3.2).

Let us now prove (3.3). First, observe that $\Delta_n \to 0$ and $\mathrm{n}\Delta_\mathrm{n}^{1+\alpha} \to \infty$ imply that $\mathrm{n}\Delta_\mathrm{n} \to \infty$. On the other hand, (2.8) implies

$$Z_T \to \mu + \zeta_\infty \quad (3.5)$$

almost surely and in $L^2(\Omega)$ as $\mathrm{T} \to \infty$.

Thus, by using (2.7), $\{\zeta_t, t \geq 0\}$ is Gaussian and (3.5), we obtain for every $\mathrm{p} \geq 0$,

$$\left(E\left[\left|Z_{t_i}^2 - Z_{t_{i-1}}^2\right|^p\right]\right)^{\frac{1}{p}} \leq \left(E\left[\left|\left(Z_{t_i} - Z_{t_{i-1}}\right)\left(Z_{t_i} + Z_{t_{i-1}}\right)\right|^p\right]\right)^{\frac{1}{p}}$$

$$\leq C(\mu,\theta,H)\left(E\left[\left|Z_{t_i} - Z_{t_{i-1}}\right|^p\right]\right)^{\frac{1}{p}}$$

$$\leq C(\mu,\theta,H)\left(\left|e^{-\theta t_i} - e^{-\theta t_{i-1}}\right| + \left(E\left[\left|\zeta_{t_i} - \zeta_{t_{i-1}}\right|^p\right]\right)^{\frac{1}{p}}\right)$$

$$\leq C(p,\mu,\theta,H)\left(e^{-\theta t_i}\left|e^{\theta\Delta_n} - 1\right| + \left(E\left[\left|\zeta_{t_i} - \zeta_{t_{i-1}}\right|^2\right]\right)^{\frac{1}{2}}\right)$$

$$\leq C(p,\mu,\theta,H)\left(\Delta_n e^{-\theta t_i} + \Delta_n^H e^{-\theta i \Delta_n}\right)$$

$$\leq C(p,\mu,\theta,H)\Delta_n^H e^{-\theta t_i}, \qquad (3.1)$$

where we used $\frac{e^{\theta\Delta_n} - 1}{\Delta_n} \to 0$ and the following inequality given in [10] for every $\mathrm{i} = 1,\ldots,\mathrm{n}, \quad \mathrm{n} \geq 1$,

$$\left(E\left[\left|\zeta_{t_i} - \zeta_{t_{i-1}}\right|^2\right]\right)^{\frac{1}{2}} \leq C(\theta,H)\Delta_n^H e^{-\theta t_i} \quad .$$

Thus for every $\mathrm{p} \geq 1$,

$$\left(E[|R_n|^p]\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n-1} e^{-2\theta(n-i)\Delta_n}\left(E\left[\left|Z_{t_i}^2 - Z_{t_{i-1}}^2\right|^p\right]\right)^{\frac{1}{p}}$$

$$\leq C(p,\mu,\theta,H)e^{-\theta n\Delta_n}\Delta_n^H \sum_{i=1}^{n-1} e^{-\theta(n-i)\Delta_n}$$

$$\leq C(p,\mu,\theta,H)e^{-\theta n\Delta_n}\Delta_n^H e^{-\theta\Delta_n}\frac{1 - e^{-\theta(n-1)\Delta_n}}{1 - e^{-\theta\Delta_n}}$$

$$\leq C(p,\mu,\theta,H)\Delta_n^{H-1} e^{-\theta n\Delta_n} \quad . \qquad (3.7)$$

The last inequality comes from $\Delta_n \to 0$ and $\frac{\Delta_n}{1 - e^{-\theta\Delta_n}} \to \frac{1}{\theta}$.

Taking a constant β verifying $\frac{1-\gamma}{\beta} < \alpha < \beta$, there is $\varepsilon > 0$ such that $\alpha = \frac{\varepsilon + 1 - \gamma}{\beta - \varepsilon}$.

Hence, we can write

$$(n\Delta_n)^{\beta}\Delta_n^{1-\gamma} = n^{\varepsilon}(n\Delta_n^{1+\alpha})^{\beta-\varepsilon} \quad . \tag{3.8}$$

As a consequence, by (3.7) and (3.8),

$$(E[|R_n|^p])^{\frac{1}{p}} \leq C(p,\theta,\mu,H)\Delta_n^{\gamma-1}e^{-\theta n\Delta_n}$$

$$\leq C(p,\theta,\mu,H)\frac{1}{n^{\varepsilon}(n\Delta_n^{1+\alpha})^{\beta-\varepsilon}}\frac{(n\Delta_n)^{\beta}}{e^{\theta n\Delta_n}}$$

$$\leq C(p,\theta,\mu,H)n^{-\varepsilon}. \tag{3.9}$$

Therefore, by combining (3.9) and Lemma 2.1, the convergence (3.3) is proved. On the other hand, the convergence (3.4) is a direct consequence of (3.2), (3.3) and (3.5). □

Lemme 3.2. Define for every $n \geq 1$

$$D_n := \frac{e^{-2\theta T_n}}{n}\sum_{i=1}^{n} X_{t_{i-1}}. \tag{3.10}$$

Assume that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$, then, for every $n \geq 1$,

$$E\left({D_n}^2\right) \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha}{1+\alpha}} \tag{3.11}$$

Moreover, for every $0 \leq \delta < 1$,

$$E\left[\left((n\Delta_n)^{\delta}D_n\right)^2\right] \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha(1-H)}{1+\alpha}}. \tag{3.12}$$

As a consequence, for every $0 \leq \delta < 1$,

$$(n\Delta_n)^{\delta} \to 0 \text{ almost surely as } n \to \infty. \tag{3.13}$$

Proof. We first prove (3.11). Using (2.7) and (3.5), we have

$$E(D_n^2) = \frac{e^{-2\theta T_n}}{n^2}\sum_{i,j=1}^{n} E\left(X_{t_{i-1}}X_{t_{j-1}}\right) = \frac{e^{-2\theta T_n}}{n^2}\sum_{i,j=1}^{n} e^{\theta t_{i-1}+\theta t_{j-1}}E\left(Z_{t_{i-1}}Z_{t_{j-1}}\right)$$

$$\leq C(\theta,\mu,H)\frac{e^{-2\theta T_n}}{n^2}\sum_{i,j=1}^{n} e^{\theta t_{i-1}+\theta t_{j-1}} = C(\theta,\mu,H)\left(\frac{e^{-\theta T_n}}{n}\sum_{i=1}^{n} e^{\theta t_{i-1}}\right)^2$$

$$= C(\theta,\mu,H)\left(\frac{e^{-\theta T_n}}{n}\frac{e^{\theta n\Delta_n}-1}{e^{\theta\Delta_n}-1}\right)^2$$

$$\leq C(\theta,\mu,H)\left(\frac{1}{n\Delta_n}\,\frac{\Delta_n}{e^{\theta\Delta_n}-1}\right)^2$$

$$\leq C(\theta,\mu,H)\,\frac{1}{(n\Delta_n)^2}\quad. \qquad (3.14)$$

Setting $\gamma = \frac{\alpha}{1+\alpha}$, we obtain

$$E(D_n^2) \leq C(\theta,\mu,H)\,\frac{n^{-2\gamma}}{(n^{1-\gamma}\Delta_n)^2} = C(\theta,\mu,H)\,\frac{n^{-\frac{2\alpha}{1+\alpha}}}{(n\Delta_n^{1+\alpha})^{\frac{1}{1+\alpha}}} \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha}{1+\alpha}},$$

which proves (3.11).

For (3.12), by (3.14), we have,

$$E[((n\Delta_n)^H D_n)^2] \leq C(\theta,\mu,H)(n\Delta_n)^{-2(1-\gamma)}.$$

Thus, using similar arguments as in (3.8), we can conclude

$$E[((n\Delta_n)^H D_n)^2] \leq C(\theta,\mu,H,\alpha)n^{-\frac{2\alpha(1-H)}{1+\alpha}},$$

which implies the desired result.

Finally, the convergence (3.13) is a direct consequence of (3.12) and Lemma 2.1. □

Definition 3.1. Let $\{Z_n\}$ be a sequence of random variables defined on $(\Omega,\mathcal{F},P)$. We say $\{Z_n\}$ is tight (or bounded in probability), if for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that,

$$P(|Z_n| > M_\varepsilon) < \varepsilon\,, \quad \text{for all n.}$$

Theorem 3.3. Let $H \in (0,1)$. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$. Then, for every $q \geq 1$,

$$\Delta_n^q e^{\theta T_n}\left(\widetilde{\theta_n} - \theta\right) \text{ is not tight.} \qquad (3.15)$$

In addition if we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$, then the estimator $\widetilde{\theta_n}$ is $\sqrt{T_n}$-consistent in the sens that the sequence

$$\sqrt{T_n}\left(\widetilde{\theta_n} - \theta\right) \text{ is tight} \qquad (3.16)$$

and

$$\sqrt{T_n}\left(\widetilde{\mu_n} - \mu\right) \text{ is not tight.} \qquad (3.17)$$

Proof. Fix $q \geq 1$. From (1.6) and (2.7) we can write

$$\Delta_n^q e^{\theta T_n}\left(\widetilde{\theta_n}-\theta\right)$$

$$= \Delta_n^q e^{\theta T_n}\left(\frac{\frac{1}{2}Z_{T_n}^2 - Z_{T_n}D_n}{e^{2\theta T_n}S_n - \left(\sqrt{T_n}D_n\right)^2} - \theta\right)$$

$$= \frac{\Delta_n^q e^{\theta T_n}}{2e^{2\theta T_n}S_n - 2\left(\sqrt{T_n}D_n\right)^2}\left[\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)Z_{T_{n-1}}^2 - 2\theta\left(e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n}-1}Z_{T_{n-1}}^2\right)\right]$$

Moeover,

$$e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n}-1}Z_{T_{n-1}}^2 = e^{-2\theta T_n}\Delta_n\sum_{i=1}^n e^{2\theta t_{i-1}}Z_{t_{i-1}}^2 - \frac{\Delta_n}{e^{2\theta\Delta_n}-1}Z_{T_{n-1}}^2$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\left(\sum_{i=1}^n e^{-2\theta(T_n-t_i)}Z_{t_{i-1}}^2 - \sum_{i=1}^n e^{-2\theta(T_n-t_{i-1})}Z_{t_{i-1}}^2 - Z_{T_{n-1}}^2\right)$$

$$= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}R_n\ ,$$

where R_n is given by (3.1).

Thus we obtain

$$\Delta_n^q e^{\theta T_n}\left(\widetilde{\theta_n}-\theta\right) = \frac{\Delta_n^q e^{\theta T_n}}{2e^{2\theta T_n}S_n}\left[\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)Z_{T_{n-1}}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)R_n\right]. \quad (3.18)$$

According to (3.6), we get

$$\left(E\left[\left(\Delta_n^q e^{\theta T_n}\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right)\right)^2\right]\right)^{\frac{1}{2}} \leq C(\theta,\mu,H)\Delta_n^{q+H} \to\ 0\ .$$

(3.19)

We also have

$$\Delta_n^q e^{\theta T_n}\left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) = \Delta_n^{q+1}e^{\theta T_n}\left(\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\ \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\right) \to \infty \qquad (3.20)$$

since

$$\Delta_n^{q+1}e^{\theta T_n} = \left(n\Delta_n^{q+\alpha}\right)^{\frac{q+1}{\alpha}}\frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\alpha}}} \to \infty \ \text{ and } \ \left(\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\ \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\right) \to \theta.$$

Furthermore, by (3.7),

$$\left(E\left[\left(\Delta_n^q e^{\theta T_n} R_n\right)^2\right]\right)^{\frac{1}{2}} \leq \mathrm{C}(\theta,\mu,\mathrm{H})\Delta_n^{q+H-1} \rightarrow 0\,. \qquad (3.21)$$

Combining (3.18), (3.19), (3.20), (3.21) and (3.4), we conclude that for every $q \geq 1$, $\Delta_n^q e^{\theta T_n}\left(\widetilde{\theta_n} - \theta\right)$ is not tight.

For $0 \leq q < 1$ we have

$$\Delta_n^q e^{\theta T_n}\left(\widetilde{\theta_n} - \theta\right) = \Delta_n^{q-1}\left(\Delta_n e^{\theta T_n}\left(\widetilde{\theta_n} - \theta\right)\right),$$

which completes the proof of (3.15), where we used the previous case and the fact that $\Delta_n^{q-1} \rightarrow \infty$.

Let us now prove (3.16). It follows from (3.18) that

$$\sqrt{T_n}\left(\widetilde{\theta_n} - \theta\right) = \frac{\sqrt{T_n}}{2e^{-2\theta T_n} S_n}\left[\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) Z_{T_n-1}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) R_n\right].$$

Combining this with

$$\left(E\left[\left(\sqrt{T_n}\left(Z_{T_n}^2 - Z_{T_{n-1}}^2\right)\right)^2\right]\right)^{\frac{1}{2}} \leq \mathrm{C}(\theta,\gamma)\Delta_n^{\gamma}\sqrt{T_n}e^{-\theta T_n} \rightarrow 0\,,$$

$$\sqrt{T_n}\left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) = \sqrt{n\Delta_n^3}\left(\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\,\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\right) \rightarrow 0\,,$$

$$\left(E\left[\left(\sqrt{T_n}R_n\right)^2\right]\right)^{\frac{1}{2}} \leq \mathrm{C}(\theta,\gamma)\Delta_n^{\gamma-1}\sqrt{T_n}e^{-\theta T_n} = \mathrm{C}(\theta,\gamma)\frac{T_n^{\frac{1}{2}+\frac{1-\gamma}{\alpha}} e^{-\theta T_n}}{\left(n\Delta_n^{1+\alpha}\right)^{\frac{1-\gamma}{\alpha}}} \rightarrow 0\,,$$

and the convergence (3.4), we deduce that

$$\sqrt{T_n}\left(\widetilde{\theta_n} - \theta\right) \rightarrow 0 \qquad (3.22)$$

in probability, which proves (3.16).

Now it remains to prove (3.17). Using (1.6) and (1.7), we can show that $\widetilde{\theta_n}$ and $\widetilde{\mu_n}$ satisfy

$$\widetilde{\theta_n}\widetilde{\mu_n}T_n = \frac{X_{T_n}\left(\sum_{i=1}^n X_{t_{i-1}}^2 - \frac{X_{T_n}}{n}\sum_{i=1}^n X_{t_{i-1}}\right)}{\sum_{i=1}^n X_{t_{i-1}}^2 - \frac{1}{n}\left(\sum_{i=1}^n X_{t_{i-1}}\right)^2}$$

$$= X_{T_n} - \widetilde{\theta_n}\Delta_n \sum_{i=1}^n X_{t_{i-1}}.$$

Combining this with (1.1), we obtain

$$\mathrm{T_n}\widetilde{\theta_n}(\widetilde{\mu_n}-\mu)$$

$$=\mu T_n\left(\theta-\widetilde{\theta_n}\right)+\theta\int_0^{T_n}X_t\,dt+B_{T_n}^H-\widetilde{\theta_n}\Delta_n\sum_{i=1}^{n}X_{t_{i-1}}$$

$$=\mu T_n\left(\theta-\widetilde{\theta_n}\right)+\widetilde{\theta_n}\left(\int_0^{T_n}X_t\,dt-\Delta_n\sum_{i=1}^{n}X_{t_{i-1}}\right)+\left(\theta-\widetilde{\theta_n}\right)\int_0^{T_n}X_t\,dt+B_{T_n}^H.$$

Thus, we obtain

$$\sqrt{\mathrm{T_n}}(\widetilde{\mu_n}-\mu)$$

$$=\frac{\mu\sqrt{\mathrm{T_n}}}{\widetilde{\theta_n}}\left(\theta-\widetilde{\theta_n}\right)+\frac{1}{\sqrt{T_n}}\left(\int_0^{T_n}X_t\,dt-\Delta_n\sum_{i=1}^{n}X_{t_{i-1}}\right)+\frac{\left(\theta-\widetilde{\theta_n}\right)}{\widetilde{\theta_n}\sqrt{T_n}}\int_0^{T_n}X_t\,dt+\frac{B_{T_n}^H}{\widetilde{\theta_n}\sqrt{T_n}}$$

$$:=\mathrm{A_n}+\mathrm{B_n}+\mathrm{C_n}+\mathrm{D_n}\ .$$

Theorem 3.2 and the convergence (3.22) imply that $\mathrm{A_n}\to 0$ in probability.

We can write $C_n=\frac{\left(\theta-\widetilde{\theta_n}\right)}{\widetilde{\theta_n}\sqrt{T_n}}\int_0^{T_n}X_t\,dt=\frac{\sqrt{T_n}\left(\theta-\widetilde{\theta_n}\right)}{\widetilde{\theta_n}}\left(\frac{1}{T_n}\int_0^{T_n}X_t\,dt\right).$

Then, Theorem 3.2 and the convergence (3.22) imply that $\frac{\sqrt{T_n}\left(\theta-\widetilde{\theta_n}\right)}{\widetilde{\theta_n}}\to 0$ in probability. Moreover, using l'Hôpital rule,

$$\lim_{T_n\to\infty}\frac{1}{T_n}\int_0^{T_n}X_t\,dt=\lim_{T_n\to\infty}X_{T_n}=\lim_{T_n\to\infty}\left(\mu\left(1-e^{-\theta T_n}\right)+\zeta_{T_n}\right)=\mu+\zeta_\infty.$$

Hence $\mathrm{C_n}\to 0$ in probability.

Recall that $\mathrm{E}[(B_t^H-B_s^H)^2]=|\mathrm{t}-\mathrm{s}|^{2\mathrm{H}}$; $\mathrm{t},\mathrm{s}\geq 0$.

Then for $\mathrm{H}\in\left]0,\frac{1}{2}\right[$, we have almost surely, as $T_n\to\infty$

$$\frac{B_{T_n}^H}{\sqrt{T_n}}\to 0,\quad \text{by Borel-Cantelli Lemma.}$$

Combining this with Theorem 3.2 we obtain that $D_n:=\frac{B_{T_n}^H}{\widetilde{\theta_n}\sqrt{T_n}}\to 0$ in probability.

$$B_n := \frac{1}{\sqrt{T_n}}\left(\int_0^{T_n} X_t\,dt - \Delta_n \sum_{i=1}^{n} X_{t_{i-1}}\right)$$
$$= \frac{e^{\theta T_n}}{\sqrt{T_n}}\left(e^{-\theta T_n}\int_0^{T_n} X_t\,dt - e^{-\theta T_n}\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}\right) \qquad (3.23)$$

By lemma 2.3, we have $e^{-\theta T_n}\int_0^{T_n} X_t\,dt \to \frac{1}{\theta}(\mu + \zeta_\infty)$ almost surely.

We also have

$$\mathrm{E}\left[\left(e^{-\theta T_n}\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}\right)^2\right] = \Delta_n^2 e^{-2\theta T_n} \sum_{i,j=1}^{n} \mathrm{E}\left(X_{t_{i-1}} X_{t_{j-1}}\right) = \Delta_n^2 e^{-2\theta T_n} \sum_{i,j=1}^{n} e^{\theta t_{i-1}+\theta t_{j-1}} \mathrm{E}\left(Z_{t_{i-1}} Z_{t_{j-1}}\right).$$

Then, by using the same arguments as in Lemma 3.2, we obtain

$$\mathrm{E}\left[\left(e^{-\theta T_n}\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}\right)^2\right] \leq C(\mu,\theta,H)\Delta_n^2 e^{-2\theta T_n}\left(\frac{e^{\theta n \Delta_n}-1}{e^{\theta \Delta_n}-1}\right)^2 \leq C(\mu,\theta,H)\Delta_n^2 \to 0. \quad (3.24)$$

Combining (2.10), (3.23), (3.24), and the fact that $\frac{e^{\theta T_n}}{\sqrt{T_n}} \to \infty$, we conclude that $B_n \to \infty$.

Consequently, the convergence (3.17) is proved. Thus the desired results are obtained. □

Theorem 3.2. Assume that $0 < H < 1$. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to 0$ for some $\alpha > 0$. Then as $n \to \infty$,

$$\widetilde{\theta_n} \to \theta \quad \textit{almost surely.} \qquad (3.25)$$

Proof. *We can write*

$$\widetilde{\theta_n} = \frac{\frac{1}{2}X_{T_n}^2 - \frac{X_{T_n}}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2 - \frac{\Delta_n}{n}\left(\sum_{i=1}^{n} X_{t_{i-1}}\right)^2}$$
$$= \frac{\frac{1}{2}e^{-2\theta T_n} X_{T_n}^2 - Z_{T_n} D_n}{e^{-2\theta T_n} S_n - \left(\sqrt{n\Delta_n}\, D_n\right)^2}.$$

Thus, according to (2.9) , (3.4) , (3.5) and (3.13), we can deduce that

$\widetilde{\theta_n} \to \theta$ *almost surely as* $n \to \infty$. □

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