Unifying Inflation-Dark energy through scalar field and Quadratic Teleparallel model constrained by observational data

Abstract

The present work addresses two most tenacious enigmatic phases in the dynamical evolution of the universe. Quadratic Teleparallel T^2 model coupled with scalar field as only universe content is constrained with observational data to provide significant results on inflationary scenario and the late time expansion of the universe powered by the dark energy. Mathematical method of functions separation is used to solve the Friedmann-like equations induced by quadratic model. Firstly, we obtain scalar field and its potential whose expressions have permitted to compute the inflationary observable as e.fold number function. For suitable choice of model parameters, numerical analysis leads to results in agreement with Planck data and BICEP2 experiment. Secondly, the energy density and the pressure of the scalar field are provided versus the redshift z in a context where the Hubble parameter obtained from Friedmann-like equation resolution, is constrained with the current observational data. Under this consideration, the state equation parameter $\omega_\phi(z) \longrightarrow -1$ leading to the conclusion that the scalar field behaves like the ΛCDM model when the quadratic model is constrained with observational data.

Keywords: *Inflation, Dark Energy, slow-roll, scalar factor, scalar field, teleparallel.*

1 Introduction

Several cosmological investigations, especially cosmological observation have given the evidence of the current expansion acceleration of the universe [1]-[4]. Such scenario is powered by the so-called dark energy whose widely accepted candidate for its explanation is the cosmological constant [5]. This choice is supported by observational approach promoted by the standard model of cosmology. When theoretically searching for dark energy nature, several approaches have been introduced with goal of confirming observational predictions.

As alternative ways to cosmological constant, several dark energy models, basing on the scalar field have been adopted to try and explain the remarkable observation of our accelerating Universe. In general, scalar field is introduced to explore the dynamical feature of the dark energy [6]-[8]. An interesting brief review on these models has been performed in [7] from which we can cite quintessence, K-essence, tachyon, phantom and dilatonic models. Under these approaches, it is hoped an state equation which varies versus times with the possibility of crossing phantom barrier ($\omega = -1$) [9]-[11]. Furthermore, multi-component nature of the dark energy including cosmological constant and scalar field is also explored in these work and showed to fit more observational data.

Moreover, others sources sustain the idea that dark energy may have a geometrical origin, i.e., that there is a connection between Dark Energy and a non-standard behavior of gravitation on cosmological scales has resulted in it becoming a very active area of research over the past few years (see for example [12]-[14]). The current acceleration of the universe expansion has also profited explanation from the modification of standard theory of gravitation: General Relativity based on the non-vanishing curvature connexion and its equivalent theory called the teleparallel theory based on non-vanishing torsion connexion [15]. The most addressed modified theories are f (R)[16], f (G) [17], f (T)[18] etc... It is important to note here that in addition to the dark energy problem for which they were initiated, these theories produce very interesting results in the study of inflation and also the structures formation [19]-[23] in the universe. Especially, in the context of inflation description, the scalar field is added not only to provide theoretical representation of the inflationary observable [24] and to compare them to observational data, but also, to provide gravitational Lagrangian density that mimics the same cosmological expansion as the scalar field-driven inflation of General Relativity. Recently, the introduction of scalar field in modified theory have further enriched the debates on the expansion of the universe and especially the exit from inflation [25]-[26].

The inflation studying and the ΛCDM model give to the standard model of cosmology all the necessary tools to better reflect the realities of observational data. The challenge then lies in finding models that unify inflation and dark energy via the scalar field, as well as the transition between these phases. This is precisely the problem that this paper attempts to address. The problem will be addressed in the framework of f (T). As brief motivation on this theory, the f (T) theory leads to gravitational second-order field equations, the same as for GR, while it is of the fourth order in the context of f (R). From this point of view, this theory is better adapted to deal with cosmological enigmas in modified theories of gravity. By the way, it exists one form of this theory which specially retains a lot of attention: the quadratic f (T) model which is the similar formulation of Starobinskymodel in f (R) background [27],[28]. For example, the author in [29] has explained how the quadratic form of the scalar torsion can provide an origin for late accelerated phase of the universe in the Friedmann-Roberson-Walker background. Furthermore, a gravitational model which can support simultaneously the inflation and dark energy must be able to provide an exit from inflation. An meaningful example can be seen in [30] where the trace-anomaly driven inflation related to quadratic model produces de Sitter inflation with graceful exit. So, quadratic f (T) model remains viable model to provide a unified way to address the inflation scenario and the dark energy problem via the scalar field. To avoid an arbitrary description, we also challenge the constraint of the model parameters to the observational data.

The present paper is organized as follows: in the section Sec.2, we introduce the f (T) theory by establishing the main equations. The section Sec.3 is devoted to the inflationary scenario description from the main equations and comparison to observational data. The sections Sec.4 addresses dark energy and the cosmological scope of the obtained results. The paper is ended by the section Sec.3 which presents the conclusion.

2 Main equations in the coupling modified teleparallel theory and scalar field

The modified teleparallel theory $f(T)$ action is expressed as [31]

$$S = \frac{1}{4\kappa^2}\int d^4x h f(T) + \int d^4x h \mathcal{L}_M, \tag{1}$$

where $h = |\det(h^a{}_\mu)|$ is equivalent to $\sqrt{-g}$ in General $\kappa^2 = \frac{16\pi G}{c^4}$, $\mathcal{L}_M$ Relativity,is the Lagrangian of the matter field. Then, the variation of this action with respect to the tetrads $h^a{}_\mu$ gives

$$\frac{1}{h}\partial_\mu(hS_a{}^{\mu\nu})f_T(T) - h_a{}^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu}f_T(T) + A^i{}_{a\mu}S_i{}^{\mu\nu}f_T(T) + S_a{}^{\mu\nu}\partial_\mu(T)f_{TT}(T) + \frac{1}{4}h_a{}^\nu f(T) = \frac{1}{4\kappa^2}T_a^\nu, \tag{2}$$

where $f_T(T) = df(T)/dT$, $f_{TT}(T) = d^2f(T)/dT^2$ and T_a^ν represents the energy-momentum tensor. In this study, we consider a universe described by the Friedmann-Lemaitre-Robertson-Walker metric given by

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{3}$$

where $a(t)$ denotes the scale factor. The scalar torsion related to the metric Eq.(3) is given by

$$T = -6H^2(t), \tag{4}$$

where $H(t)$ is the Hubble parameter. In the present work, we suppose that the universe is filled with perfect fluid powered by the scalar field φ. In the context of Friedmannn-Lemaitre-Robertson-Walker metric (31), the appropriated form of the energy momentum tensor of perfect fluid is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \tag{5}$$

where $g_{\mu\nu}$and u_ν, are the metric tensor and the 4-vector characterizing a co-mobile observer, respectively, and hand $\rho$and p are the global energy density and the pressure of universe content, respectively. Under these previous considerations, one can extract the Friedmannn- like equations of covariant modified Telleparallel theory

$$\kappa^2\rho = 6H^2 f_T + \frac{1}{4}f \quad \text{and} \quad \kappa^2 p = 48\dot{H}H^2 f_{TT} - (2\dot{H} + 6H^2)f_T - \frac{1}{4}f \tag{6}$$

In the present description, the only component of the universe content is supposed to be the scalar field. Indeed, the energymomentum tensor of the scalar field coming from the Noether theorem is given by

$$\mathcal{T}_{\mu\nu} = \epsilon\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{\epsilon}{2}\partial_\beta\phi\partial^\beta\phi - V(\phi)\right] \tag{7}$$

Here, $V(\varphi)$ is the potential of the scalar field. By making using the previous metric, we deduce from (7), the energy-density and the pressure of the scalar field like several works such as [32]-[35].

$$\rho = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi)\ and\ p = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi) \qquad (8)$$

So the system of equations traducing the interaction between the scalar field and the geometry in the framework of the modified theory are

$$\kappa^2\left(\frac{\epsilon}{2}\dot{\phi}^2 + V(\phi)\right) = 6H^2 f_T + \frac{1}{4}f, \qquad (9)$$

$$\kappa^2\left(\frac{\epsilon}{2}\dot{\phi}^2 - V(\phi)\right) = 48\dot{H}H^2 f_{TT} - (2\dot{H} + 6H^2)f_T - \frac{1}{4}f \qquad (10)$$

(11)

The conservation equation $\dot{\rho} + 3H(\rho + p) = 0$, in the present context, leads to the following equation called Klein-Gordon equation [33]

$$\epsilon\ddot{\phi} + 3H\epsilon\dot{\phi} + V'(\phi) = 0 \qquad (12)$$

By adding the equations Eq.(9) and Eq.(11); we have

$$48\dot{H}H^2 f_{TT} - 2\dot{H}f_T = \kappa^2\epsilon\dot{\phi}^2 \qquad (13)$$

We are dealing here with cosmological investigation based on the following quadratic model [29].

$$f(T) = T + \lambda T^2 \qquad (14)$$

Under the algebraic function Eq.(14), the motor equation Eq.(13) becomes

$$120\lambda\dot{H}H^2 = \kappa^2\epsilon\dot{\phi}^2 \qquad (15)$$

3-Inflationary scenario from quadratic f(T) model

In attempt to describe the inflationary scenario, we introduce the following operator relating the e.folding*N* number to cosmic time *t*.

$$\frac{d}{dt} = H(N)\frac{d}{dN} \qquad (16)$$

Under this consideration, the equation Eq.(15) becomes

$$120\lambda H'(N)H(N) = \kappa^2\epsilon(\phi'(N))^2 \qquad (17)$$

Here the prime ($'$) means the derivative with e.fold number. In the same way, we express the KleinGordon equation Eq.(12) as

$$\epsilon H(N)^2\phi''(N) + \epsilon(H(N)H'(N) + 3H(N)^2)\phi'(N) + V'(\phi) = 0 \qquad (18)$$

where$V'(\phi)$ means the derivative of the potential with respect to the scalar fieldϕ. The previous equation will be solved with the goal to express the inflationary observable. We make the remark that the equation

Eq.(17) is made of two separated e.fold number functions $H(N)$ et $\phi(N)$. So, one can obtain these two functions under the following consideration

$$120\lambda H^{'}(N)H(N) = \kappa^2(\phi'(N))^2 = c \tag{19}$$

where c is a constant. So, such approach leads to two different differential equations whose solutions are given by

$$H(N) = \frac{\sqrt{cN + 120c_1\lambda}}{2\sqrt{15}\sqrt{\lambda}} \tag{20}$$

$$\phi(N) = \frac{\sqrt{c}N}{\sqrt{\epsilon\kappa}} + c_2 \tag{21}$$

where c_1 and c_2 are integration constants. The relation Eq.(21) permits to express the e.fold number as function of scalar field. By making using of Eq.(20) in the Klein-Gordon equation Eq.(18), one can extract the potential of the scalar field as follows

$$V(\phi) = -\frac{c^{3/2}\phi - 6c\kappa c_2\phi + 3c\kappa\phi^2 + 720\sqrt{c}\lambda c_1\phi}{120\kappa\lambda} + c_3 \tag{22}$$

Here, we have posed $\epsilon = 1$ (quintessence-like evolution [28]) to avoid confusion with the slow-roll parameters.

The expression for the slow-roll parameters with respect to the canonical scalar field potential V $V(\phi)$ are given by [42]

$$\varepsilon \equiv \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \quad \eta \equiv \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} \quad , \quad \xi^2 \equiv \frac{1}{\kappa^4}\frac{V'(\phi)V'''(\phi)}{(V(\phi))^2}. \tag{23}$$

For the scalar field models, the spectral index n_s, of curvature perturbations, the tensor-to-scalar ratio r of the density perturbations and the running of spectral index α_s are expressed as [43]

$$n_s - 1 \sim -6\varepsilon + 2\eta, \quad r = 16\varepsilon, \quad \alpha_s \equiv \frac{dn_s}{dln\kappa} \sim 16\varepsilon\eta - 24\varepsilon^2 - 2\xi^2. \tag{24}$$

Depending on the scalar field potential which is function of the scalar field, the previous observables can be expressed in term of scalar field. But from the fact that the scalar field is directly related to thee.fold number through Eq.(21), these observables can be expressed as e.fold number functions. Such approach makes possible the fitting of these observables with observational data. We express firstly the slow-roll parameters.

$$\begin{aligned} \varepsilon &= \frac{\left(c^{3/2} + 6c\kappa(\phi - c_2) + 720\sqrt{c}\lambda c_1\right)^2}{2\kappa^2\left(\sqrt{c}\phi\left(3\sqrt{c}\kappa(\phi - 2c_2) + c + 720\lambda c_1\right) - 120\kappa\lambda c_3\right)^2} \\ \eta &= -\frac{6c}{\kappa\left(120\kappa\lambda c_3 - \sqrt{c}\phi\left(3\sqrt{c}\kappa(\phi - 2\sigma) + c + 720\lambda s\right)\right)} \\ \xi^2 &= 0 \end{aligned} \tag{25}$$

(26)

(27)

The observables are obtained as function of the e.folds number as follows

$$
\begin{aligned}
r &= \frac{8c(6cN+c+720\lambda c_1)^2}{\left(3c^2N^2+c^2N-3c\kappa^2c_2^2+720c\lambda Nc_1+\sqrt{c}\kappa c_2(c+720\lambda c_1)-120\kappa^2\lambda c_3\right)^2}\\
\eta &= -\frac{3c(6cN+c+720\lambda c_1)^2}{\left(3c^2N^2+c^2N-3c\kappa^2c_2^2+720c\lambda Nc_1+\sqrt{c}\kappa c_2(c+720\lambda c_1)-120\kappa^2\lambda c_3\right)^2}\\
&\quad+\frac{12c}{3c^2N^2+c^2N-3c\kappa^2c_2^2+720c\lambda Nc_1+\sqrt{c}\kappa c_2(c+720\lambda c_1)-120\kappa^2\lambda c_3}+1\\
\alpha_s &= \frac{A(N)}{B(N)}
\end{aligned}
$$

(28)

(29)

(30)(31)

With

$$
\begin{aligned}
A(N) &= 6c^2(6cN+c+720\lambda c_1)^2\Big[8\Big(3c^2N^2+c^2N-3c\kappa^2c_2^2+720c\lambda Nc_1+\sqrt{c}\kappa c_2(c+720\lambda c_1)\\
&\quad-120\kappa^2\lambda c_3\Big)-(6cN+c+720\lambda c_1)^2\Big] \qquad (32)\\
B(N) &= \Big(3c^2N^2+c^2N-3c\kappa^2c_2^2+720c\lambda Nc_1+\sqrt{c}\kappa c_2(c+720\lambda c_1)-120\kappa^2\lambda c_3\Big)^4 \qquad (33)
\end{aligned}
$$

Several observational data are investigated on these parameters. The use of these data stays a viable way to constraint theoretical model. We present here the recent observations on spectral index n_s, the tensor-to-scalar ratio r and the running of spectral index α_s. The recent data of Planck satellite [4] suggested $n_s = 0.9603 \pm 0.0073(68\%CL)$,$r < 0.11(95\%CL)$, and $\alpha_s = -0.0134 \pm 0.0090(68\%CL)$ [Planck et WMAP [39]; [38]], whose negative sign is at 1.5σ. The BICEP2 experiment [4] implies $r = 0.20^{+0.07}_{-0.05}(68\%CL)$. It is mentioned that discussions exist on how to subtract the foreground, for example in [4],[37]. Recently, progress appears also in [40] to ensure the BICEP2 declarations. It has been also remarked that the representation of α_s is also given in [41].

The contribution of the quadratic model in the theoretical description of these observables is clearly showed through the figures Fig.1 to Fig.3. Indeed, the figure Fig.1 reveals that through the quadratic model, the tensor-to-scalar ratio r typically decreases with the increasing of the e.fold number N. Under this evolution, several observational data on the tensor-to-scalar ratio r can be meet. This means that as inflation progresses, the contribution of gravitational waves, carried by the tensor-to-scalar ratio r, becomes less important compared to scalar perturbations. For example, the Planck Collaboration's 2018 results in [44] and the BICEP2 experiment [4] suggestions support our theoretical prediction under quadratic model and near the end of inflation namely N = 60. The graph in right of the figure Fig.1 illustrates the fact that the teleparallel predictions are very far from those of the observational experiment. Such results strengthen the idea of modification of the teleparallel theory of gravity. In the figure Fig.2, the spectral index promoted by the quadratic model increases and near N = 60, it leads to observational data established in [44]. In the same time, the result given by the pure teleparellel (the right graph of Fig.2) on the spectral index do not satisfy any observation data. Finally, the figure Fig.2) shows that the running of spectral index predicted by the quadratic model, especially near N = 60, are consistent with the Planck data mentioned [41] and [44] whereas the evolution of this observable is chaotic in the case of pure teleparallel. To show more the consistency of the curves plotted in these figures, we extract from them some values of the observables that can be verified from works appropriately referenced. The table Tab.1 presents these values localizable in a set of values presented in the cited references. An another argument which supports the consistency of present theoretical description, comes from [45]. After providing the observational data on these inflationary observables, they also conclude that in the framework of single-field inflationary models with Einstein gravity, their results imply that slow-roll models with a concave potential $V''(\phi)$ <0, are increasingly favoured by the data. It is clear here that the observables described in the figures Fig.1-Fig.3 and the table Tab.1 are expressed from the polynomial potential (22). By the way, one has $V''(\phi) = -\frac{c}{20\lambda}$ <0 because c >0 and λ>0.

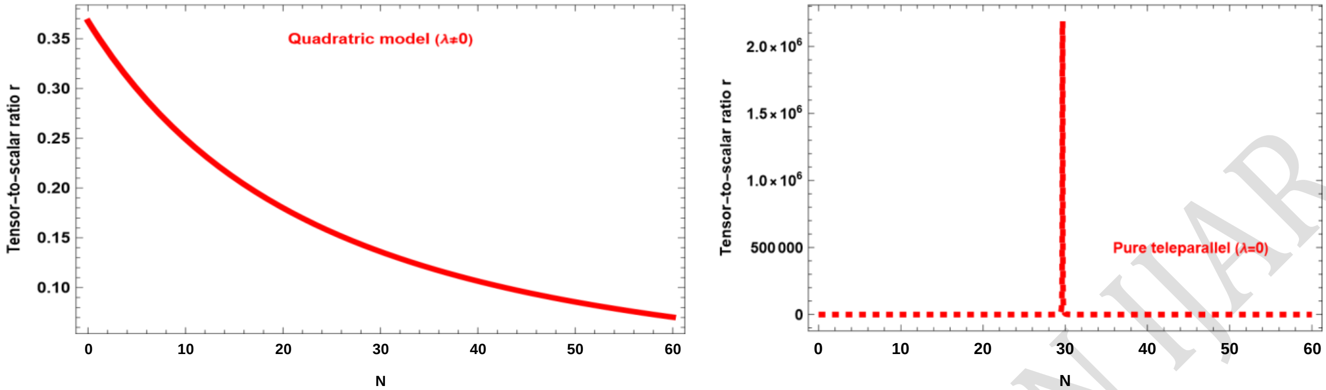

Figure 1: Evolution versus e.fold Number N of the tensor-to-scalar ratio r in the case of quadratic model (λ = 10 leading to the left panel) and the pure teleparall theory (λ = 0 leading to the right panel). The curves are obtained for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ and $\kappa = 1$.

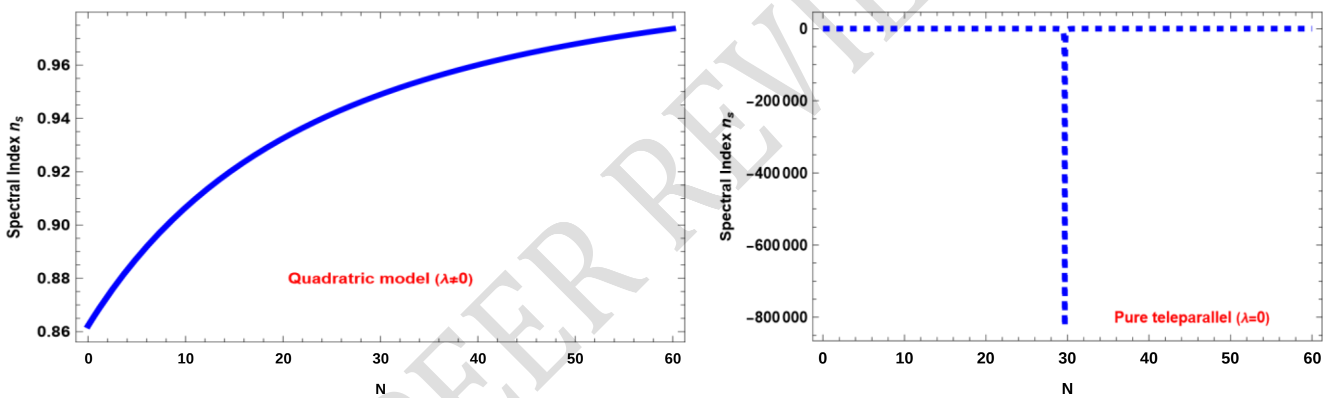

Figure 2: Evolution versus e.fold Number N of the spectral index n_s in the case of quadratic model (λ = 10 leading to the left panel) and the pure teleparall theory (λ = 0 leading to the right panel). The curves are obtained for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ and $\kappa = 1$.

N	Tensor-to-scalar ratio r	Spectral index n_s	Running of spectral index α_s	Refs.
40	0.1064	0.9601	-0.0011	[4,42]
45	0.0952	0.9643	-0.0008	[45,44]
50	0.0856	0.9679	-0.0007	[45,44]
55	0.0774	0.9709	-0.0006	[45,44]
60	0.0702	0.9736	-0.0005	[45,44]]

Table 1: Observable values from quadratic teleparallel model. These values are deduced from the figures Fig.1 to Fig.3

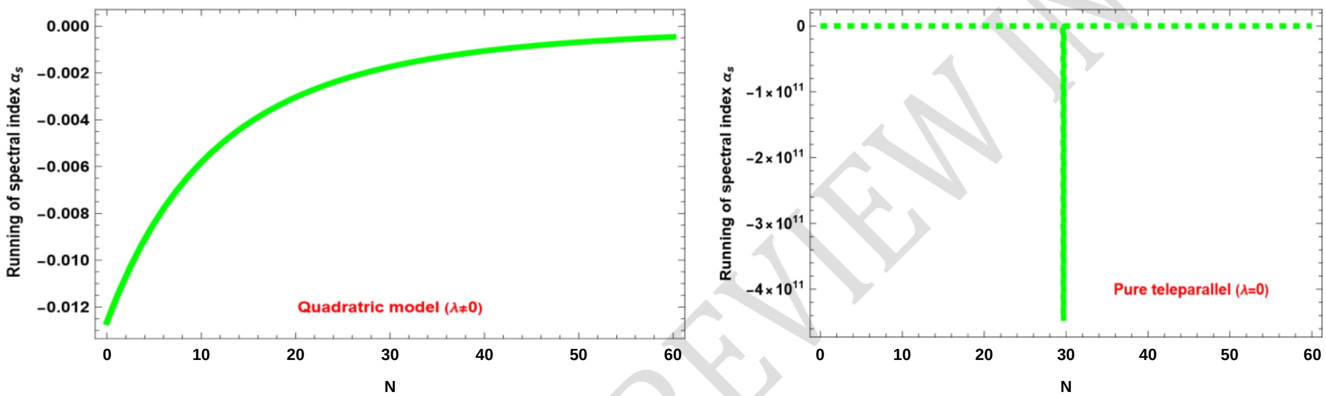

Figure 3: Evolution versus e.fold Number N of the running of spectral index α_s in the case of quadratic model (λ = 10 leading to the left panel) and the pure teleparallel theory (λ = 0 leading to the right panel). The curves are obtained for $c = 0.01$, $c_1 = -2$, $c_2 = 3$, $c_3 = 2$ and $\kappa = 1$.

4 Dark energy description from quadratic teleparallel model

The dark energy description is one of the most currently attractive subject in cosmology. Like several investigations, the introduction of the scalar field gives a really way to deal with the topic in the context of modified theory of gravity. In the present section, the Friedmann equations powered by the quadratic model with scalar field as only universe content will be solved to provide the energy density, the pressure and the state equation parameter. We recall that the sum of the two Friedmann equation has generated the equation (15) when $f(T)$ quadratic model is applied. In the same approach like the previous section, this equation is solved and gives

$$H(t) = \frac{\sqrt[3]{ct+120\lambda s}}{2\sqrt[3]{5}\sqrt[3]{\lambda}} \tag{34}$$

$$\phi(t) = \frac{\sqrt{c}t}{\kappa\sqrt{\epsilon}} + \sigma \tag{35}$$

Here, *c*, *s* and *σ*are integration constants. The resolution of the Klein-Gordon equation (12) gives

$$V(\phi) = -\frac{9\left(\sqrt{c}\kappa\sqrt{\epsilon}(\phi-\sigma)+120\lambda s\right)^{4/3}}{8\sqrt[3]{5}\kappa^2\sqrt[3]{\lambda}} + v \tag{36}$$

with *v*, the integration constant. We can now provide the energy density and the pressure of the scalar field as function of cosmic time *t*.

$$\rho(t) = \frac{c}{2\kappa^2} - \frac{9(ct+120\lambda s)^{4/3}}{8\sqrt[3]{5}\kappa^2\sqrt[3]{\lambda}} + v, \qquad p(t) = \frac{c}{2\kappa^2} + \frac{9(ct+120\lambda s)^{4/3}}{8\sqrt[3]{5}\kappa^2\sqrt[3]{\lambda}} - v \tag{37}$$

To made description based on the recent data on the cosmological parameters, we aim provide these previous quantities in term of redshift *z*. In connection with redshift, the Hubble parameter is defined by [28].

$$H(t) = -\frac{1}{1+z}\frac{dz}{dt} \tag{38}$$

By using Eq.(34) and Eq.(38) one obtains the cosmic time *t* as function redshift *z*.

$$t = 4\left(\frac{\sqrt[4]{10}\sqrt[4]{b^3\lambda - 3b^2\lambda\log(z+1) + 3b\lambda\log^2(z+1) - \lambda\log^3(z+1)}}{3^{3/4}\sqrt[4]{c}} + \frac{60\lambda}{c}\right) \tag{39}$$

Here *b* is also an integration constant. By the way, one has

$$H(z) = \frac{\sqrt[3]{\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda(b-\log(z+1))^3} + 90\lambda(s+2)}}{\sqrt[3]{30}\sqrt[3]{\lambda}}$$

$$\rho(z) = \frac{1}{10}\left(-\frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda(b-\log(z+1))^3} + 90\lambda(s+2)\right)^{4/3}}{\kappa^2\sqrt[3]{\lambda}} + \frac{5c}{\kappa^2} + 10v\right)$$

$$p(z) = \frac{1}{10}\left(\frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda(b-\log(z+1))^3} + 90\lambda(s+2)\right)^{4/3}}{\kappa^2\sqrt[3]{\lambda}} + \frac{5c}{\kappa^2} - 10v\right) \tag{40}$$

(41)

(42)

From several works like [28,44,45], it is possible to know the current (z = 0) observational value of the Hubble parameter. For example, it is estimated to $H(0) = H_0 = 70.4 \pm 1.6\text{km.s}^{-1}$. We use these values as initial condition to extract the expression of the parameter b. Such approach helps to reduce the number of the parameters and to base our theoretical description on observational data. One has

$$
\begin{aligned}
b &= \frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c} \\
H(z) &= \frac{\sqrt[3]{\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)}^{3}+90\lambda(s+2)}}{\sqrt[3]{30}\sqrt[3]{\lambda}} \\
\rho(z) &= -\frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)}^{3}+90\lambda(s+2)\right)^{4/3}}{10\kappa^2\sqrt[3]{\lambda}} \\
&\quad +\frac{c}{2\kappa^2}+v \\
p(z) &= \frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)}^{3}+90\lambda(s+2)\right)^{4/3}}{10\kappa^2\sqrt[3]{\lambda}} \\
&\quad +\frac{c}{2\kappa^2}-v
\end{aligned}
$$

(44)

(45)

(46)

(47)

We now plot these quantities versus the redshift *z*. We provide the evolution of the Hubble parameter in order to test its consistency with observational data and to compare its behaviors to those already investigated in literature. Moreover, After the energy density and the pressure depicted in figure Fig.5, the parameter of the state equation $\omega(z) = p(z)/\rho(z)$ is also provided in the figure Fig.6.

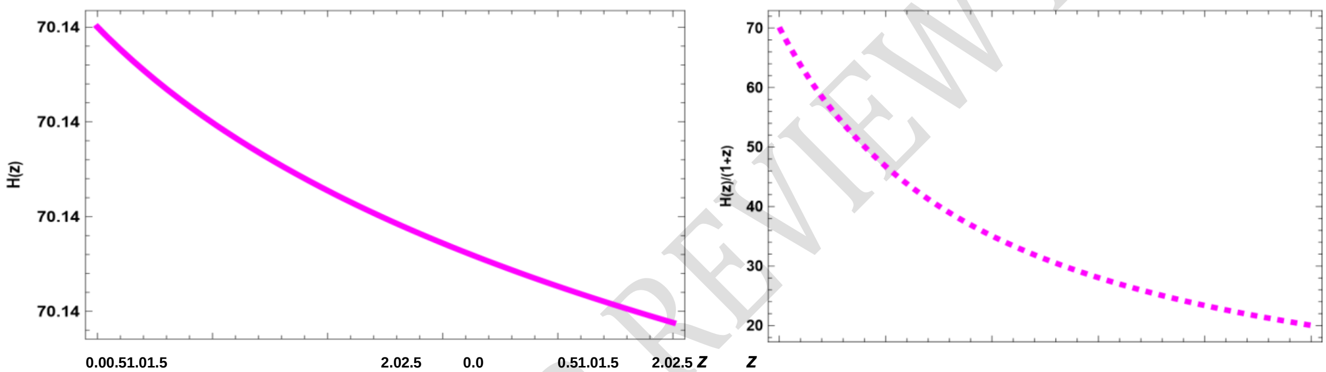

Figure 4:Evolution versus redshift z of the Hubble parameter. The curve is obtained for H_0=70.14;s = 220000;λ = −20;c = 10.

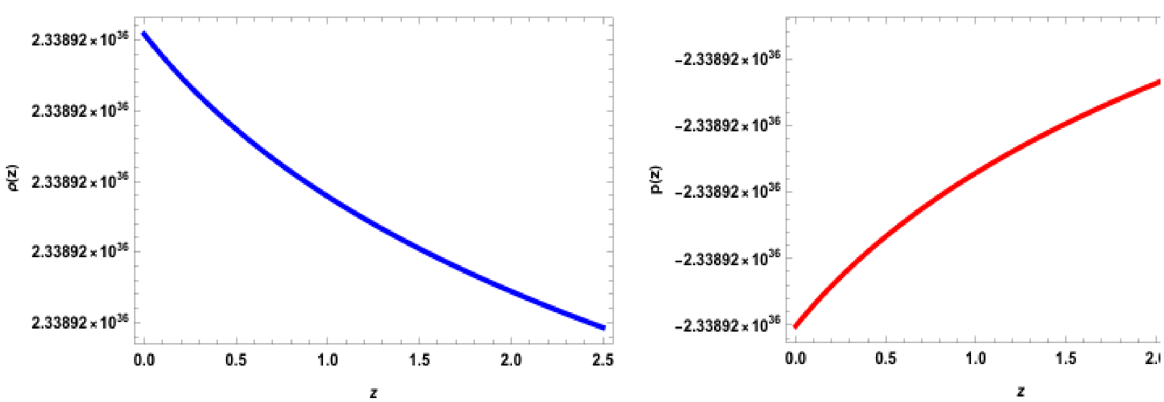

Figure5: Evolution versus redshift z of the energy density and the pressure of the scalar eld in quadratic teleparallel model constrained by observational data. The curves are obtained for H_0= 7014; s = 220000; λ = -20; c = 10; υ = 100; $k = \sqrt{\frac{1.8626}{10^{26}}}$

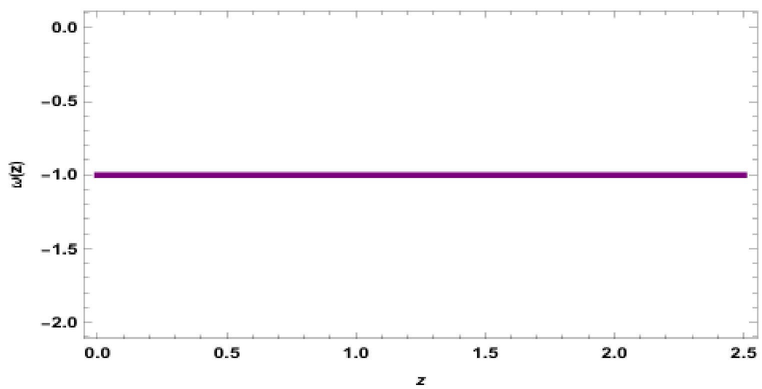

Figure 6: Evolution versus redshift *z* of the state equation parameter of the scalar field ω_ϕ. The curve is obtained for H_0= 7014; s = 220000; λ = -20; c = 10; υ = 100; $k = \sqrt{\frac{1.8626}{10^{26}}}$

4.1 Cosmological scope

One of the main goal of theoretical investigation is to defend observational prediction. Here, the Hubble parameter reconstructed in Eq.(35) is constrained to gives the present observational value of this cosmological parameter. The result is presented in the figure Fig.4 where under the choice $H_0 = H(0) =$ 70.14km.S^{-1}.Mpc^{-1} (see [28]), the hubble parameter $H(z)$ decreases slightly with the increasing of the redshift *z*. This decreasing with the redshfit aligns with the results obtained in interesting studies in [28, 44, 45]. Although this decreasing, the variation of the Hubble parameter is not significant. This means that our model promotes a constant Hubble whose value is given by observational data. Furthermore, the curve of $H(z)/(1+z)$ in Fig.4 presents a concavity facing upwards like several works [28]. This analysis on the Hubble parameter has permitted to know the conditions under which, our model can lead to the present observational value of the Hubble parameter. So, the free parameters of our model are constrained in order to deal with one of the great cosmological and astrophysical problem: the problem of dark energy. In the present investigation, it imports to recall that the considered candidate of the dark energy is the scalar field. It represents an alternative object to cosmological constant which the most plausible candidate of dark energy. Recent works relate that in dynamical dark energy models, the equation of state of the dark energy changes over time [7, 8]. These models include but are not limited to quintessence, k-essence, and phantom-type scalar field models, where generally a scalar field is coupled with the matter minimally or non-minimally with a associated potential which can generate sufficient negative pressure to drive the accelerated expansion of the universe. It suggested in current observations that the equation of state of the scalar field might have a phantom barrier ($\omega_\phi = -1$) crossing in the recent past [10, 11].

In the present work and under the value of the free parameters for which the Hubble parameter gives current observational value, we depict versus the redshift the pressure, the energy density and the equation of state parameter of the scalar field. The evolution of these quantities is presented in the figures Figs.5-6. The scalar field pressure is negative and increases with the redshift whereas the energy density decreases with the redshift. Although these variations, their values with respect to the redshift are very near their present value ($z = 0$). Consequently, the equation of state of the Hubble parameter is practically constant and gives ($\omega_\phi = -1$). As conclusion, the scalar field behaves like the constant cosmology Λ. The quadratic teleparallel model makes the scalar field behaving like the constant cosmology. Such result is consistent with the approached followed in this section where the free parameters are constrained to give results in accordance with observational data.

To reinforce all these results, it will be interesting to follow the evolution of the scalar potential and compare it to existing results in the context of the accelerated expansion of the universe. Firstly, the scalar field in Eq.(36) can be expressed as redshift function. One has:

$$V(z)=\frac{1}{10}\left(10v-\frac{30^{2/3}\left(\sqrt[4]{30}c^{3/4}\sqrt[4]{\lambda\left(\frac{30\lambda\left(-H_0^3+3s+6\right)^{4/3}}{c}-\log(z+1)\right)^3}+90\lambda(s+2)\right)^{4/3}}{\kappa^2\sqrt[3]{\lambda}}\right) \quad (48)$$

Recent works [8, 46] indicate that in a universe dominated by dark energy, the potential must remain

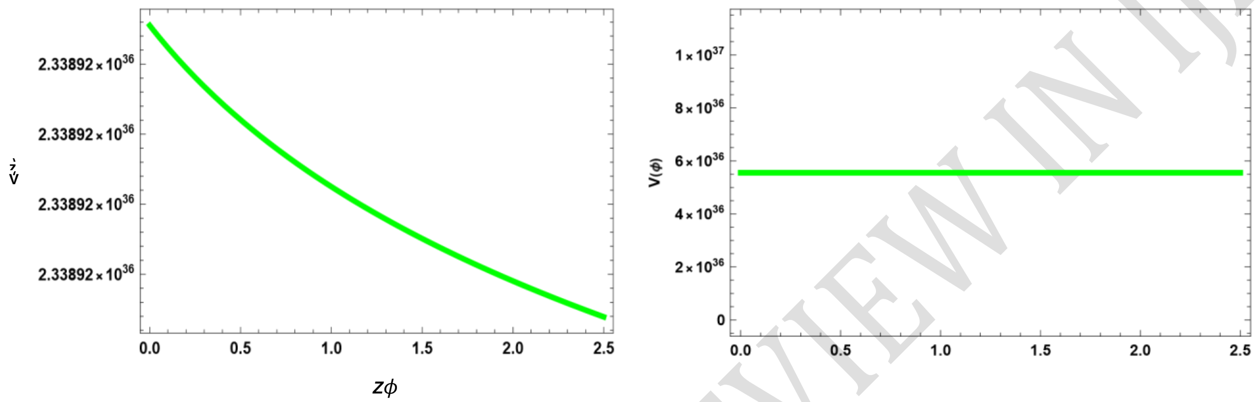

Figure 7: Evolution versus the redshift z (left panel) and versus the scalar field ϕ (right panel) of the scalar field potential. The graphs are obtained for $H_0 = 70.14; s = 220000; \lambda = -20; c = 10; v = 100;$ $k = \sqrt{\frac{1.8626}{10^{26}}}$

very flat and vary extremely slowly with *z*, except for dynamic quintessence models where a gentle decay is possible. When constraining the quadratic teleparallel model in order to give results aligned with current observational data, the quadratic teleparallel model makes quintessence scalar field giving results near the constant cosmology. First, we obtain from the panel in left of Fig.7 that the scalar field potential decays with the redshift. Moreover, this variation is very low. This low variation of the scalar field potential is clearly shown through the right panel of Fig.7 when depicting this potential versus the scalar field. So, our results are also consistent with observational data and the model describes the current expansion of the universe with scalar field behaving like the constant cosmology.

4 Conclusion

Thepresent investigationisdevotedtothecosmological implicationof thequadraticteleparallel withattemptofprovidingaunifiedwaytodescribe the inflationandthedarkenergydominating era. Bytheway, the scalar field isusedas theuniversecontentandthemetric that satisfies the cosmologicalprinciplehasbeenchosen.AllourresultsaremadefromtheresolutionoftheFriedmann equationgivenbythequadraticmodel. Theresolutionschemeprovides theHubbleparameterand thescalar fieldexpression. FromtheconservationequationnamelytheKlein-Gordanequation, it is sopossibletogetthepotentialofthescalar field.Firstly,withthegoalofdescribingtheinflationaryscenario,weprovidethepreviouscosmological quantities intermofe.foldnumberN. So,weintroducetheslow-rollparameterandtheinflationary observablewhichhavepermittedtoconstrainedthequadraticparameterandtheintegrationconstants toobservational data. Indeed, dependingonthe scalar fieldpotential, the spectral indexn_s, the tensor-to-scalar ratio r and the runningof spectral index α_s are calculatedand their theoretical representationsarenumericallydepictedversusthee.foldnumber.For $\lambda = 20$(quadratic model),thespectral index$n_s$notonlyincreasesbutalsorevolvesaround096whenthee.foldnumber 13 tends towards to 60 as predicted by most of observational data like Planck Satellite and WMAP. These data support also the running of spectral index α_s theoretically established in this work. Furthermore, ns < 1 supports the fact that the density fluctuation of the universe is very remarkable at large scale. In the same time, the tensor-to-scalar ratio r meets the value established from BICEP2 experiment. As showed in the figures Figs.1-3, it is not possible to obtain the same results through the pure teleparallel model because of the chaotic behavior of this observable when $\lambda = 0$. Secondly, instead of e.fold number, we base our description on the redshift z to challenge the current acceleration of universe expansion. The Hubble parameter, the energy density, the pressure and the potential of the scalar field are all expressed as redshift function. We constraint here the free parameters of our model by making the Hubble parameter leading to its current value estimated by viable observational data. Numerical analysis shows that our model leads to constant state equation parameter whose value coincides with those predicted by the most plausible candidate of dark energy, namely, the CDM model. As conclusion, the scalar field behaves like ACDM model in quadratic model constrained with observational data.The numerical study of the scalar field potential versus the redshift has consolidated this interesting result where the potential varies slowly or is practically constant in a universe dominated by the dark energy.

Our present work attempts to explain inflation and dark energy through a single model (quadratic model) without taking into consideration the transition between the two eras. In a future work, we project to explore the conditions on the quadratic parameter which ensure a stable transition from inflation to radiation era and to current acceleration expansion.

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