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RESEARCH ARTICLE

On Obtaining Some Special Pythagorean Triangles with their Areas as Pentagonal Numbers

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Abstract

Pythagorean Triangles which were first studied by the Pythagoreans around 400 B.C., remains one of the fascinated topics for those who just adore numbers. In this paper, Special Pythagorean Triangles are obtained with their areas as Pentagonal numbers using software Mathematica. Existence of Pythagorean triangles with two consecutive sides and their areas as pentagonal numbers is also investigated. Few interesting observations are made. Various 3D graphs of such Triangles as Pythagorean triplets are plotted.

INTRODUCTION

The famous theorem of Pythagoras, which is known as *The Pythagorean Theorem*, has been a source of inspiration for those who are fascinated by numbers. The knowledge of Mathematics has been enhanced by the solutions of many problems related to it. Gopalan and Janaki [1] have found special Pythagorean Triangles with their perimeters as a pentagonal Numbers, Darbari [2] has investigated the cases where two sides are consecutive. These gave the motivation to explore the existence of Pythagorean triangles with their areas as pentagonal numbers and to consider special cases with two consecutive sides.

2. Method of Analysis:

Let the Pythagorean Equation be

$\mathbf{X^2 + Y^2 = Z^2}$, **(2.1)**

Then the primitive solutions are given by [3]

$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2$ **(2.2)**

for some integers **m, n** of opposite parity such that **m > n > 0** and (m, n) = 1.

2.1 Area is a Pentagonal number:

Definition 2.1: A natural number p is called a pentagonal number if it can be written in the form $\beta(3\beta - 1)/2$, $\beta \in \mathbb{N}$.

If the area of the Pythagorean Triangle (X, Y, Z) is a pentagonal number A, then

$\frac{1}{2} X Y = \beta(3\beta - 1)/2 = A.$ **(2.3)**

By virtue of equation (2.2), equation (2.3) becomes

$2(m^2 - n^2)mn = \beta(3\beta - 1),\ \beta \in \mathbb{N}.$

Or, $(m^2 - n^2)mn = \beta(3\beta - 1)/2$ **(2.4)**

$\Rightarrow \beta = \frac{1}{6}\left(1 \pm \sqrt{1 + 24m^3n - 24n^3m}\right)$

Discarding the second value of β as it is negative, we get $\beta = \frac{1}{6}\left(1 + \sqrt{1 + 24m^3n - 24n^3m}\right)$ **(2.5)**

Solving equation (2.5) for 0 < m < 1000 & 0 < n <1000 & 0 < β <1000000 , we get 24 solutions. First five of them are as follows:

S. N	m	n	β	X	Y	Z	X^2	Y^2	Z^2	A
1	5	2	12	21	20	29	441	400	841	210
2	6	1	12	35	12	37	1225	144	1369	210
3	6	5	15	11	60	61	121	3600	3721	330
4	25	23	192	96	1150	1154	9216	1322500	1331716	55200
5	39	31	672	560	2418	2482	313600	5846724	6160324	677040

Table 2.1: Primitive Pythagorean Triangles (X, Y, Z) with (1/2)XY = β (3β - 1)/2

2.2 Hypotenuse and one leg are consecutive: In Pythagorean triangles if one leg and hypotenuse are consecutive, then in such cases, m = n + 1. (2.6)

This gives equation (2.4) as $(2n+1)(n+1)n = \frac{\beta(3\beta-1)}{2}$

$\Rightarrow \beta = \frac{1 \pm \sqrt{48n^3+72n^2+24n+1}}{6}$

$\Rightarrow \beta = \frac{1 + \sqrt{48n^3+72n^2+24n+1}}{6}$ **(2.7)**

as β is natural number. Solving equation (2.7) by using the software Mathematica for $n < 10^4$ for $\beta < 10^{15}$, we get only two solutions! These special Pythagorean Triangles (X, Y, Z) with their area as pentagonal numbers and in each of which one leg and hypotenuse are consecutive, are given below:

S.N.	**n**	**β**	**X**	**Y**	**Z**	**X^2**	**Y^2**	**$X^2 + Y^2 = Z^2$**	**XY/2 = β (3β - 1)/2**
1	**5**	**15**	**11**	**60**	**61**	**121**	**3600**	**3721**	**330**
2	**182**	**2847**	**365**	**66612**	**66613**	**133225**	**4437158544**	**4437291769**	**12156690**

Table 2.2: Primitive Pythagorean Triangles (X, Y, Z) with (1/2)XY = β (3β - 1)/2 & Z = Y +1

2.3 Two legs are consecutive: In the Primitive solutions of Pythagorean triangle, one of the legs is even. As Y = 2mn, Y is obviously even. If two legs are consecutive then, X is odd. Since $X - Y = \pm 1$, two cases arise.

Case 1: Let X= Y + 1. Then by equation (2.3), we have

$$\frac{(Y+1)Y}{2} = \frac{\beta(3\beta-1)}{2} = A \qquad (2.8)$$

$$\Rightarrow 4m^2n^2 + 2mn = \beta(3\beta-1)$$

Solving equation (2.8) using software Mathematica for $m < 10^{15}$, $n < 10^{15}$ and $\beta < 10^{30}$, we get only one solution, which is given in the following table:

S.N.	m	n	β	X	Y	Z	X^2	Y^2	$X^2 + Y^2 = Z^2$	A = β(3β -1)/2
1	5	2	12	21	20	29	441	400	841	210

Table 2.3: Primitive Pythagorean Triangles (X, Y, Z) with (1/2)XY = β (3β - 1)/2 & X = Y +1

Case 2: Let X = Y - 1. Then by equation (2.3), we have

$$\frac{Y(Y-1)}{2} = \frac{\beta(3\beta-1)}{2} = A \qquad (2.9)$$

$$\Rightarrow 2mn(2mn-1) = 4m^2n^2 - 2mn = \beta(3\beta-1)$$

Again solving equation (2.9) using software Mathematica for $m < 10^{15}$, $n < 10^{15}$ and $\beta < 10^{30}$, we get no solution!

3. 3D Plot: For $0 < m < 1000$ & $0 < n < 1000$ & $0 < \beta < 1000000$, we get 24 solutions. Plotting these as ListPlot3D using software Mathematica, we get the following graph (Figure 1).

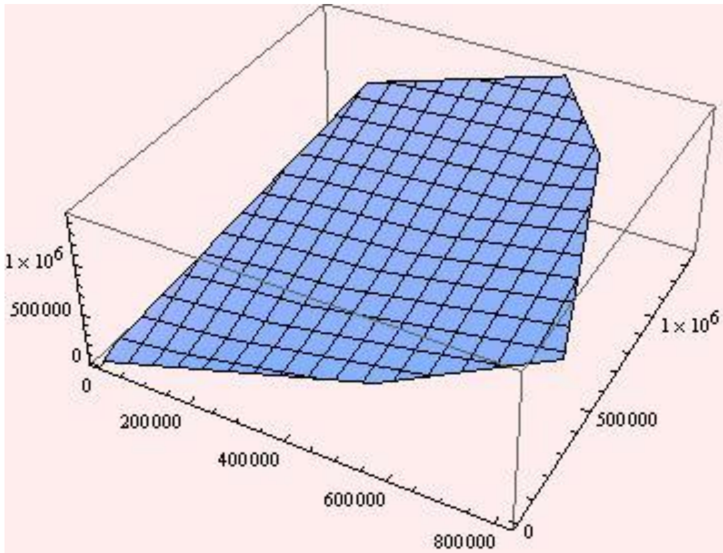

Figure 3.1

4. Observations and conclusion: We observe that

1. $X + Y + Z = 0 \pmod 2$.
2. $(X + Y + Z)(X + Y - Z) = 0 \pmod 8$
3. $(Y + Z - X) = 0 \pmod 2$
4. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z)$.
5. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.

In conclusion, other special Pythagorean Triangle can be obtained which satisfy the conditions other than discussed in the above problem.

References:

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