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RESEARCH ARTICLE

SOME RESULTS OF RIESZ REPRESENTATION FOR FUZZY NORMED SPACES

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Abstract

The main aim of this paper is to consider the fuzzy norm , define the fuzzy normed spaces , and prove some theorems in these spaces and study some basic results on finite dimensional fuzzy normed spaces.

INTRODUCTION

The notion of fuzzy norm on a linear space was introduced by Katsaras [7] in 1984. Later on many other Mathematicians like Felbin [5] in 1992 ,Cheng and Mordeson [4] in 1994 , Bag and Samanta [2] in 2003 etc , have given different definitions of fuzzy normed spaces . In this paper we have been able to establish some important results involving compactness of finite dimensional fuzzy normed linear spaces including Riesz Lemma .

2.Preliminaries

Definition (2.1) : [6] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if $*$ is satisfies the following conditions :

(i) $*$ is commutative and associative ;

(ii) $a * 1 = a$ for all $a \in [0,1]$;

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all ($a, b, c, d \in [0,1]$). If $*$ is continuous then it is called continuous t-norm.

Definition (2.2) :[11] Let X be a non-empty set, $*$ be a continuous t-norm on I=[0,1]. A function $N : X \times (0,1) \longrightarrow [0,1]$ is called a fuzzy norm function on X if satisfies the following axioms for all $x, y \in X, t, s > 0$:

(N1) $N(x,t) > 0$;

(N2) $N(x,t) = 1 \iff x = 0$;

(N3) $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$;

(N4) $N(x,t) * N(y,s) \leq N(x+y, t+s)$;

(N5) $N(x,.): (0,\infty) \rightarrow [0,1]$ is continuous;

(N6) $Lim_{t\rightarrow\infty} N(x,t) = 1$.

$(X, N, *)$ is said to be a fuzzy normed space.

Definition (2.3) : [2] Let $(X, N,*)$ be a fuzzy normed linear space .Let $\{x_n\}$ be a sequence in X .Then $\{x_n\}$ is said to be convergent if $x \in X$ such that $\lim_{n\to\infty} N(x_n - x\,, t) = 1\ \forall t > 0$. In this case x is called the limit of the sequence $\{x_n\}$ and is denoted by $lim x_n$.

Definition (2.4) : [2] Let $(X, N,*)$ be a fuzzy normed linear space . A subset B of X is said to be closed if for any sequence $\{x_n\}$ in B converges to x i.e.$\lim_{n\to\infty} N(x_n - x\,, t) = 1\ \forall t > 0$ implies that $x \in$ B.

Definition(2.5) : [9] Let $(X, N,*)$ and $(Y, N,*)$ be two fuzzy normed spaces. Then the function $f : X \to Y$ is said to be continuous at $x_\circ \in X$ if for all $\varepsilon \in (0,1)$ and all $t > 0$ there is exist $\delta \in (0,1)$ and $s > 0$ such that for all $x \in X$ $N(x - x_\circ\,, s\,) > 1 - \delta$ implies $N(f(x) - f(x_\circ)\,, t\,) > 1 - \varepsilon$.

The function f is called continuous function if it continuous at every point of X.

Definition (2.6) : [3] Let $(X, N,*)$ be a fuzzy normed linear space. We define a set $B(x, \alpha, t)\ as\ B(x, \alpha, t) = \{\, y : N(x - y\,, t) > 1 - \alpha \,\}$.

Definition (2.7) : [10] Let $(X, N,*)$ be a fuzzy normed linear space and $B \subset X$. B is said to be fuzzy bounded if for each r , $0 < r < 1$, $\exists\ t > 0$ such that $N(x, t) > 1 - r \quad \forall\ x \in B$.

Theorem (2.8) : [1] Let f be linear functional of fuzzy normed linear space X in to another fuzzy normed linear space Y. Then the following statements are equivalent :

1- f is continuous .
2- f is continuous at origin .
3- f is bounded.

Theorem (2.9) : [8] Let X be linear space over a field F.

(1) If $x \epsilon X$, and a function $T_x : X' \longrightarrow F$ defined by $T_x(f) = f(x)$ for all $f \epsilon X'$, then T_x is linear function , i.e. $T_x \epsilon X''$, and it is called Evaluation Functional Induced by x.

(2) If the function $\psi: X \longrightarrow X''$ defined by $\psi(x) = T_x$ for all $x \epsilon$X, then ψ injection linear function and ψ is called Canonical Function.

Definition (2.10) : Let X be a fuzzy normed linear space over a field F. We define X^{**} as :

$X^{**} = (X^*)^* = \{ f: X^* \to F\ ,\ f$ is bounded (continuous) linear functional$\}$

X^{**} is called the second dual space.

Definition (2.11) : Let $(X, N,*)\ and\ (Y, N,*)$ be fuzzy normed spaces over $F\ and\ f: X \to Y$ be linear function . We define

$N(f, t) = \inf\ \{\, N(f(x), t): x \in X \,\}\ for\ all\ t > o$.

Theorem (2.12) : Let $(X, N,*)\ and\ (Y, N,*)$ be fuzzy normed spaces . Then $N(f, t)$ is defined in Definition (2.11) is a norm.

Proof : We check the items in Definition (2.2) .It is easy to see that (N_1) , $(N_2), (N_3), (N_5)\ and\ (N_6)$ are true. We consider (N_4) :

$$\begin{aligned} N(f,t) * N(g,t) &= \{\inf\{N(f(x),t): x \in X\} * \{\inf\{N(g(x),s): x \in X\} \\ &= \inf\{\, N(f(x),t) * N(g(x),s) : x \in X \,\} \\ &\le \inf\{\, N\big((f+g)(x)\,, t+s\,\big) : x \in X \,\} \\ &= N(\, f+g\,, t+s\,)\,. \end{aligned}$$

3. Main results

Theorem (3.1) : Let $(X, N,*)$ be a fuzzy normed space over a field F.

(1) $If\ x \in X\ and\ T_x: X^* \to F$ defined as $T_x(f) = f(x)$ for all $f \in X^*$, then $T_x \in X^{**}\ and\ N(T_x, t) = N(f, t)$.

(2) $If\ \psi : X \to X''$ defined as $\psi(x) = T_x$ for all $x \in X$, then ψ is one-to-one linear function.

Proof : (1) T_x is linear (see theorem (2.9)).

To prove T_x is continuous .

$X^* = \{\, f: X \to F\ , f$ is bounded (continuous) linear function $\}$.

Since f is continuous at every point of X, hence f is continuous at $x_0 \in X$. Then for

all $\varepsilon \in (0,1)$ for all $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that for all $x \in X$
$N(x - x_0, s) > 1 - \delta \Rightarrow N(f(x) - f(x_0), t) > 1 - \varepsilon$
$\Rightarrow N(T_x - T_{x_\circ}), t) > 1 - \varepsilon$.
Therefore T_x is a continuous at x_0 . Since x_0 is an arbitrary point
Then T_x is a continuous function , hence $T_x \in X^{**}$.
$N(T_x, t) = \inf\{ N(T_x(f), t) : x \in X\}$
$= \inf \{ N(f(x), t) : x \in X\}$
$= N(f, t)$.

(2) see theorem (2.9).

Theorem (3.2) : Let $\{x_1, x_2, \dots, x_n\}$ be a linear independent set of vectors in a fuzzy normed linear space $(X, N, *)$ with $*$ is t-norm at (1,1). Then there is $c > 0$ and $\delta \in (0,1)$ such that for any set of scalars $\{ \lambda_1, \lambda_2, \dots, \lambda_n \}$;

$$N\left(\sum_{i=1}^{n} \lambda_i x_i , c \sum_{i=1}^{n} |\lambda_i|\right) < 1 - \delta \dots\dots\dots\dots\dots (1)$$

Proof : Let $s = |\lambda_1|, |\lambda_2|, \dots, |\lambda_n|$. If $s = 0$ then $\lambda_i = 0 \ \forall i = 1,2, \dots, n$ and the relation (1) holds for any $c > 0$ and $\delta \in (0,1)$.
Next we suppose that $s > 0$. Then (1) is equivalent to
$N(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n , c) < 1 - \delta \dots\dots\dots\dots\dots (2)$
For some $c > 0$ and $\delta \in (0,1)$, and for all scalars $\alpha' s$ with $\sum_{i=1}^{n} |\alpha_i| = 1$.
If possible suppose that (2) does not hold. Thus for each $c > 0$ and $\delta \in (0,1)$, $\exists$ a set of scalars $\{ \alpha_1, \alpha_2, \dots, \alpha_n \}$ with $\sum_{i=1}^{n} |\alpha_i| = 1$ for which $\{\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n , c) \geq 1 - \delta$.
Then for $c = \delta = \frac{1}{m}$, $m = 1,2, \dots$, $\exists$ a set of scalars
$\left\{\alpha_1^{(m)}, \alpha_2^{(m)}, \dots, \alpha_n^{(m)}\right\}$ with $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ such that
$N\left(y_m, \frac{1}{m}\right) \geq 1 - \frac{1}{m}$ where $y_m = \alpha_1^{(m)} x_1 + \alpha_2^{(m)} x_2 + \cdots + \alpha_n^{(m)} x_n$.
Since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$, we have $0 \leq |\alpha_i^{(m)}| \leq 1$ for $i = 1,2, \dots, n$.
So for each fixed j the sequence $\{\alpha_i^{(m)}\}$ is bounded and hence $\{\alpha_1^{(m)}\}$ has convergent subsequence . Let α_1 denote the limit of the subsequence and let $\{ y_1, m\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_1, m\}$ has a subsequence $\{y_2, m\}$ for which the corresponding subsequence of scalars $\{\alpha_2^{(m)}\}$ converges to α_2.continuing in this way , after n steps we obtain a subsequence $\{y_n, m\}$ where
$y_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i$ with $\sum_{i=1}^{n} |\gamma_i^{(m)}| = 1$ and $\gamma_i^{(m)} \rightarrow \gamma_i$ as $m \rightarrow \infty$.
Let $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Thus we have
$\lim_{m\to\infty} N(y_n , m - y , t) = 1 \quad \forall t > 0 \dots\dots\dots\dots\dots (3)$
Now for $k > 0$, choose m such that $\frac{1}{m} < k$.
We have $\left(1 - \frac{1}{m}\right) * N\left(0, k - \frac{1}{m}\right) \leq N\left(y_{n,m}, \frac{1}{m}\right) * N\left(0, k - \frac{1}{m}\right) \leq$
$N\left(y_{n,m} + 0 , \frac{1}{m} + k - \frac{1}{m}\right) = N\left(y_{n,m}, k\right)$.
i.e. $\left(1 - \frac{1}{m}\right) * N\left(0 , k - \frac{1}{m}\right) \leq N\left(y_{n,m}, k\right)$
i.e. $\lim_{m\to\infty} N\left(y_{n,m}, k\right) \leq 1$
i.e. $\lim_{m\to\infty} N\left(y_{n,m}, k\right) = 1 \dots\dots\dots\dots\dots (4)$
Now $N\left(y - y_{n,m}, k\right) * N\left(y_{n,m} k\right) \leq N\left(y - y_{n,m} + y_{n,m}, k + k\right) = N(y, 2k)$
$\Rightarrow \lim_{m\to\infty} N\left(y - y_{n,m}, k\right) * \lim_{m\to\infty} N\left(y_{n,m} k\right) \leq N(y \leq ,2k)$
(by continuity of t-norm at (1,1)) $\Rightarrow 1 * 1 \leq N(y, 2k)$ by (3)&(4)
$\Rightarrow 1 = 1 * 1 = N(y, 2k)$.
Since $k > 0$ is arbitrary , by (N2) it follows that $y = 0$.
Again since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ and $\{ x_1, x_2, \dots, x_n \}$ are linear independent set of vectors , so $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$.

Thus we arrive at a contradiction and the lemma is proved.

Theorem (3.3) : (**Riesz Lemma**) Let M be closed proper subspace of a fuzzy normed linear space $(X, N,*)$ and let λ be a real number such that $0 < \lambda < 1$.Then there exists a vector $x_\lambda \in X$ such that $N(x_\lambda, 1) > 0$ and $N(x_\lambda - x, \lambda) = 0$ for all $x \in M$.

Proof :Since M is proper subspace of X , $\exists\, v \in X - M$.

Denote $d = \wedge_{x \in M} \wedge \{ t > 0 : N(v - x, t) > 0 \}$.

We claim that $d > 0$, i.e. $\wedge_{x \in M} \wedge \{ t > 0 : N(v - x, t) > 0 \} = 0 \Rightarrow$

for a given $\varepsilon > 0$, $\exists\, x(\varepsilon) \in Y$ such that $\wedge \{ t > 0 : N(v - x, t) > o \} < \varepsilon$

$\Rightarrow N(v - x, \varepsilon) > 0$.

Choose $\alpha \in (0,1)$ such that $N(v - x, \varepsilon) > 1 - \alpha$. i.e. $y \in B(v, 1 - \alpha, \varepsilon)$.

Since $\varepsilon > 0$ is arbitrary , it follows that v is in the closure of M.

Since M is closed , it implies that $v \in M$ which is a contradiction . Thus $d > 0$.

We now take $\lambda \in (0,1)$. So $\frac{d}{\lambda} > d$. Thus for some $x_\circ \in M$,

we have $d \leq \wedge \{ t > 0 : N(v - x_\circ, t) > 0\} < K' < \frac{d}{\lambda}$ ………(1)

Let $x_\lambda = \frac{v - x_\circ}{k'}$. Now $(x_\lambda, 1) = N(\frac{v - x_\circ}{k'}, 1)$.

i.e. $N(x_\lambda, 1) = N(v - x_\circ, k')$ ………(2)

Now $\wedge \{ t > 0 : N(v - x_\circ, t) > 0 \} < k' \Rightarrow N(v - x_\circ, k') > 0$.

From (2) we have $N(x_\lambda, 1) > 0$.

Now for $x \in M$, $\wedge \{ t > 0 : N(x_\lambda - x, t) > 0\} =$

$\wedge \{t > 0 : N(v - x_\circ - k'x, k't) > 0 \} =$

$\frac{1}{k'} \wedge \{s > 0 : N(v - x_\circ - k'x, s) > 0 \}$.

i.e. $\wedge \{t > 0 : N(x_\lambda - x, t) > 0 \} \geq \frac{d}{k'}$ ($since \quad x_\circ + k'x \in M$)

$\Rightarrow \wedge \{ t > 0 : N(x_\lambda - x, t) > 0 \} > \lambda$ by (1)

i.e. $N(x_\lambda - x, \lambda) \leq 0 \Rightarrow N(x_\lambda - x, \lambda) = 0 \quad \forall\, x \in M$.

Definition (3.4) : [2] Let $(X, N,*)$ be a fuzzy normed linear space . A subset B of X is said to be compact if any sequence $\{x_n\}$ in B has a subsequence converging to an element of B .

Theorem (3.5) : Let $(X, N,*)$ be a fuzzy normed linear space and $x \neq 0$. If suppose that $A = \{ x \in X : N(x, 1) > 0\}$ is compact , then X is finite dimensional.

Proof : If possible suppose that dim $X = \infty$. Take $x_1 \in X$ such that $N(x_1, 1) > 0$. Suppose X_1 is the subspace of X generated by x_1 . Since dim $X_1 = 1$, it is aclosed and proper subset of X .Thus by the Lemma (3.3)

$\exists\, x_2 \in X$ $such\ that$ $N(x_2, 1) > 0$ and $N(x_2 - x_1, \frac{1}{2}) = 0$.

The elements x_1, x_2 generate a two dimensional proper closed subspace of X.

By the Lemma (3.3) , $\exists\, x_3 \in X$ with $N(x_3, 1) > 0$ such that

$N\left(x_3 - x_1, \frac{1}{2}\right) = 0$, $N\left(x_3 - x_2, \frac{1}{2}\right) = 0$.

Proceeding in the same way , we obtain a sequence $\{x_n\}$ of elements $x_n \in A$ such that $N(x_n, 1) > 0$ and $N\left(x_n - x_m, \frac{1}{2}\right) = 0$ $(m \neq n)$. It follows that neither the sequence $\{x_n\}$ nor its any subsequence converges. This contradicts the compactness of A . Hence dim X is finite .

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