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RESEARCH ARTICLE

Cost Minimization in Fuzzy DEA Models

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Abstract

Data Envelopment Analysis is a powerful managerial tool to evaluate the relative efficiency of each Decision Making Unit (DMU) and is widely used in ranking DMUs. For a classical DEA the set of entities involved should be crisp in nature. But often in a real evaluation problem input and output data of entities often fluctuate or are not known precisely. This kind fluctuations or vagueness in the data can very well be modeled by fuzzy numbers. Based on the fundamental CCR model a fuzzy DEA model is developed here to evaluate efficiency of a DMU when the input cost is fuzzy. The proposed fuzzy DEA model is solved by fuzzy linear programming method by Li Xiaghong to obtain optimal solution. Analysis of results provides information about the efficiency of DMU for variations in the cost of input variables.

Introduction

DEA is an increasingly popular methodology for Data analysis which finds extensive use in industry , society, and even in education. It is a mathematical programming technique for measuring performance efficiency of organizations popularly named as DMUS. Traditionally evaluation of efficiency is done using the concept of production function which provides a relationship between input and output. But if the production function is unavailable evaluation of efficiency becomes complex. Most of the DEA articles make an assumption that input and output data are crisp ones without any variation. In fact inputs and outputs are ever changeful. It is very difficult to evaluate the efficiency of DMU with varying inputs and outputs by conventional DEA models. Some researchers have proposed several methods to deal with the variation in the data. The methods by Peijun Guo et al (2001) Saowanee Lertworasirikul et al (2003), Yuh-Yuan Guh (2001), Ahmad Reza Jafarian-Moghadam et al (2011), Feng Yang et al (2010), Ying-Ming Wang et al (2009), Konstantinos Triantis et al (1998) are such methods.

In this paper a cost minimization DEA model for evaluating the efficiency of DMU with the given fuzzy input costs is developed and a technique for getting its solution is evolved as well.

Introduction to Fuzzy Linear Programming

A classical linear programming problem (LPP) will be of the form

$$\text{Max } Z = CX$$
$$\text{Subject to } AX \leq b$$
$$X \geq 0$$

Where $X=(x_1,x_2 \,......... \, x_n)^T$, $C=(c_1 \,....... \, c_n)$, $b=(b_1 \,...... \, b_m)^T$, $A = (a_{ij})_{m\times n}$ are crisp values. But in real decision making problems, usually coefficients of LP are inexact. In addition the decision maker also allows some violation in the accomplishment of the constraint. In these cases fuzzy linear programming (FLP) provides the flexibility. There are different models of FLP available in the literature. The method by Nasseri et al, Delgado et al, Campos et al, Li Xiaoghong, Herrera et al are only a few of them.

If the given FLP is such that the coefficients are fuzzy numbers, then Li Xiaoghong suggested a method by which FLP can be converted into a multi-objective linear programming problem. But Nassri et al uses a fuzzy ranking procedure to solve such problems.

DEA Models

The most frequently used DEA models are the CCR model named after Charnes, Cooper and Rhodes (1978)] and the BCC model named after Banker, Charnes and Cooper (1984). After these works, DEA has been extensively used for the performance evaluation of many firms. Different DEA models and its solution methods are explained by Subash C. Ray (2004). Several journal papers are also available in the literature explaining the different DEA models and its applications. The use of linear programming can clearly be viewed from these papers. The articles by Peijun Guo et al (2001), MP. Estellita Lins et al (2004), Feng Yang et al (2010), A. Makui et al (2008), Brandon Y.H. Wong et al (2009) and Mir Mozaffar Masoumi et al are only a few of them. Basically all these models are the continuation of the basic models by CCR (1978) and BCC (1984).

Suppose that there are n DMUs each of which consumes the same type of inputs and produces the same type of outputs. Let m be the number of inputs and r be the number of outputs. All inputs and outputs are assumed to be non-negative, but atleast one input and one output are positive.

In the CCR model the multiple inputs and multiple outputs of each DMU are aggregated into a single virtual input and a single virtual output. The CCR model is a fractional mathematical programming problem of the form

$$\text{Max } \frac{v_1 y_{1o} + v_2 y_{2o} + \cdots + v_r y_{ro}}{u_1 x_{1o} + u_2 x_{2o} + \cdots + u_m x_{mo}}$$

$$\text{Subject to } 0 \leq \frac{v_1 y_{1i} + v_2 y_{2i} + \cdots + v_r y_{ri}}{u_1 x_{1i} + u_2 x_{2i} + \cdots + u_m x_{mi}} \leq 1;$$

$$u_1, u_2 \cdots u_m \geq 0;$$

$$v_1, v_2 \cdots v_r \geq 0.$$

Where DMU_i is the i^{th} DMU for $i = 1,2.\dots n$ and DMU_o is the target DMU. x_{ji} is the amount of input j consumed by DMU_i, y_{ki} is the amount of output k produced by DMU_i, u_j is the weight of input j and v_k is the weight of output k for $i = 1,2,\dots.n,\ j = 1,2,\dots.m,\ \&\ k = 1,2,\dots.r.$ The objective of this fractional linear programming problem is to determine the input weights and output weights that maximize the ratio of a virtual output to a virtual input for DMU_o. The constraints restrict the ratio of virtual outputs to virtual inputs for every DMU to be less than or equal to 1.

This fractional linear problem can be converted to simple linear programming problem by normalizing the numerator or denominator of the objective function. By normalizing the denominator (that is by introducing the constraint $u_1 x_{1o} + u_2 x_{2o} + \cdots + u_m x_{mo} = 1$ into the DEA problem) the problem becomes output maximization DEA program. Similarly by normalizing the numerator (that is by introducing the constraint $v_1 y_{1o} + v_2 y_{2o} + \cdots + v_r y_{ro} = 1$ into the DEA problem) the problem becomes the input minimization problem. In the input oriented CCR model, a DMU is inefficient if it not is possible to reduce any input without increasing any other inputs and achieve the same level of output. So under the assumption that all inputs and outputs have nonzero worth, DMU_o will be efficient if the optimal objective function ratio equal to 1. If the ratio is less than 1, it is possible to produce the given output using smaller quantity of inputs. The efficiencies of all DMUs can be obtained by solving the problem n times, once for each DMU as the target DMU.

Using matrix notation the output maximization CCR DEA can be written as

$$\begin{aligned} &max\ \ V^T y_o \\ &s.t.\ \ U^T x_o = 1 \\ &-U^T X + V^T Y \leq 0 \\ &U, V \geq 0. \end{aligned}$$

The dual of this LPP can be written as

$$\begin{aligned} &min\ \ \theta \\ &s.t.\ \ \theta x_o - X\lambda \geq 0 \\ &Y\lambda \geq y_o \\ &\lambda \geq 0 \end{aligned}$$

Where θ is a free variable.

The CCR model is developed on the assumption of constant returns to scale (CCR) of DMUs. From the economic theory, there are three types of returns to scale

- Constant returns to scale (CRS): an increase in the amount of inputs consumed leads to a proportional increase in the amount of output consumed.
- Increasing returns to scale (IRS): an increase in the amount of inputs consumed leads to a larger than proportional increase in the amount of output consumed.
- Decreasing returns to scale (DRS): an increase in the amount of inputs consumed leads to a smaller than proportional increase in the amount of output consumed.

IRS and DRS are referred to as "variable returns to scale (VRS)" because the outputs produced increase more or less than proportionally to the increase in the inputs. Under constant returns to scale (CRS) assumption the output maximization and input minimization problems are equivalent. The CCR DEA models are also called CRS DEA models.

There are several extensions of CCR DEA models available in the literature. In an important extension of CCR DEA approach Banker, Charnes and Cooper (BCC) generalized the CRS DEA models to handle Variable Returns to Scale (VRS) models (1984). These models are called BCC DEA models. The input oriented measure of technical efficiency of DMUo under VRS due to BCC is given by

$$\begin{aligned} & min \quad \theta \\ & s.t. \quad \theta x_o - X\lambda \geq 0 \\ & \qquad Y\lambda \geq y_o \\ & \qquad e\lambda = 1 \\ & \qquad \lambda \geq 0 \end{aligned}$$

Where e is a row vector with elements equal to 1.

Using this BCC model, the input oriented technical efficiency under VRS is $TE_o^V = \theta^*$.

Cost Minimization Model in DEA

Consider the production possibility set under VRS for the input vector x^j and the output vector y^j

$$T^V = \{(x,y): x \geq \sum_{j=1}^{N} \lambda_j x^j \,; y \leq \sum_{j=1}^{N} \lambda_j y^j ; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0, j = 1,2, \ldots ..N\}$$

And the corresponding input requirement set for any input vector y

$$V(y) = \left\{(x): x \geq \sum_{j=1}^{N} \lambda_j x^j \,; y \leq \sum_{j=1}^{N} \lambda_j y^j ; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0, (j = 1,2 \ldots .N)\right\}$$

Then for a target output bundle y^0 and at a given input price vector w^0, the minimum cost under VRS assumption is $C^* = \left\{\min w^{0^T} : x \in V(y^0)\right\}$

Minimum cost can obtain by solving the DEA LP problem

$$\begin{aligned} & \min \sum \lambda_j x_{ij} \leq x^i ; \; i = 1,2, \ldots ..n \\ & \sum \lambda_j y_{rj} \geq y_{r0} ; \; r = 1,2, \ldots .m \\ & \sum \lambda_j = 1 \\ & \lambda_j \geq o ; \;\; j = 1,2, \ldots ..N \end{aligned}$$

The optimal solution of this problem yields the cost minimizing input bundle $x^* = (x_1^*, x_2^*, \ldots \ldots .x_n^*)$ and the objective function value shows the minimum cost. The overall cost efficiency (γ) measures the factor by which the cost can be scaled down if the firm uses the optimum input bundle and performs at the full technical efficiency. The two components of cost efficiency (γ) are, (1) input oriented technical efficiency $(\beta)$and (2) allocative efficiency(α), where $\alpha = \frac{\gamma}{\beta} \cdot$ The two distinct sources of cost inefficiency are technical inefficiency in the form of wasteful use of inputs, and allocative inefficiency due to selection of an inappropriate input mix.

<u>The example for cost minimization with two inputs and one output</u>

Table: Output and input quantity data for cost minimization.

Firm	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Output (y)	12	8	17	5	14	11	9
Input1 (x_1)	8	6	12	4	11	8	7
Input2 (x_2)	7	5	8	6	9	7	10

This is a one-output two-input example. Consider firm 5. Inorder to find the cost efficiency of the firm, let the input prices be w_1=10 and w_2=5. Then the actual cost of the firm is 155. Then the cost minimization DEA problem is

$$\begin{aligned} & min\ C = 10x_1 + 5x_2 \\ \text{s. t.} \quad & 8\lambda_1 + 6\lambda_2 + 12\lambda_3 + 4\lambda_4 + 11\lambda_5 + 8\lambda_6 + 7\lambda_7 \leq x_1 \\ & 7\lambda_1 + 5\lambda_2 + 8\lambda_3 + 6\lambda_4 + 9\lambda_5 + 7\lambda_6 + 10\lambda_7 \leq x_2 \\ & 12\lambda_1 + 8\lambda_2 + 17\lambda_3 + 5\lambda_4 + 14\lambda_5 + 11\lambda_6 + 9\lambda_7 \geq 14 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = 1 \\ & \lambda_1, \lambda_2, \cdots\cdots\cdots\cdots \lambda_7 \geq 0, \qquad x_1, x_2 \geq 0 \end{aligned}$$

The optimal solution is $x_1^* = 9 \cdot 6$, $x_2^* = 7 \cdot 4$, $\lambda_1^* = 0 \cdot 6$, $\lambda_3^* = 0 \cdot 4$, $\lambda_j^* = 0$, for $j \neq 1{,}3$ with $C^* = 133$. Thus the cost efficiency is $\gamma = \frac{133}{155} = 0 \cdot 85806$. The measure of technical efficiency using the input oriented BCC DEA for this firm is $\beta = 0 \cdot 87273$ and hence the allocative efficiency is $\alpha = \frac{\gamma}{\beta} = \frac{0 \cdot 85806}{0 \cdot 87273} = 0 \cdot 9832$.

Fuzzy Cost Minimization Model in DEA

If the cost vector associated with the input quantities are fuzzy numbers, then the cost minimization DEA model becomes

$$min \ \sum_i^n \tilde{w}_i \tilde{x}_i$$

s.t. $\sum_1^N \lambda_j x_{ij} \leq \tilde{x}_i$, i =1,2,.....n,

$\sum_1^N \lambda_j y_{rj} \geq y_{oj}$ r = 1,2,.......m

$\sum_1^N \lambda_i = 1$

$\lambda_j \geq 0$ j = 1,2,......N

Here if the costs associated with the input quantities $\tilde{c}_j$ are triangular fuzzy numbers and the method by Li Xiaoghong is used to solve this FLP, then the optimal solution will be a triangular fuzzy number. Using this solution the minimum cost can calculate and using this minimum cost the efficiencies of each DMU can evaluate.

For an example consider the problem discussed in the last section. Assume the cost vector (c_1, c_2) of the problem as triangular fuzzy number. Let $\tilde{c}_1 = \langle 10, \quad 9, \quad 12 \rangle$ and $\tilde{c}_2 = \langle 5, \quad 4, \quad 5.5 \rangle$. Then the DEA cost minimization becomes

$$min \quad \tilde{\boldsymbol{C}} = \tilde{c}_1 \tilde{x}_2 + \tilde{c}_2 \tilde{x}_2$$

s. t. $8\lambda_1 + 6\lambda_2 + 12\lambda_3 + 4\lambda_4 + 11\lambda_5 + 8\lambda_6 + 7\lambda_7 \leq \tilde{x}_1$

$7\lambda_1 + 5\lambda_2 + 8\lambda_3 + 6\lambda_4 + 9\lambda_5 + 7\lambda_6 + 10\lambda_7 \leq \tilde{x}_2$

$12\lambda_1 + 8\lambda_2 + 17\lambda_3 + 5\lambda_4 + 14\lambda_5 + 11\lambda_6 + 9\lambda_7 \geq 14$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = 1$

$\lambda_j \geq 0$ j = 1,2,.....7

By the method of Li Xiaoghong this FLP can be converted to a multi-objective crisp LPP as follows

$$min \quad \tilde{\boldsymbol{C}} = \langle 10x_1 + 5x_2, \quad 9\underline{x}_1 + 4\underline{x}_2, \quad 12\overline{x}_1 + 5 \cdot 5\overline{x}_2 \rangle$$

$s \cdot t \cdot$ $8\lambda_1 + 6\lambda_2 + 12\lambda_3 + 4\lambda_4 + 11\lambda_5 + 8\lambda_6 + 7\lambda_7 \leq x_1$

$8\lambda_1 + 6\lambda_2 + 12\lambda_3 + 4\lambda_4 + 11\lambda_5 + 8\lambda_6 + 7\lambda_7 \leq \underline{x}_1$

$8\lambda_1 + 6\lambda_2 + 12\lambda_3 + 4\lambda_4 + 11\lambda_5 + 8\lambda_6 + 7\lambda_7 \leq \overline{x}_1$

$7\lambda_1 + 5\lambda_2 + 8\lambda_3 + 6\lambda_4 + 9\lambda_5 + 7\lambda_6 + 10\lambda_7 \leq x_2$

$7\lambda_1 + 5\lambda_2 + 8\lambda_3 + 6\lambda_4 + 9\lambda_5 + 7\lambda_6 + 10\lambda_7 \leq \underline{x}_2$

$7\lambda_1 + 5\lambda_2 + 8\lambda_3 + 6\lambda_4 + 9\lambda_5 + 7\lambda_6 + 10\lambda_7 \leq \overline{x}_2$

$12\lambda_1 + 8\lambda_2 + 17\lambda_3 + 5\lambda_4 + 14\lambda_5 + 11\lambda_6 + 9\lambda_7 \geq 14$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 = 1$

$\lambda_j \geq 0$ j = 1,2,.....7, $x_1, \underline{x}_1, \overline{x}_1, x_2, \underline{x}_2, \overline{x}_2 \geq 0$.

The optimum solution to this LPP is given by

$\langle x_1^* \quad \underline{x}_1^* \quad \overline{x}_1^* \rangle = \langle 9 \cdot 6, \quad 9 \cdot 6, \quad 9 \cdot 6 \rangle$

$\langle x_2^* \quad \underline{x}_2^* \quad \overline{x}_2^* \rangle = \langle 7 \cdot 4, \quad 7 \cdot 4, \quad 7 \cdot 4 \rangle$

$\lambda_1^* = 0 \cdot 6$, $\lambda_3^* = 0 \cdot 4$, $\lambda_j^* = 0$, for $j \neq 1{,}3$ with minimum cost $\tilde{\boldsymbol{C}}^* = \langle 113, \quad 116, \quad 155 \cdot 9 \rangle$

Using this minimum cost, the cost efficiency of the firm is $\tilde{\gamma} = \langle 0.858064, \quad 0.859259, \quad 0.858953 \rangle$. The technical efficiency $\beta = 0 \cdot 87273$.

Therefore the allocative efficiency is $\alpha = \frac{\gamma}{\beta} = \langle 0.9832, \quad 0.9846, \quad 0.9842 \rangle$. Thus the presence of fuzziness will not affect much more on the efficiency of a firm.

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