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RESEARCH ARTICLE

On the method of Green function for solving some boundary value problems

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Abstract

In this paper we use the method of Green function to solve some boundary value problem subject to Sturm-Liovil operator.and proving some properties of Green function in these problems.

(I) Green's function of the Sturm-Liovile equation

Consider $u = u(x), x \in [a, b]$,satisfay the equation

$$Lu(x) = f(x) \qquad (1)$$

With acanonical boundary conditions at $a\ and\ b$, where L,is Sturm-Liovile operator given by

$$L = \frac{1}{w(x)}[\frac{d}{dx}p(x)\frac{d}{dx} - q(x)] \qquad (2)$$

Now we know that if β_m , $m = 1,2, \ldots$. are the eigen value of L and $e(x)_m$ are the corresponding igen vector's

$$i.e \quad Le(x)_m = \beta_m . e(x)_m \qquad (3)$$

then the solution of (1) can be written as

$$u(x) = \sum_m u_m\ e_m \qquad (4)$$

There for from (1) we have

$$Lu(x) = f(x) = L\sum_m u_m\ e_m$$

That means

$$\sum_m u_m\ Le_m = f(x)$$

Define

$$< f, g >= \int_a^b f(x)^* w(x)\, g(x) dx$$

Where $w(x)$ is the wight function for the scaler product now we have the follwing

$< e_m, f > = \int_a^b e^*{}_m\ fwdx = \sum_m u_m\ \beta_m e^*{}_m w e_m$ and henc

$$u_m = \beta^{-1}{}_m \int_a^b e^*{}_m\ fwdx \qquad (5)$$

Substitute (5) in(4)
We get

$$u(x) = \sum_m \frac{1}{\beta_m}(\int_a^b e_m{}^*\ w(x) f(x) dx) e_m$$

Or

$$u(x) = \int_a^b G(x,\tau) f(\tau) w(\tau) d\tau$$

Where

$$G(x,\tau)=\sum_m e(\tau)_m{}^* e_m(x)\frac{1}{\beta_m} \quad (6)$$

This function is known as Green function .from (6) we have

$$Lu(x)=f(x)=\int_a^b L.G(x,\tau)f(\tau)w(x)d\tau$$

Or

$$\int_a^b L^\sim G(x,\tau)[f(\tau)]^\sim d\tau = f^\sim(x) \quad (7)$$

Where

$$L^\sim=[\frac{d}{dx}p(x)\frac{d}{dx}-q(x)] \ ,f^\sim(x)=f(x)w(x)$$

From equation (7) we can put

$$L^\sim G(x,\tau)=\delta(x,\tau)=\delta(x-\tau) \quad (8)$$

Then we can say that Green function has a discontinuoty at $x=\tau$, therfor the integration can be divided into two parts as follows

$$\int_a^b G(x,\tau)f(\tau)w(\tau)d\tau=\int_a^{x-0} G(x,\tau)f(\tau)w(\tau)d\tau+\int_{x+0}^b G(x,\tau)f(\tau)w(\tau)d\tau$$

Now by substituting with the differential equation (8) and using Libintz rule we have

$$p(x)[G_x(x,x-0)-G_x(x,x+0)]f^\sim(x)+\int_a^{x-0} L^\sim G(x,\tau)f^\sim(\tau)d\tau+$$
$$\int_{x+0}^b L^\sim G(x,\tau)f^\sim(\tau)d\tau=f^\sim(x) \quad (9)$$

Clearly by comparing we can say that (9) can be satysfyed if

$$L^\sim G(x,\tau)=0 \quad ,x\neq\tau \quad (10)$$

and

$$p(x)[G_x(x,x-0)-G_x(x,x+0)]=1 \quad (11)$$

In addition we mention that the function $G(x,\tau)$ is continous at $x=\tau$ and then

$$G(x,x-0)=G(x,x+0) \quad (12)$$

The equations (10,11,12) defined the Green function $G(x,\tau)$ for the problem with its canonical boundary conditions

Let the solution for $G(x,\tau)$ represented by the following form

$$G(x,\tau)=\begin{cases}A(\tau)U(x), & x<\tau\\ B(\tau)V(x), & x>\tau\end{cases} \quad (13)$$

Where

$$L^\sim U(x)=0,\quad L^\sim V(x)=0 \quad (14)$$

And $U(x)$ satisfy the boundary condition at the point $x=a$,and $V(x)$ satisfy the boundary condition at the point $x=b$

Then from (10,13)we get

$$p(x)\left[A(x)\frac{d}{dx}U(x)-B(x)\frac{d}{dx}V(x)\right]=1 \quad (15)$$
$$A(x)U(x)=B(x)V(x) \quad (16)$$

Solving for $A(x),B(x)$ we get finally

$$G(x,\tau)=\begin{cases}cV(\tau)U(x), x<\tau\\ cU(\tau)V(x), x>\tau\end{cases} \quad (17)$$

The constant c in equation (17), is given by

$$\frac{1}{c}=p(x)\left[U(x)\frac{d}{dx}V(x)-V(x)\frac{d}{dx}U(x)\right]=p(x)w(x) \quad (18)$$

Where

$w(x)$ is Wronskian of the two independent solutions $U(x),V(x)$ .the constancy of $p(x)w(x)$ means that Green function is symmetric

i.e $G(x,\tau)=G(\tau,x)$

(II) Green functions of time-dependent linear PDE

Consider a one-dimensional heat equation

$$\gamma u_t=u_{xx} \quad (19)$$

Where $u=u(x,t)$, and $x\in(-\infty,\infty)$ .We want to find asolution for the problem according to the initial condition

$$u(x,0)=q(x) \quad (20)$$

We are going to define the differential operator $O^\wedge$ where $O^\wedge=\gamma\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}$ then(19) written as

$$O^{\wedge}u(x,t) = 0 \quad (21)$$

We assume a solution in the form

$$u(x,t) = \int G(x,\tau)q(\tau)d\,\tau \quad (22)$$

Clearly from the boundary condition (20) we have

$$u(x,0) = q(x) = \lim_{t\to 0} \int_{-\infty}^{\infty} G(x,\tau,t)q(\tau)d\tau$$

$$\therefore \int_{-\infty}^{\infty} G(x,\tau,0)q(\tau)d\tau = q(x)$$

That means the function $G(x,\tau,0)$ has a singuler point at $x=\tau$ like the kroneker-delta function then

$$G(x,\tau,0) = \delta(x-\tau) \quad (23)$$

And generally the form of Green function should be the following

$$G(x,\tau,t) = G(x-\tau,t) \quad (24)$$

Therfor the boundary condition written as

$$\lim_{t\to 0} \int G(x^{\sim},t)q(x-x^{\sim})dx^{\sim} = q(x), \forall q(x) \quad (25)$$

Now we return to solve equation (19) and find Green function in this case .We are going to use the method of scale invariance and the self-similarity.

Let $u(x,t)$ be the solution of (19) .Consider the scaling transformation as

$$\begin{cases} u \to \mu u = u^{\sim} \\ x \to \mu^{\alpha} x \\ t \to \mu^{\sigma} t \end{cases} \quad (26)$$

Where $\mu, \alpha, and\ \sigma$ are real numbers. Then the heat equation written as

$$\gamma u^{\sim}{}_{t} = \mu^{\sigma-2\alpha} u^{\sim}{}_{xx} \quad (27)$$

Which means that the function $u^{\sim}(x,t)$ is a solution of (19) iff

$$\sigma = 2\alpha \quad (28)$$

For this reason $G^{\sim}(x,t) = \mu G(\mu^{\alpha}x, \mu^{2\alpha}t)$ (29)

Replacing in equation (25)the variables $x\,,t$ by $\mu^{\alpha}x,\ \mu^{2\alpha}t$ respectively .we write equation (25) as

$$\lim_{t\to 0} \int G(\mu^{\alpha}x, \mu^{2\alpha}t)q^{\sim}(\mu^{\alpha}x)dx = \mu^{-\alpha}q^{\sim}(0) \quad (30)$$

Or

$$\lim_{t\to 0} \int G^{\sim}(x,t)q^{\sim}(x)dx = \mu^{1-\alpha}q^{\sim}(0) \quad (31)$$

Comparing (31) with (25) ,we make a remarkable observation that

$$\alpha = 1 \quad (32)$$

Hence , we stablished the self-similarType equation here.ity of the Green function ie

$$\mu G(\mu x, \mu^2 t) = G(x,t) \quad (33)$$

Let $G(x,t) = f(\theta,t)$ where θ is dimension less variable satisfy $\theta = \frac{\gamma}{t}x^2$ then $\mu f(\mu\theta, \mu^2 t) = f(\theta,t)$

Lemma:

If a function $f(t)$ satisfy $\mu f(\mu^{\alpha}t) = f(t)\ \forall \mu\ and\ \alpha$ then

$$f(t) \propto t^{\frac{-1}{\alpha}} \quad (34)$$

Using the previous lemma we can write $G(\theta,t) = f(\theta,t) \propto \frac{g(\theta)}{\sqrt{t}}$ (35)

Where $g(\theta)$ is arbitrary function depend on θ

Substituting in heat equation we have

$$\left(\frac{1}{2} + \theta\frac{d}{d\theta}\right)[4\frac{dg}{d\theta} + g] = 0 \quad (36)$$

Equation (36) can be written as

$$\left(\frac{1}{2} + \theta\frac{d}{d\theta}\right)\varphi(\theta) = 0 \quad (37)$$

Where $\varphi(\theta) = 4\frac{dg}{d\theta} + g$,the solution of (37) is given by $\varphi(\theta) = \frac{A}{\sqrt{\theta}}$, but at $\theta \to \infty$ $\varphi(\theta)$ must be finite therefore the constant $A = 0$ and hence $\varphi(\theta) = 0$ equation (37) leads for

$$4\frac{dg}{d\theta} + g = 0\,,\ \text{then}\ (\theta) = c\exp(-\frac{\theta}{4})$$

And we can get the constant c with the normalization condition ,therefore $c = \frac{1}{2\sqrt{\pi}}$

The final representation for Green function is

$$G(x,\tau,t) = \frac{1}{2}\sqrt{\frac{\gamma}{\pi t}}\exp(-\frac{\gamma}{4t}(x-\tau)^2) \qquad (38)$$

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