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On Hankel Transform

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Introduction

Hankel Transform is one of the integral transform. In this paper we intreduce some of it properties and use this transform for solving some boundary value problems .

Definition :

We can intreduce Hankel transform definition by using Fourier transform as

$$\mathcal{F}\{ f(x,y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\vec{k}\cdot\vec{r})} \; f(x,y)\,dx\,dy$$

and

$$\mathcal{F}^{-1}\{ F(k,l)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\,i(\vec{k}\cdot\vec{r})} \; F(k,l)\,dk\,dl$$

where $\vec{r} = (x,y)$, $\vec{k} = (k,l)$, intreducing the polar coordinates

$$x = r\cos\theta \;\; , \;\; y = r\sin\theta \;\; , \;\; k = \vec{k}\cos\emptyset \;\; , \;\; l = \vec{k}\sin\emptyset$$

we have

$$F(k,\emptyset) == \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{2\pi} e^{-i\,r\,k\,\cos(\theta-\emptyset)} \; r\;dr\,d\theta\;\; f(r,\theta)$$

let $f(r,\theta) = e^{-i\,n\,\theta} f(r)$ then

$$F(k,\emptyset) == \frac{1}{2\pi}\int_{-\infty}^{\infty} f(r)\;r\;dr\int_{0}^{2\pi} e^{i[n\theta \,-\, r\,k\,\cos(\theta-\emptyset)]}\;d\theta$$

subistitute $\alpha - \frac{\pi}{2} = \theta - \emptyset$ then

$$F(k,\emptyset) == \frac{1}{2\pi}\int_{-\infty}^{\infty} f(r)\;r\;dr\int_{\frac{\pi}{2}-\emptyset}^{2\pi+\frac{\pi}{2}-\emptyset} e^{i\left[n(\alpha-\frac{\pi}{2}+\emptyset)\,-\,r\,k\,\sin(\alpha)\right]}\;d\alpha$$

using the integral representation for Bessel function

$$J_n(k\,r) = \int_{\emptyset_0}^{2\pi+\emptyset_0} e^{i[\alpha\,n\,-\,r\,k\,\sin(\alpha)]}\;d\alpha$$

then

$F(k,\emptyset) = e^{i\,n(\emptyset-\frac{\pi}{2})}\;\tilde{f}(k)$ where $\tilde{f}(k)$ is called Hankel transform of $f(r)$ defind as

$$\mathcal{H}_n\{ f(r)\} = \tilde{f}(k) = \int_{0}^{\infty} r\,J_n(k\,r)\;f(r)\;dr \qquad (1)$$

Similarly we can prove that

$$\mathcal{H}_n^{-1}\{\tilde{f}(k)\} = f(r) = \int_0^\infty k\, J_n(k r)\, \tilde{f}(k)\; dk$$

and finaly we can get

$$f(r) = \int_0^\infty \int_0^\infty k\, P\, J_n(k r)\; J_n(k P)\, f(P)\;\; dk\;\, dP \qquad (2)$$

In particular Hankel transform of zero order $(n = 0)$ and order one $(n = 1)$ are very usful for the solution of problems involving Laplaces equations in an axisymmetric clynderical geometry .
Hankel transform of derivatives :
we can prove that

$$\mathcal{H}_n\{f^{\backslash}(r)\} = \frac{k}{2n}\left[(n-1)\,\tilde{f}_{n+1}(k) - (n+1)\,\tilde{f}_{n-1}(k)\right] \qquad (3)$$

where $\tilde{f}_n(k) = \mathcal{H}_n\{f(r)\}$
Proof :

$$\mathcal{H}_n\{f^{\backslash}(r)\} = \int_0^\infty r\, J_n(k r)\, f^{\backslash}(r)\, dr \; = r\, J_n(k r)\, f(r)\;\Big|_0^\infty - \int_0^\infty f(r)\, \frac{d}{dr}(r\, J_n(k r))\; dr$$

and if we have the condition that $r\, f(r) \;\to 0$ as $r \to \infty\;,\; r \to 0$
then we get

$$\mathcal{H}_n\{f^{\backslash}(r)\} = -\int_0^\infty f(r)\, \frac{d}{dr}(r\, J_n(k r))\; dr$$

$$i.e \quad \mathcal{H}_n\{f^{\backslash}(r)\} = -\int_0^\infty r\, f(r)\, \frac{d}{dr}(\,J_n(k r)) + f(r)\, J_n(k r)\;\; dr$$

from the properties of Bessel function

$$J_n^{\backslash}(k r) = \frac{k r}{2}\,(J_{n-1} - J_{n+1}) \qquad (i)$$

$$J_n = \frac{k r}{2\,n}\,[J_{n+1} - J_{n-1}] \qquad (ii)$$

then by subistitution

$$\mathcal{H}_n\{f^{\backslash}(r)\} = -\int_0^\infty [(n-1)\, J_{n+1} - (n+1)\, J_{n-1}]\, \frac{k r}{2\,n}\;\; dr$$

$$= \frac{k}{2\,n}\left[(n-1)\,\tilde{f}_{n+1}(k) - (n+1)\,\tilde{f}_{n-1}(k)\right]$$

which complete the Proof
Clearly for $n = 1$ we get

$$\mathcal{H}_1\{f^{\backslash}(r)\} = -k\, \tilde{f}_0(k) \qquad (4)$$

Now we can use (3) to calculate (H . T) of $f^{\backslash\backslash}(r)$ as

$$\mathcal{H}_n\{f^{\backslash\backslash}(r)\} = \frac{k}{2n}\left[(n-1)\,\mathcal{H}_{n+1}\{f^{\backslash}(r)\} - (n+1)\,\mathcal{H}_{n-1}\{f^{\backslash}(r)\}\right]$$

and after some simplification we get

$$\mathcal{H}_n\{f^{\backslash\backslash}(r)\} = \frac{k^2}{4}\left[\frac{n-1}{n+1}\,\tilde{f}_{n+2} - \frac{2\,(n^2 - 3)}{n^2 - 1}\,\tilde{f}_n + \frac{n+1}{n-2}\,\tilde{f}_{n-2}\right] \qquad (5)$$

We can take the Hankel transform for Bessel differential operator

$$\Delta_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \qquad as$$

$$\mathcal{H}_n\{\Delta_n f(r)\} = -k^2\, \mathcal{H}_n\{f(r)\} \qquad (6)$$

The Proof :

$$\mathcal{H}_n\{\Delta_n f(r)\} = \int_0^\infty r\left[\frac{1}{r}\frac{d}{dr}\left(r\,\frac{df}{dr}\right) - \frac{n^2}{r^2}\, f\right] J_n(k r)\;\, dr$$

$$= \int_0^\infty \left[\frac{d}{dr}\left(r\,\frac{df}{dr}\right) - \frac{n^2}{r}\, f\right] J_n(k r)\;\, dr$$

integrating by parts we have

$$\mathcal{H}_n\{\Delta_n f(r)\} = \int_0^\infty \left[k^2\, J_n^{\backslash\backslash}(k r) + \frac{k}{r}\; J_n^{\backslash}(k r) - \frac{n^2}{r^2}\, J_n(k r)\right] r\, f(r)\;\, dr$$

$$= -k^2 \int_0^\infty r\, f(r)\, J_n(k r)\; dr = -k^2\, \mathcal{H}_n\{ f(r) \}$$

Specially for $n = 0$ equation (6) leads to

$$\mathcal{H}_0\{ \Delta_0 f(r) \} = \mathcal{H}_0 \left\{ \frac{d^2}{dr^2} f(r) + \frac{1}{r}\frac{d}{dr} f(r) \right\} = -k^2\, \mathcal{H}_0\{ f(r) \} \qquad (6)$$

Some Applications :

(i) Consider : $\mathcal{H}_0\{ e^{-a r^2} \} = \int_0^\infty r\, e^{-a r^2} J_0(k r)\, dr$

let $r^2 = t \;\Rightarrow\; \mathcal{H}_0\{ e^{-a r^2} \} = \mathcal{L}\{ J_0(k\sqrt{t}) \}$

$$= \frac{1}{2}\int_0^\infty e^{-a t} \sum_{s=0}^\infty \frac{(-1)^s}{(s!)^2} \left(\frac{k\sqrt{t}}{2}\right)^{2s} dt$$

$$= \frac{1}{2}\sum_{s=0}^\infty \frac{(-1)^s}{(s!)^2}\; \mathcal{L}\left(\frac{k\sqrt{t}}{2}\right)^{2s} = \frac{1}{2}\sum_{s=0}^\infty \frac{(-1)^s}{(s!)^2} \left(\frac{k^2}{4}\right)^s \frac{\Gamma(s+1)}{a^{s+1}}$$

$$= \frac{1}{2a} \sum_{s=0}^\infty \left(\frac{k^2}{4a}\right)^s \frac{1}{s!} = \frac{1}{2a}\, e^{-\frac{k^2}{4a}}$$

That means Hankel transform for a Gaussian function is a Gaussian . Which lead to proving the Haizenberg uncertinaty principle in Quantum mechanics .

(ii) The Laplace equation in the region $z > 0$, with a symmetric Dirichlet condition at the boundary let the boundary value problem given by

$$\left\{ \begin{array}{c} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{\partial^2 v}{\partial z^2} = 0 \quad , \quad z > 0 \quad , \quad 0 < r < \infty \\ v(r, 0) = f(r) \end{array} \right\} \qquad (8)$$

Using Hankel transform of order 0 we get

$$\frac{\partial^2 V}{\partial z^2}(s, z) - s^2 V(s, z) = 0 \qquad (9)$$

and the boundary condition yeilds to

$$V(s, 0) = \int_0^\infty r\, f(r)\, J_0(sr)\; dr \qquad (10)$$

the solution is

$$V(s, z) = e^{-s z} \int_0^\infty r\, f(r)\, J_0(sr)\; dr \qquad (11)$$

so that

$$v(r, z) = \int_0^\infty s\, e^{-s z} J_0(sr)\, ds \int_0^\infty p\, f(p)\, J_0(sp)\; dp$$

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