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Inequality of Bernstein type for polynomials of hyperbolic crosses in a mixed norm

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Abstract

Introduction

We prove the inequality of Bernstein type for polynomials with spectra of hyperbolic crosses in a mixed norm $L_{\vec{p}}$, $\mathbf{1} < \vec{\boldsymbol{p}} < +\infty$.. Inequality of Bernstein type plays an important role for the proof of Jackson - Nicholas, inverse theorems and a number of other useful applications.

V.N.Temlyakov proved Bernstein's inequality for polynomials whose harmonics lie in hyperbolic crosses, when the norm of a polynomial and its derivative are measured in metric of spaces $L_{\boldsymbol{p}}$, $1 < p \le +\infty$, [1]. Following his scheme from [1] we prove that the Bernstein inequality for polynomials with spectra of hyperbolic crosses has a similar look in mixed norm, too.

First, we present the necessary notation and auxiliary statements.

Let R_d - d be dimensional Euclidean space of points $\mathbf{x} = (x_1, x_2, ..., x_d)$ with real coordinates, $\pi_d \equiv [-\pi, \pi]^d - d$ -dimensional cube. We say that $f \in L_{\mathbf{p}}(\pi_d)$, if $f(\mathbf{x}) = f(x_1, ..., x_d)$ is a measurable function - periodic in each variable, such that $\| f \|_{\mathbf{p}} < \infty$, where

$\mathbf{p} = (p_1, p_2, ..., p_d)$, $1 \le p_i < \infty$, $i = 1, 2, ..., d$ and

$$\| f \|_{\mathbf{p}} \equiv \left(\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \left(f\left(x_1, ..., x_d\right)\right)^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 ... \right)^{\frac{p_d}{p_{d-1}}} dx_d \right)^{\frac{1}{p_d}}$$ (if $p_1 = ... = p_d = p$, then

$$\| f \|_{\mathbf{p}} \equiv \| f \|_p = \left(\int_{\pi_d} | f(x)|^p \, dx \right)^{\frac{1}{p}}$$).

Let: $\mathbf{k} = (k_1, ..., k_d)$, k_j be integers, $\mathbf{s} = (s_1, ..., s_d)$, s_j −positive numbers, $j = 1, ..., d$;

$$\delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{|\mathbf{k}| \in \rho(\mathbf{s})} \hat{f}(\mathbf{k}) e^{i(\mathbf{k}, \mathbf{x})}$$, where $|\mathbf{k}| = (|k_1|, ..., |k_d|)$;

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{\pi_d} f(\mathbf{x}) e^{-i(\mathbf{k},\mathbf{x})} d\mathbf{x};$$

e -some set of integers from $[1;d]$ and $\boldsymbol{\gamma} = (\gamma_1,...,\gamma_d)$ is vector, where $\gamma_j = 1$ when $j \in e$ and $\gamma_j > 1$ when $j \notin e$;

$\mathbf{r} = (r_1,...,r_d)$ and $r = \min_{i=1,...,d} r_i > 0.$

We define the following set: $\Gamma(N,\boldsymbol{\gamma}) = \left\{ \mathbf{k} : \ k_j > 0, \ j = 1,...,d, \ \prod_{j=1}^{d} k_j^{\gamma_j} \le N \right\}$

(the set of all $\mathbf{k}$, that $|\mathbf{k}| \in \Gamma(N,\boldsymbol{\gamma})$, is called hyperbolic cross);

Through $T(N,\boldsymbol{\gamma})$ we denote the set of polynomials of the form:

$$\sum_{|\mathbf{k}| \in \Gamma(N,\boldsymbol{\gamma})} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}.$$

In case $\boldsymbol{\gamma} = \mathbf{1}$ $(\mathbf{r} = \mathbf{0})$ vector notation $\boldsymbol{\gamma}$ $(\mathbf{r})$ is not written at all.

Let $\mathbf{r} = r\,\boldsymbol{\gamma}, \ \ r \ge 0, \ U_N^{\mathbf{r}}(\mathbf{x},\boldsymbol{\alpha}) = 2^d \sum_{K \in \Gamma(N,\boldsymbol{\gamma})} \prod_{j=1}^{d} k_j^{r_j} \cos\left(k_j x_j + \frac{\alpha_j \pi}{2} \right).$

For polynomials $t(\mathbf{x}) \in \mathrm{T}(N,\boldsymbol{\gamma})$ the record $t^{(\mathbf{r})}(\mathbf{x},\boldsymbol{\alpha})$ will indicate convolution $t(\mathbf{x})$ with $U_N^{\mathbf{r}}(\mathbf{x},\boldsymbol{\alpha})$, , i.e. $t^{(\mathbf{r})}(\mathbf{x},\boldsymbol{\alpha}) = (2\pi)^{-d} \int_{\pi_d} t(\mathbf{x}-\mathbf{y}) U_N^{\mathbf{r}}(\mathbf{y},\boldsymbol{\alpha}) d\mathbf{y}$ (when $\boldsymbol{\alpha} = \mathbf{0}, \ \ \mathbf{r} = \mathbf{0}$ we have $t^{(\mathbf{0})}(\mathbf{x},\mathbf{0}) \equiv t(\mathbf{x})$).

Let m be some set of integers from $[1,d]$ and $\tilde{t}(\mathbf{x})$ indicates polynomial, which is conjugate to $t(\mathbf{x})$ by variables x_j, $j \in m$, then

$\tilde{t}_m(\mathbf{x}) = t^{(\mathbf{0})}(\mathbf{x},\boldsymbol{\alpha}^m)$, where $\alpha_j^m = \begin{cases} -1, & where \ \ j \in m, \\ 0, & where \ \ j \notin m. \end{cases}$

Through $C(\alpha,\beta,...)$ we denote some positive values, different, generally speaking, in various formulas and depending only on the parameters indicated in parentheses. With the positive *A* and any *B* record $B \underset{\alpha,\beta,...}{<<} A$ will mean $|B| \le C(\alpha,\beta,...)A.$

For positive *A* and *B*, record A ≍ B means $A \underset{\alpha,\beta,...}{<<} B \underset{\alpha,\beta,...}{<<} A.$

Here are auxiliary statements on the basis of which, the theorem will be proved.

Lemma I ([2, p. 238]). When $1 < p_i < \infty, \ \ i = 1,...,d$, for any $f \in L_{\mathbf{p}}(\pi_d)$,

$$\|f\|_{\mathbf{p}} \asymp \left\| \left\{ \sum_{\mathbf{s}} \left| \delta_{\mathbf{s}(f,\mathbf{x})} \right|^2 \right\}^{\frac{1}{2}} \right\|_{\mathbf{p}}$$ are valid.

Lemma I is a generalization of the Littlewood-Paley theorem to the case of a mixed norm.
The following implies from Lemma I

Corollary. Let $1 < p_i < \infty, \ \ i = 1,...,d, \ \ f(x) \in L_{\mathbf{p}}(\pi_d)$ and $S_l^{\boldsymbol{\gamma}}(f) = \sum\limits_{(\boldsymbol{\gamma},\mathbf{s}) \le l+1} \delta_{\mathbf{s}}(f, \mathbf{x}).$

Then inequality $\left\| S_l^{\boldsymbol{\gamma}}(f) \right\|_{\mathbf{p}} \underset{\mathbf{p}}{<<} \left\| f \right\|_{\mathbf{p}}$ is valid.

Indeed, applying Lemma I twice, we have

$$\left\| S_l^{\boldsymbol{\gamma}}(f) \right\|_{\mathbf{p}} = \left\| \sum_{(\boldsymbol{\gamma},\mathbf{s}) \le l+1} \delta_{\mathbf{s}}(f, \mathbf{x}) \right\|_{\mathbf{p}} \underset{\mathbf{p}}{<<} \left\| \left\{ \sum_{(\boldsymbol{\gamma},\mathbf{s}) \le l+1} \left| \delta_{\mathbf{s}}(f, \mathbf{x}) \right|^2 \right\}^{\frac{1}{2}} \right\|_{\mathbf{p}} \underset{\mathbf{p}}{<<}$$

$$\underset{\mathbf{p}}{<<} \left\| \left\{ \sum_{(\boldsymbol{\gamma},\mathbf{s}) \le l+1} \left| \delta_{\mathbf{s}}(f, \mathbf{x}) \right|^2 \right\}^{\frac{1}{2}} \right\|_{\mathbf{p}} \underset{\mathbf{p}}{<<} \left\| f \right\|_{\mathbf{p}}.$$

Lemma 2.(Marcinkiewicz theorem[2, p. 239]). Let for multiple sequence $\{\lambda_{\mathbf{k}}\} \equiv \{\lambda_{k_1,...,k_d}\}$ there is a number M, independent of $s_i, \ \ i = 1,...,d$ and such that

$$\sum_{|k_1| = 2^{s_1 - 1}}^{2^{s_d} - 1} \ldots \sum_{|k_d| = 2^{s_d - 1}}^{2^{s_d} - 1} \left| \Delta_1 \ldots \Delta_d \lambda_{k_1,...,k_d} \right| \le M, \quad s_i = 1, 2, ..., \quad i = 1, ..., d,$$

where $\Delta_j \tau_{k_1,...,k_d} = \tau_{k_1,...,k_{j+1},...,k_d} - \tau_{k_1,...,k_d}$. Then for any function

$f(\mathbf{x}) = \sum\limits_{\mathbf{k}} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})} \in L_{\mathbf{p}}(\pi_d), \ \ 1 < p_i < \infty, \ \ i = 1,...,d$ function

$$\Lambda f = \sum_{\mathbf{k}} \lambda_{\mathbf{k}} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})} \in L_{\mathbf{p}}(\pi_d)$$

and

$$\left\| \Lambda f \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{<<} \left\| f \right\|_{\mathbf{p}}. \tag{1}$$

Theorem. Let $\mathbf{p} = (p_1,..., p_d), \ \ 1 < p_i < \infty, \ \ i = 1,...,d; \ \ \ \mathbf{r} = (r_1,...,r_d),$ $0 < r_i < \infty, \ \ i = 1,...,d; \ \ \ \mathbf{r} = r\,\boldsymbol{\gamma},$ where $r = \min\limits_{i=1,...,d} r_i.$

Then, for arbitrary $\boldsymbol{\alpha} = (\alpha_1,...,\alpha_d)$ we have an inequality

$$\sup_{t \in T(N,\boldsymbol{\gamma})} \frac{\left\| t^{(r\gamma)}(\mathbf{x}, \boldsymbol{\alpha}) \right\|_{\mathbf{p}}}{\left\| t(\mathbf{x}) \right\|_{\mathbf{p}}} \underset{\mathbf{p},M}{<<} N^r.$$

Proof. Consider the case $\boldsymbol{\alpha} = \mathbf{0}$, as when $1 < p_i < \infty, \ \ i = 1,...,d$, by multi-dimensional analogue the Riesz theorem on the conjugate function, see [2, p.241], the operator of the trigonometric conjugation is a bounded operator from $L_{\mathbf{p}}(\pi_d)$ to $L_{\mathbf{p}}(\pi_d)$ .Then, by the orthogonality of the trigonometric system and from definition $t^{(\mathbf{r})}(\mathbf{x}, \boldsymbol{\alpha})$ follows that

$$t^{(r\gamma)}(\mathbf{x}, \mathbf{0}) = \frac{1}{(2\pi)^d} \int_{\pi_d} t(\mathbf{x} - \mathbf{y}) U^{(r\gamma)}(\mathbf{y}, \mathbf{0}) d\mathbf{y} =$$

$$= \frac{1}{(2\pi)^d} \int_{\pi_d} \sum_{|\mathbf{k}| \in \Gamma(N,\boldsymbol{\gamma})} \hat{t}(\mathbf{k}) e^{i(\mathbf{k}, \mathbf{x} - \mathbf{y})} 2^d \sum_{\mathbf{k} \in \Gamma(N,\boldsymbol{\gamma})} \prod_{j=1}^{d} k_j^{r\gamma_j} \cos(k_j y_j) d\mathbf{y} =$$

$$= \sum_{|\mathbf{k}| \in \Gamma(N,\boldsymbol{\gamma})} \hat{t}(\mathbf{k}) \prod_{j=1}^{d} \left| k_j^{r\gamma_j} \right| e^{i(\mathbf{k},\mathbf{x})}. \qquad (2)$$

One-time sequence $\lambda_{k_i}^{r\gamma_i} \equiv |k_i|^{r\gamma_i} 2^{-r\gamma_i s_i}$ and

$$\left(\lambda_{k_i}^{r\gamma_i}\right)^{-1} \equiv |k_i|^{-r\gamma_i} 2^{r\gamma_i s_i}, \qquad 2^{s_i-1} \le |k_i| < 2^{s_i}, \quad s_i = 1,2,..., \quad i = 1,...,d$$

are multipliers of Marcinkiewicz, i.e for some relations M_i,

ratios $\left|\lambda_{k_i}^{r\gamma_i}\right| \le M_i$, $\sum_{|k_i|=2^{s_i-1}}^{2^{s_i}-1} \left|\lambda_{k_i}^{r\gamma_i} - \lambda_{k_i+1}^{r\gamma_i}\right| \le M_i$, $s_i = 1,2,...,$ are done,

and therefore for the multiple sequence $\lambda_{k_1,...,k_d} \equiv \prod_{i=1}^{d} \lambda_{k_i}^{r\gamma_i}$,

inequality $\sum_{|k_1|=2^{s_1-1}}^{2^{s_1}-1} \dots \sum_{|k_d|=2^{s_d-1}}^{2^{s_d}-1} \left|\Delta_1 \dots \Delta_d \lambda_{k_1,...,k_d}\right| \le M, \quad s_i = 1,2,..., \quad i = 1,...,d,$, where

$\Delta_1 \dots \Delta_d \lambda_{k_1,...,k_d} = \prod_{i=1}^{d} \left(\lambda_{k_i+1}^{r\gamma_i} - \lambda_{k_i}^{r\gamma_i}\right)$. ,will take place. Hence, by Lemma 2, we have

$$\Lambda f \;\square\; \sum_{\mathbf{k}} \prod_{j=1}^{d} \lambda_{k_j}^{r\gamma_j} \hat{f}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})} \in L_{\mathbf{p}}(\pi_d) \quad \text{and} \quad \|\Lambda f\|_{\mathbf{p}} \underset{\mathbf{p},M}{<<} \|f\|_{\mathbf{p}}.$$

Similarly, we can show that $\Lambda^{-1} f \;\square\; \sum_{\mathbf{k}} \prod_{j=1}^{d} \left(\lambda_{k_j}^{r\gamma_j}\right)^{-1} \hat{f}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})} \in L_{\mathbf{p}}(\pi_d)$ and $\left\|\Lambda^{-1} f\right\|_{\mathbf{p}} \underset{\mathbf{p},M}{<<} \|f\|_{\mathbf{p}}$..

Therefore, by (1), we have

$$\|f\|_{\mathbf{p}} = \left\| \sum_{\mathbf{k}} \hat{f}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})} \right\|_{\mathbf{p}} = \left\| \sum_{\mathbf{k}} \prod_{j=1}^{d} \lambda_{k_j}^{r\gamma_j} \prod_{j=1}^{d} \left(\lambda_{k_j}^{r\gamma_j}\right)^{-1} \hat{f}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})} \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{<<}$$

$$\underset{\mathbf{p},M}{<<} \left\| \sum_{\mathbf{k}} \prod_{j=1}^{d} \left(\lambda_{k_j}^{r\gamma_j}\right)^{-1} \hat{f}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})} \right\|_{\mathbf{p}} \equiv \left\|\Lambda^{-1} f\right\|_{\mathbf{p}}. \qquad (3)$$

Let $t(\mathbf{x}) \equiv t(\mathbf{x},\mathbf{0}) \in T(N,\boldsymbol{\gamma})$.. Then, by Lemma I, the correlation

$$\|t(\mathbf{x},\mathbf{0})\|_{\mathbf{p}} \asymp \left\| \left(\sum_{\mathbf{s}} \left|\delta_{\mathbf{s}}(\mathbf{t},\mathbf{x})\right|^2 \right)^{\frac{1}{2}} \right\|_{\mathbf{p}}, \qquad (4)$$

takes place, and by (2), (3) and (4) we obtain

$$\left\|t^{(r\gamma)}(\mathbf{x},\mathbf{o})\right\|_{\mathbf{p}} = \left\| \sum_{|\mathbf{k}| \in \Gamma(N,\boldsymbol{\gamma})} \hat{t}(\mathbf{k}) \prod_{j=1}^{d} |k_j|^{r\gamma_j} e^{i(\mathbf{k},\mathbf{x})} \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{<<} \left\| \sum_{|\mathbf{k}| \in \Gamma(N,\boldsymbol{\gamma})} \hat{t}(\mathbf{k}) \prod_{j=1}^{d} \left(\lambda_{k_i}^{r\gamma_i}\right)^{-1} \prod_{j=1}^{d} |k_j|^{r\gamma_j} e^{i(\mathbf{k},\mathbf{x})} \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{<<}$$

$$\underset{\mathbf{p},M}{\ll} \left\| \left(\sum_{\mathbf{s}} \left| \sum_{|k_1|=2^{s_1}-1}^{2^{s_1}-1} \cdots \sum_{|k_1|=2^{s_d-1}}^{2^{s_1}-1} \hat{t}(\mathbf{k}) \prod_{j=1}^{d} |k_j|^{r\gamma_j} \prod_{i=1}^{d} |k_i|^{-r\gamma_i} 2^{r\gamma_i s_s} e^{i(\mathbf{k},\mathbf{x})} \right|^2 \right)^{\frac{1}{2}} \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{\ll}$$

$$\underset{\mathbf{p},M}{\ll} \left\| \left(\sum_{\mathbf{s}} 2^{2r(\gamma,\mathbf{s})} \left| \sum_{|k_1|=2^{s_1-1}}^{2^{s_1}-1} \cdots \sum_{|k_d|=2^{s_d-1}}^{2^{s_d}-1} \hat{t}(\mathbf{k}) e^{i(\mathbf{k},\mathbf{x})} \right|^2 \right)^{\frac{1}{2}} \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{\ll} N^r \left\| \left(\sum_{\mathbf{s}} \left| \delta_{\mathbf{s}}(\mathbf{t},\mathbf{x}) \right|^2 \right)^{\frac{1}{2}} \right\|_{\mathbf{p}} \underset{\mathbf{p},M}{\ll} N^r \|t\|_{\mathbf{p}}.$$

The theorem is proved.

Note that when $p_1 = \ldots = p_d = p$, i.e. $\| f \|_{\mathbf{p}} \equiv \| f \|_p = \left(\int_{\pi_d} | f(\mathbf{x})|^p \, d\mathbf{x} \right)^{\frac{1}{p}}$, where $1 < p < \infty$, the inequality (*) of the theorem coincides with the inequality established by V.N.Temlyakov [1].

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