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The generalized complex exponent and its application for finding sums

Bimurat Sagindykov
Candidate (Ph. D.) of Physical and Mathematical sciences, Kazakh National Technical University,
Almaty, Kazakhstan
bimurat55@gmail.com

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Abstract

In this paper we describe the generalized complex exponent. The method used in paper allows to find some finite sums for exponential trigonometric series.

Introduction

A generalized complex exponent is a mathematical function defined by the relation $f(z) = e^z$, where z is a generalized complex number $z = x + py$, $p^2 = -\theta_0 + p\theta_1$. The generalized complex exponent defined as the analytic continuation of the real variable exponent $x : f(z) = e^x$.

Let define a formal expression $e^z = e^{x+py} = e^x \cdot e^{py}$. The expression determined on the real axis by this way will be coincided with a classic real exponent. For complete construction it is necessary to prove analyticity of e^z function, i.e. to show that e^z function can be transformed into convergent series. Let show it

$$f(z) = e^z = e^x \cdot e^{py} = e^{py} \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

where $e^{py} = I(\theta_0, \theta_1, y) + pK(\theta_0, \theta_1, y)$, and θ_0, θ_1 are real numbers. It is easy to prove the convergence of the series:

$$\left|e^{py} \sum_{n=0}^{\infty} \frac{x^n}{n!}\right| = |e^{py}| \cdot \left|\sum_{n=0}^{\infty} \frac{x^n}{n!}\right| \leq e^{\left|\frac{\theta_1}{2} y\right|} \cdot e^{|x|}.$$

The series convergence absolutely everywhere, by this means that a sum of the series in each point will determine the value of the $f(z) = e^z$ analytic function.

The only generalization of real numbers with the preservation of the known laws of arithmetic are complex numbers. Therefore we consider only the internal structure of complex numbers. The generalized complex number z can be presented as $z = x + py$, where $p^2 = -\theta_0 + p\theta_1$. Let consider special cases to make a term be in accord with the name.

If $\theta_0 = 1$, $\theta_1 = 0$, $p^2 = -1$, $p = i$ a generalized complex number corresponds to a complex number of $z = x + iy$ form.

If $\theta_0 = 0$, $\theta_1 = 0$, $p^2 = 0$ then we go to a dual number.

If $\theta_0 = -1$, $\theta_1 = 0$, $p^2 = 1$ then we get a double number.

Changing control parameters θ_0, θ_1 we obtain different theories.

Before presenting generalized complex numbers in the form $z = x + py$ we define addition, multiplication, and conjugation by the following formulas :

Addition $z_1 + z_2 = (x_1 + x_2) + p(y_1 + y_2)$;

Multiplication $z_1 \cdot z_2 = (x_1 x_2 - \theta_0 y_1 y_2) + p(x_1 y_2 + y_1 x_2 + \theta_1 y_1 y_2)$;

Conjugate $\bar{z} = x + \theta_1 y - py$.

Product of $z\cdot\bar{z}=x^2+\theta_1 xy+\theta_0 y^2$ gives a non-negative real number. As a consequence, it defines the norm of z generalized complex number. Thus

$$|z|^2=z\cdot\bar{z}=x^2+\theta_1 xy+\theta_0 y^2 \tag{1}$$

The right hand side of (1) is a quadratic form of two variables x, y.

Relating to an invariant of the quadratic form generalized complex numbers are divided into three types: elliptic, hyperbolic and parabolic complex numbers. Let $p^2=-\theta_0+p\theta_1$ then numbers are divided into determined types depending on θ_0, θ_1.

If $D=\frac{\theta_1^2}{4}-\theta_0<0$, we have an elliptic type.

If $D=\frac{\theta_1^2}{4}-\theta_0>0$, we get a hyperbolic system of numbers.

If $D=\frac{\theta_1^2}{4}-\theta_0=0$ we have a parabolic system.

Let notice Euler formula for generalized complex numbers

$$e^{pt}=I(\theta_0,\theta_1,y)+pK(\theta_0,\theta_1,y)=$$
$$=\begin{cases} e^{\frac{\theta_1}{2}t}\left[\left(\cos\sqrt{-D}t-\frac{\theta_1}{2\sqrt{-D}}\sin\sqrt{-D}t\right)+p\frac{1}{\sqrt{-D}}\sin\sqrt{-D}t\right], D<0;\\ e^{\frac{\theta_1}{2}t}\left[\left(1-\frac{\theta_1}{2}t\right)+pt\right], D=0;\\ e^{\frac{\theta_1}{2}t}\left[\left(\cosh\sqrt{D}t-\frac{\theta_1}{2\sqrt{Dt}}\sinh\sqrt{D}t\right)+p\frac{1}{\sqrt{D}}\sinh\sqrt{D}t\right], D>0.\end{cases} \tag{2}$$

The true nature of (2) will be defined through the paper. The easiest way to prove this formula is using theory of different equations.

The conjugate of e^{pt} in a formula (2) gives

$$\overline{e^{pt}}=\overline{I(t)+pK(t)}=I(t)+\theta_1K(t)-pK(t) \text{ or}$$
$$e^{(\theta_1-p)t}=I(t)+\theta_1K(t)-pK(t). \tag{3}$$

Multiplying (2) and (3) we can easily get the basic trigonometric identity for generalized complex numbers

$$e^{\theta_1 t}=I^2(t)+\theta_1 I(t)K(t)+\theta_0K^2(t). \tag{4}$$

2 Addition formulas

According to the accepted agreement $e^{p(m+n)t}=I[(m+n)t]+pK[(m+n)t]$. On the other side $e^{p(m+n)t}=e^{pmt}\cdot e^{pnt}=[I(mt)+pK(mt)]\cdot[I(nt)+pK(nt)]$. Separating the real and imaginary parts relating to p we have

$$\begin{aligned} I_{m+n}&=I_mI_n-\theta_0K_mK_n,\\ K_{m+n}&=I_mK_n+K_mI_n+\theta_1K_mK_n,\end{aligned} \tag{5}$$

where $K_{m+n}=K[(m+n)t]$.

Example 1.

Let $\theta_0=-1$, $\theta_1=0$, $D=\frac{\theta_1^2}{4}-\theta_0=1>0$, $I_m=I(mt)=\cosh mt$, $K_m=K(mt)=\sinh mt$. Then from (2) and (5) it follows next addition formulas

$$\cosh(m+n)t=\cosh mt\cosh nt+\sinh mt\sinh nt,$$
$$\sinh(m+n)t=\cosh mt\sinh nt+\sinh mt\cosh nt.$$

Setting $m=n$ in (5) we get a formula of double argument

$$\begin{aligned} I_{2n}&=I_n^2-\theta_0K_n^2,\\ K_{2n}&=2I_nK_n+\theta_1K_n^2.\end{aligned} \tag{6}$$

Example 2.

Let $\theta_0=1$, $\theta_1=0$, $D=\frac{\theta_1^2}{4}-\theta_0=-1<0$, $\sqrt{-D}=1$, $I_n=\cos nt$, $K_n=\sin nt$. Then

$$\cos 2nt=\cos^2 nt-\sin^2 nt,$$
$$\sin 2nt=2\sin nt\cdot\cos nt.$$

3 Finding sums of some exponential – trigonometric series

Let $f(z)$ be a function of the generalized complex variable $z=x+py$, and analytic for $|z|\le 1$, where $|z|^2=z\cdot\bar{z}=(x+py)(x+\theta_1y-py)=x^2+\theta_1xy+\theta_0y^2$. It is known from these conditions that for $|z|\le 1$ the $f(z)$ function can be expanded in a formal power series

$$f(z)=c_0+c_1z+c_2z^2+\cdots+c_nz^n+\cdots \tag{7}$$

Assume that the coefficients of (7) are real numbers. Setting $z = e^{px} = I(x) + pK(x)$ for any x we have

$$\begin{aligned} f(e^{px}) &= c_0 + c_1[I(x) + pK(x)] + c_2[I(2x) + pK(2x)] + \cdots \\ &+ c_n[I(nx) + pK(nx)] + \cdots = \\ &= c_0 + c_1 I(x) + c_2 I(2x) + \cdots + c_n I(nx) + \cdots \\ &+ p[c_1 K(x) + c_2 K(2x) + \cdots + c_n K(nx)]. \end{aligned} \tag{8}$$

Separating the real and imaginary parts in (8) we present $f(e^{px})$ in next form

$$f(e^{px}) = \varphi(x) + p\psi(x),$$

where $\varphi(x)$ and $\psi(x)$ are real functions. It is obvious from (8) that

$$\varphi(x) = c_0 + c_1 I(x) + c_2 I(2x) + \cdots + c_n I(nx) + \cdots$$
$$\psi(x) = c_1 K(x) + c_2 K(2x) + \cdots + c_n K(nx) + \cdots$$

This fact can be used to get the sum of some exponential trigonometric series.

Example 3.

It is known that for any z

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{m=0}^{\infty} \frac{z^m}{m!},$$

Then in account of (8)

$$e^{e^{px}} = \sum_{m=0}^{\infty} \frac{e^{pmx}}{m!} = \sum_{m=0}^{\infty} \frac{I(mx) + pK(mx)}{m!}.$$

From the other side

$$e^{e^{px}} = e^{I(x)+pK(x)} = e^{I(x)} \cdot e^{pK(x)} = e^{I(x)}\left[I\big(K(x)\big) + pK\big(K(x)\big)\right],$$

and therefore

$$\begin{aligned} e^{I(x)} I\big(K(x)\big) &= \sum_{m=0}^{\infty} \frac{I(mx)}{m!}, \\ e^{I(x)} K\big(K(x)\big) &= \sum_{m=0}^{\infty} \frac{K(mx)}{m!}. \end{aligned} \tag{9}$$

Example 4.

Let $\theta_0 = -1$, $\theta_1 = 0$, $p^2 = -\theta_0 + p\theta_1$, $D = \frac{\theta_1^2}{4} - \theta_0 = 1 > 0$, $I(x) = \cosh x$, $K(x) = \sinh x$. Then according to (9)

$$e^{\cosh x} \cosh(\sinh x) = 1 + \cosh x + \frac{\cosh 2x}{2!} + \cdots + \frac{\cosh mx}{m!} + \cdots,$$
$$e^{\cosh x} \sinh(\sinh x) = \sinh x + \frac{\sinh 2x}{2!} + \cdots + \frac{\sinh mx}{m!} + \cdots.$$

Example 5.

Find the sum of series $\sum_{k=0}^{\infty} \frac{e^{pkx}}{a^k}$. Given series is a geometrical progression (convergent for $\left|\frac{e^{px}}{a}\right| < 1$) with a common ratio $q = \frac{e^{px}}{a}$. It follows that

$$\sum_{k=0}^{\infty} \frac{e^{pkx}}{a^k} = \frac{1}{1 - \frac{e^{px}}{a}} = \frac{a}{a - e^{px}}.$$

According to the accepted argument $e^{px} = I(x) + pK(x)$ and $e^{pkx} = I(kx) + pK(kx)$. Then

$$\sum_{k=0}^{\infty} \frac{1}{a^k}[I(kx) + pK(kx)] = \frac{a}{[a - I(x)] - pK(x)} =$$
$$\frac{a - I(x) - \theta_1 K(x) + pK(x)}{I^2(x) + \theta_1 I(x)K(x) + \theta_0 K^2(x) + a^2 - 2aI(x) - \theta_1 aK(x)}.$$

From here

$$\sum_{k=0}^{\infty} \frac{I(kx)}{a^k} = \frac{a - I(x) - \theta_1 K(x)}{I^2(x) + \theta_1 I(x)K(x) + \theta_0 K^2(x) + a^2 - 2aI(x) - \theta_1 aK(x)},$$
$$\sum_{k=0}^{\infty} \frac{K(kx)}{a^k} = \frac{K(x)}{I^2(x) + \theta_1 I(x)K(x) + \theta_0 K^2(x) + a^2 - 2aI(x) - \theta_1 aK(x)}.$$

Example 6.

Let $\theta_0 = 2,\ \theta_1 = 2,\ a = 2,\ D = \frac{\theta_1^2}{4} - \theta_0 = 1 - 2 = -1 < 0,\ \sqrt{-D} = 1,\ I(x) = e^x(\cos x - \sin x),\ K(x) = e^x \sin x,\ I^2(x) + \theta_1 I(x)K(x) + \theta_0 K^2(x) = e^{\theta_1 x} = e^{2x}$.

Then

$$\sum_{k=0}^{\infty} \frac{e^{kx}\sin kx}{2^k} = \frac{e^x \sin x}{e^{2x} - 4\cos x \cdot e^x + 4},$$

$$\sum_{k=0}^{\infty} \frac{e^{kx}\cos kx}{2^k} = \frac{2 - e^x \cos x}{e^{2x} - 4\cos x \cdot e^x + 4}.$$

These expansions are true for any x when $\left|\frac{e^{px}}{2}\right| < 1$ or $|e^{px}| < 2$ or $e^{\frac{\theta_1}{2}x} < 2$ i.e. when $e^x < 2$.

4 Find finite sums and series

Find the sum of the series

$$\sum_{k=1}^{n} e^{pkx} = \sum_{k=1}^{n}[I(kx) + pK(kx)] = \sum_{k=1}^{n} I(kx) + p\sum_{k=1}^{n} K(kx).$$

$$\sum_{k=1}^{n} e^{pkx} = \sum_{k=1}^{n}(e^{px})^k = \frac{e^{p(n+1)x} - e^{px}}{e^{px} - 1} =$$
$$= \frac{I_{n+1} + pK_{n+1} - I_1 - pK_1}{I_1 - 1 + pK_1} = \frac{(I_{n+1} - I_1) + p(K_{n+1} - K_1)}{(I_1 - 1) + pK_1} = \tag{10}$$
$$= \frac{[(I_{n+1} - I_1) + p(K_{n+1} - K_1)][(I_1 - 1) + \theta_1 K_1 - pK_1]}{(I_1 - 1)^2 + \theta_1(I_1 - 1)K_1 + \theta_0 K_1^2}.$$

From here

$$\sum_{k=1}^{n} I(kx) = \frac{(I_1 - 1)(I_{n+1} - I_1) + \theta_1 K_1(I_{n+1} - I_1) + \theta_0 K_1(K_{n+1} - K_1)}{e^{\theta_1 x} - 2I_1 - \theta_1 K_1 + 1},$$

$$\sum_{k=1}^{n} K(kx) = \frac{K_{n+1}I_1 - I_{n+1}K_1 + K_1 - K_{n+1}}{e^{\theta_1 x} - 2I_1 - \theta_1 K_1 + 1},$$

where $e^{\theta_1 x} = I_1^2 + \theta_1 I_1 K_1 + \theta_0 K_1^2$.

Example 7.

Let $\theta_0 = 1$, $\theta_1 = 0$, $D = \frac{\theta_1^2}{4} - \theta_0 = -1 < 0$, $\sqrt{-D} = 1$, $I_1 = I(x) = \cos x$, $K_1 = K(x) = \sin x$. Then

$$\sum_{k=1}^{n} I(kx) = \cos x + \cos 2x + \cdots + \cos nx = \cos\frac{n+1}{2}x \cdot \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}},$$

$$\sum_{k=1}^{n} K(kx) = \sin x + \sin 2x + \cdots + \sin nx = \sin\frac{n+1}{2}x \cdot \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

Example 8.

Let $\theta_0 = \theta_1 = 2,\ D = \frac{\theta_1^2}{4} - \theta_0 = -1 < 0,\ \sqrt{-D} = 1,\ I_1 = I(x) = e^x(\cos x - \sin x),\ K_1 = K(x) = e^x \sin x$. Then

$$e^x \sin x + e^{2x}\sin 2x + \cdots + e^{nx}\sin nx = \frac{e^{(n+2)x}\sin nx - e^{(n+1)x}\sin(n+1)x + e^x \sin x}{e^{2x} - 2e^x\cos x + 1},$$

$$e^x \cos x + e^{2x}\cos 2x + \cdots + e^{nx}\cos nx = \frac{e^{(n+2)x}\cos nx - e^{(n+1)x}\cos(n+1)x + e^x\cos x - e^{2x}}{e^{2x} - 2e^x\cos x + 1}.$$

Knowing depression formulas we can find a finite sum of the following series $\sum_{k=1}^{n} K^2(kx)$, $\sum_{k=1}^{n} I^2(kx)$ and $\sum_{k=1}^{n} I(kx)K(kx)$, where

$$K_k^2 = K^2(kx) = \frac{2I_{2k} + \theta_1 K_{2k} - 2e^{\theta_1 kx}}{4D},$$
$$I_k^2 = \frac{(\theta_1^2 - 2\theta_0)I_{2k} + \theta_0\theta_1 K_{2k} - 2\theta_0 e^{\theta_1 kx}}{4D},$$
$$I_k \cdot K_k = \frac{\theta_1 e^{\theta_1 kx} - 2\theta_0 K_{2k} - \theta_1 I_k}{4D}.$$
$$\sum_{k=1}^{n} a^k I(kx) = \sum_{k=1}^{n} a^k I_k = Re \sum_{k=1}^{n} (ae^{px})^k =$$
$$= \frac{a^{n+2}(I_{n+1}I_1 + \theta_1 I_{n+1}K_1 + \theta_0 K_1 K_{n+1}) - a^{n+1}I_{n+1} - a^2 e^{\theta_1 x} + aI_1}{a^2 e^{\theta_1 x} - a(2I_1 + \theta_1 K_1) + 1},$$
$$\sum_{k=1}^{n} a^k K(kx) = \sum_{k=1}^{n} a^k K_k = Im \sum_{k=1}^{n} (ae^{px})^k = \frac{a^{n+2}(K_{n+1}I_1 - K_1 I_{n+1}) - a^{n+1}K_{n+1} + aK_1}{a^2 e^{\theta_1 x} - a(2I_1 + \theta_1 K_1) + 1}.$$

Example 9.

Let $\theta_0 = 1$, $\theta_1 = 0$, $D = \frac{\theta_1^2}{4} - \theta_0 = -1 < 0$, $\sqrt{-D} = 1$, $I_1 = \cos x$, $K_1 = \sin x$. Then

$$\sum_{k=1}^{n} a^k \sin kx = \frac{a^{n+2}\sin nx - a^{n+1}\sin(n+1)x + a\sin x}{a^2 - 2a\cos x + 1}.$$

Similarly, we can find the sum of cosines

$$\sum_{k=1}^{n} a^k \cos kx = \frac{a^{n+2}\cos nx - a^{n+1}\cos(n+1)x - a\cos x + 1}{a^2 - 2a\cos x + 1}.$$

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