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RESEARCH ARTICLE

A stochastic consideration to compare *T*-transitive proximity relations

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Abstract

Garmendia and Recasens proposed three methods to approximate a proximity relation *R* by a *T*- transitive one where *T* is a continuous Archimedean *t*-norm are given. However, to obtain the closest approximation *E* to *R* with respect to the Euclidean distance could be very expensive. Indeed, the calculation of *E* becomes then a nonlinear programming problem. In this paper, we show that weighted quasi-arithmetic means of *T*-indistinguish abilities can be an relatively efficient way to find the closest *E* for *T*the Łukasiewicz*t*-norm without solving complicate nonlinear programming problems. We also give formal proofs of some useful propositions and theorems.

Introduction

Transitivity of fuzzy relations is one of the most important properties that can be required in manyapplications, such as database management systems[15,19], fuzzy clustering,[9,25−27] expert systems,decision making and artificial intelligence.[21] Analogously to transitivity of crisp relations, the transitiveproperty of fuzzy relations can be understood as a threshold on the degree of the relation (for example, a degree of equality) between two elements, when a degree of relation between thoseelementsand a third element of a universe of discourse is known[7]. The classical concept of transitivity is generalized in fuzzy logic by the *T*-transitivity property of fuzzy relations, where *T* is a triangular norm[18].The most common used transitive fuzzy relations are *T*-indistinguishabilities(reflexive, symmetricand *T*-transitive fuzzy relations) since they generalize the concepts of (crisp) equivalence relationand equality and are useful to represent the ideas of similarity and neighborhoodas well[22].

Sometimes our knowledge is modeled in a proximity relation *R* (a reflexive and symmetric fuzzyrelation) defined on a finite universe *X*, and we want to compute a *T*-indistinguishability*E*from thatknowledge to impose some coherence or to generate a similarity. Of course, it is desirable that *E*isasclose as possible to *R*. Garmendia and Recasens[8]have developed several methods to find such *E*.

Moreover, they demonstrate how to apply these methods to obtain the closest *T*-indistinguishabilityto a given proximity with respect to the Euclidean distance. However, trying to acquire the closest*E* to *R* can be very expensive. Indeed, if *n* is the cardinality of the universe *X*, the transitivityof*T*-indistinguishabilities can be modeled by $3\binom{n}{3}$inequalities and they lay in the region of the$\binom{n}{2}$-dimensional space defined by them[8]. That is, the calculation of *E*becomes then a nonlinearprogramming problem.

In our approach, the preliminaries section contains definitions and formal proofs of the propositions and theorems which are needed in the following sections. The above results lead to the main discussion of the paper: by excluding the nonlinear programming method, we show that $m_f^{p,1-p}(\overline{R_{t_1}}, \underline{R_{t_1}})$ is a reasonable choice statistically in comparison with those of $\overline{R_{t1}}^{((p))}and\underline{R_{t1}}^{((p))}$with respect tothe Euclidean distance. Hence, it suggests that$m_f^{p,1-p}(\overline{R_{t_1}}, \underline{R_{t_1}})$can be an efficient way for finding the closest *E* to *R* to avoid complicate nonlinear programming problems.

2. Preliminaries

Let X and Ybe two ordinary finite non-empty sets. A fuzzy relation R between X and Y, denoted by$R(X,Y)$, is defined as a fuzzy subset of $X\times Y$ (see Zadeh[29]). That is, $R(X,Y)$ is an expression given by:

$$R(X,Y) = \{< (x,y), \mu_R(x,y) > | x \in X, y \in Y\},$$

where

μ_R:$X\times Y\rightarrow$ [0,1].

In fact, the fuzzy relation $R(X,Y)$ is associated with a membership function $\mu_R(x,y)$ assumingvalues in the interval [0, 1] for all (x, y) in $X \times Y$. The value of $\mu_R(x,y)$ represents the strength ofthe relationship between x and y. In cluster analysis, we are only interested in relations on a singleset X, i.e. $R(X) = R(X,X)$, a membership function $\mu_R(x,y)$from $X \times X$into [0, 1].

Definition1. Let R and Sbe two fuzzy relations on a finite set X. Then

$$R \le S \Leftrightarrow \mu_R(x,y) \le \mu_s(x,y), \forall(x,y) \in X \times X.$$

The t-norm was defined as a general form of the fuzzy intersection. Zimmermann[30]listed somespecified t-norms as follows:

1. $t_\omega(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$ (drastic product);
2. $t_1(x,y) = \max\{0, x+y-1\}$ (bounded difference);
3. $t_{1.5}(x,y) = xy/(2-(x+y-xy))$(Einstein product);
4. $t_2 = xy$(algebraic product);
5. $t_{2.5} = xy/(x+y-xy)$(Hamacher product);
6. $t_3(x,y) = \min\{x,y\}$ (minimum).

In the above list, the t-norms,t_1, $t_2$and$t_3$are most commonly used. It is seen that the t_1-norm isthe Łukasiewicz$t$-norm, thet_2-norm is the product t-norm, and the t_3-norm is the minimum t-norm.A fuzzy relation R on a finite set X generally has reflexive and symmetric properties defined asfollows.

Definition 2. (*Proximityrelation*) A fuzzy relation Ron a finite set X is called a proximity relation if it satisfies

1. (reflexivity)$\mu_R(x,x) = 1 \quad \forall x \in X$, and
2. (symmetry)$\mu_R(x,y) = \mu_R(y,x) \quad \forall x,y \in X$.

In real cases, a proximity relation is not applicable. A T-transitivity property for a fuzzy relation R is required for real applications.

Definition 3. (*T-transitivity*) A fuzzy relation Ron a finite set X is called T-transitivity if $\mu_R(x,z) \ge T(\mu_R(x,y), \mu_R(y,z))$for all $x,y,z \in X$, where T stands for a t-norm.

Let $R_T(X)$be a set of all fuzzy relations R with T-transitivity on X. We have the result with $R_{t_3}(X) \subseteq R_{t_2}(X) \subseteq R_{t_1}(X)$because we can demonstrate that $t_1 \le t_2 \le t_3$. In this sense, thecondition of t_3-transitivity is more restrictive than that of t_1-transitivity.

Definition 4. (*Similarity*, Zadeh[29]) A similarity is a reflexive, symmetric and minimum-transitivefuzzy relation.

Definition 5.(*T-indistinguishability*, Trillasand Valverde[22]) Given a t-norm T, a fuzzy relation Eon a finite set X is called a T-indistinguishability if it satisfies

1. (reflexivity) $\mu_E(x,x) = 1 \quad \forall x \in X$;
2. (symmetry) $\mu_E(x,y) = \mu_E(y,x) \quad \forall x,y \in X$;
3. (T-transitivity) $\mu_E(x,z) = T(\mu_E(x,y), \mu_E(y,z)) \quad \forall x,y,z \in X$.

It is obvious that a T-indistinguishability E is a similarity relation when T is the minimum$t$-norm.

Definition 6. (*T-transitive closure*, Naessens et al.[16]) Given a t-norm T and a fuzzy relation Ron a finite set X. The T-transitive closure of R is the smallest T-indistinguishability$\overline{R_T}$ on X satisfying$R \le \overline{R_T}$.

Definition 7.(max-*T composition*) Let *R* and *S*be two fuzzy relations on a finite set *X* and *T*be a *t*-norm. The max-*T* composition of *R* and *S* is the fuzzy relation $R\circ_T S$on *X*defined for all$x, z \in X$ by

$$\mu_{R\circ_T S}(x,z) = \max_{y\in X}\{T(\mu_R(x,y),\mu_s(y,z))\}.$$

Since the max-*T* composition is associative, we can define for $n \in N$ the nth power $R_T^{(n)}$of afuzzy relation *R*:

$$R_T^{(n)} = \underbrace{R\circ_T R\circ_T \ldots R\circ_T R}_{n\ times}$$

Proposition 1. (Naessens et al.,[16] Campo et al.[5]) Let *T* be a *t*-norm and *R*be a proximity relationon a finite set *X* of cardinality *n*. Then the *T*-transitive closure of *R* is

$$\overline{R_T} = \max_{s\in\{1,2,\ldots,n-1\}} R_T^{(s)}$$

In order to make an easier notation, we will replace $\mu_R(x,y)$ by *R*(*x*,*y*) later in the paper.

Proposition 2. Let *R* be a fuzzy relation on *X* and$\overline{R_{t_i}}$ is the t_i-transitive closure of *R*, *i* = 1, 2,3.Then $R\le \overline{R_{t_1}} \le \overline{R_{t_2}} \le \overline{R_{t_3}}$.

Proof. Let Ω_{t_i}be the set of t_i-transitive relations which are greater than or equal to *R*, *i* = 1, 2, 3.
Because$\overline{R_{t_i}}$be the t_i-transitive closure of *R*, then $\overline{R_{t_i}}(x,y) = inf_{Q\in\Omega_{t_i}}\{Q(x,y)\}\ \forall x,y\in X,\ i = 1,2,3$.If $A\in \Omega_{t_3}$, then $A \ge R$ and $A(x,z)\ge t_3(A(x,y),A(y,z))$for all $x,y,z \in X$.By $t_3 \ge t_2 \ge t_1$, onegets $A(x,z)\ge t_2(A(x,y),A(y,z))$. That is, $A\in \Omega_{t_2}$. This implies that $\Omega_{t_3} \subseteq \Omega_{t_2}$.

Similarly, itcan be verified that$\Omega_{t_2} \subseteq \Omega_{t_1}$. According to$\Omega_{t_3} \subseteq \Omega_{t_2} \subseteq \Omega_{t_1}$, we have $R(x,y)\le inf_{Q\in\Omega_{t_1}}\{Q(x,y)\} \le inf_{U\in\Omega_{t_2}}\{U(x,y)\} \le inf_{V\in\Omega_{t_3}}\{V(x,y)\}\ \ \forall x,y \in X$.Thus, we prove that $R\le \overline{R_{t_1}} \le \overline{R_{t_2}} \le \overline{R_{t_3}}$.□

Example 1. Let *X* be a set of cardinality 4 and *R* be the proximity relation on *X* given by

$$R = \begin{bmatrix} 1 & 0.8 & 0.2 & 0.4 \\ 0.8 & 1 & 0.7 & 0.1 \\ 0.2 & 0.7 & 1 & 0.6 \\ 0.4 & 0.1 & 0.6 & 1 \end{bmatrix}.$$

After max-*T* compositions, we get the following $\overline{R_{t_i}}$

$$\overline{R_{t_1}} = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.4 \\ 0.8 & 1 & 0.7 & 0.3 \\ 0.5 & 0.7 & 1 & 0.6 \\ 0.4 & 0.3 & 0.6 & 1 \end{bmatrix}, \overline{R_{t_2}} = \begin{bmatrix} 1 & 0.8 & 0.56 & 0.4 \\ 0.8 & 1 & 0.7 & 0.42 \\ 0.56 & 0.7 & 1 & 0.6 \\ 0.4 & 0.42 & 0.6 & 1 \end{bmatrix}, \overline{R_{t_3}} = \begin{bmatrix} 1 & 0.8 & 0.7 & 0.6 \\ 0.8 & 1 & 0.7 & 0.6 \\ 0.7 & 0.7 & 1 & 0.6 \\ 0.6 & 0.6 & 0.6 & 1 \end{bmatrix}$$

Thus, we can obtain that $R\le \overline{R_{t_1}} \le \overline{R_{t_2}} \le \overline{R_{t_3}}$.

Definition 8. (TrillasandValverde[22],Garmendia et al.[6]) Theresiduation (or quasi-inverse) $\vec{T}$of a *t*-norm *T* is a function from [0, 1] ×[0, 1] into [0,1] defined as follows:

$$\vec{T}(x|y) = \sup\{\alpha\in[0,1]|\ T(x,\alpha)\le y\}.$$

Proposition 3. Let $T_1$and T_2 be any two *t*-norms. If $T_1 \le T_2$then$\overrightarrow{T_2} \le \overrightarrow{T_1}$.

Proof. Assume that$\overrightarrow{T_2}(x|y) = k$. We then have $T_2(x,k) \le y$. Because of$T_1 \le T_2$, we can get$T_1(x,k) \le T_2(x,k) \le y$. Then $k \le \overrightarrow{T_1}(x|y) = sup\{\alpha\in[0,1]|\ T_1(x,\alpha)\le y\}$, i.e. $\overrightarrow{T_2}(x|y) \le \overrightarrow{T_1}(x|y)$.

Theorem 1. (*Representation Theorem*, Valverde[23]) Let *E* be a fuzzy relation on a set *X* and let*T* be a *t*-norm. Then *E* is a *T*-indistinguishability on the set *X* if and only if there is a family$\{h_j\}_{j\in J}$offuzzy subsets of *X* such that for all $x,y \in X$

$$E(x,y) = inf_{j\in J}\vec{T}\left(h_j(x)\vee h_j(y)\middle|\ h_j(x)\wedge h_j(y)\right)$$

where

$$h_j(x) \vee h_j(y) = \max\{h_j(x), h_j(y)\}, h_j(x) \wedge h_j(y) = \min\{h_j(x), h_j(y)\}.$$

In general, $\{h_j\}_{j\in J}$ is called a generating family of E[23].In particular, given a proximity relation$R$ on X, we can use the representation theorem to build the T-indistinguishability$\underline{R_T}$generated bythe set of the columns of R(i.e. the fuzzy subsets$R(x, \cdot),\ x \in X$)[8].

Example 2. Given a proximity relation R on X, let $R(\cdot, j)$be the$j$th column of R for $j \in X$

1. If T is the t_1-norm, then$\underline{R_{t_1}}(x,y) = inf_{j\in X}\{1 - |R(x,j) - R(y,j)|\}$for all $x, y \in X$.
2. If T is the t_2-norm, then $\underline{R_{t_2}}(x,y) = inf_{j\in X}\{R(x,j)/R(y,j), R(y,j)/R(x,j)\}$for all $x, y \in X$, where$z/0$ is assumed to be 1.
3. If T is the t_3-norm, then $\underline{R_{t_3}}(x,y) = inf_{j\in X}\{R(x,j) \leftrightarrow R(y,j)\}$for all $x, y \in X$, where

$$a \leftrightarrow b = \begin{cases} \min\{a,b\} \text{if } a \neq b, \\ 1 \quad \text{otherwise.} \end{cases}$$

Proposition 4.(Valverde[23]) Let R be a proximity relation on X and for any t-norm T, $\underline{R_T} \leq R$.

Proposition 5. Let R be a proximity relation on X, then

$$\underline{R_{t_3}} \leq \underline{R_{t_2}} \leq \underline{R_{t_1}} \leq R \leq \overline{R_{t_1}} \leq \overline{R_{t_2}} \leq \overline{R_{t_3}}$$

It is trivial by Proposition 2, Proposition 3 and Proposition4.

Example 3. Let us consider the same proximity relation as example 1

$$R = \begin{bmatrix} 1 & 0.8 & 0.2 & 0.4 \\ 0.8 & 1 & 0.7 & 0.1 \\ 0.2 & 0.7 & 1 & 0.6 \\ 0.4 & 0.1 & 0.6 & 1 \end{bmatrix}.$$

After using the representation theorem, we get

$$\underline{R_{t_3}} = \begin{bmatrix} 1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 1 \end{bmatrix}, \underline{R_{t_2}} = \begin{bmatrix} 1 & 0.25 & 0.2 & 0.125 \\ 0.25 & 1 & 0.1667 & 0.1 \\ 0.2 & 0.1667 & 1 & 0.1429 \\ 0.125 & 0.1 & 0.1429 & 1 \end{bmatrix}, \underline{R_{t_1}} = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.3 \\ 0.5 & 1 & 0.4 & 0.1 \\ 0.2 & 0.4 & 1 & 0.4 \\ 0.3 & 0.1 & 0.4 & 1 \end{bmatrix}.$$

Thus, by combining the results of Examples 1 and 3, we have that

$$\underline{R_{t_3}} \leq \underline{R_{t_2}} \leq \underline{R_{t_1}} \leq R \leq \overline{R_{t_1}} \leq \overline{R_{t_2}} \leq \overline{R_{t_3}}.$$

Definition 9. (Aczél[1],Klement[13]) Given a continuous monotonic map $f : [0,1] \longrightarrow [-\infty, \infty]$ and p, $q$positive values with $p+q$= 1, the weighted quasi-arithmetic mean $m_f^{p,q}$ generated by f and weights$p$ and $q$is defined for all $x,y \in [0,1]$by

$$m_f^{p,q}(x,y) = f^{-1}(p \cdot f(x) + q \cdot f(y))$$

where$m_f^{p,q}$ is continuous if and only if $Range(f) \neq [-\infty, \infty]$.

Definition 10. (*Archimedean t-norm*) Let$T: [0,1] \times [0,1] \to [0,1]$is an Archimedean t-norm iffthere exists an additive generator f, which is a decreasing bijection$f: [0,1] \to [0,B]$ $(B \in (0,\infty])$such that

$$T(x,y) = \begin{cases} f^{-1}\left(f(x) + f(y)\right) \text{ if } f(x) + f(y) \leq B, \\ 0 \quad \text{otherwise.} \end{cases}$$

The t_1-norm and t_2-norm are both Archimedean t-norms, and

1. If T is the t_1-norm, then$f(x) = 1 - x$.
2. If T is the t_2-norm, then $f(x) = -\ln x$.

Proposition 6. (Garmendiaand Recasens[8]) Let T be an Archimedean t-norm with additive generator $f, p \in [0,1]$and let A and B be two T-indistinguishabilities on X of cardinality n. The weightedquasi-arithmetic mean $m_f^{p,(1-p)}$with weights p and 1 - p of A and B is a T-indistinguishability.

Proof. Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$. Because *f* is an additive generator with respect to *T* inwhich *A* and *B* are two *T*-indistinguishabilities, *f* should be a decreasing function with *f*(1) = 0and $f^{-1}(0) = 1$. Furthermore, by the property of *T*-indistinguishability, we have that

(1*) $a_{ii} = b_{ii} = 1, \forall i = 1,2,\dots,n$;

(2*) $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}, \forall i \neq j$;

(3*) Since $a_{ij} \geq f^{-1}(f(a_{ik}) + f(a_{kj}))$ and $b_{ij} \geq f^{-1}(f(b_{ik}) + f(b_{kj})), \forall i \neq j \neq k$;and *f* is decreasing,we have that

$f(a_{ij}) \leq f(a_{ik}) + f(a_{kj})$ and $f(b_{ij}) \leq f(b_{ik}) + f(b_{kj}), \forall i \neq j \neq k$.

We next prove the reflexivity, symmetry and *T*-transitivity properties as follows.

(reflexivity):

$$m_f^{p,(1-p)}(a_{ii}, b_{ii}) = f^{-1}(pf(a_{ii}) + (1-p)f(b_{ii}))$$
$$= f^{-1}(pf(1) + (1-p)f(1)) = f^{-1}(0) = 1.$$

(symmetry):

$$m_f^{p,(1-p)}(a_{ij}, b_{ij}) = f^{-1}(pf(a_{ij}) + (1-p)f(b_{ij}))$$
$$= f^{-1}\left(pf(a_{ji}) + (1-p)f(b_{ji})\right) = m_f^{p,(1-p)}(a_{ji}, b_{ji}).$$

(*T*-transitivity):

$$T(m_f^{p,(1-p)}(a_{ik}, b_{ik}), m_f^{p,(1-p)}(a_{kj}, b_{kj}))$$
$$= T(f^{-1}(pf(a_{ik}) + (1-p)f(b_{ik})), f^{-1}\left(pf(a_{kj}) + (1-p)f(b_{kj})\right))$$
$$= f^{-1}(f(f^{-1}(pf(a_{ik}) + (1-p)f(b_{ik}))) + f(f^{-1}(pf(a_{kj}) + (1-p)f(b_{kj}))))$$
$$= f^{-1}\left(pf(a_{ik}) + (1-p)f(b_{ik}) + pf(a_{kj}) + (1-p)f(b_{kj})\right)$$
$$= f^{-1}\left(p\left(f(a_{ik}) + f(a_{kj})\right) + (1-p)\left(f(b_{ik}) + f(b_{kj})\right)\right)$$
$$\leq f^{-1}\left(pf(a_{ij}) + (1-p)f(b_{ij})\right) \text{(by } f \text{ is decreasing and 3*)}$$
$$= m_f^{p,1-p}(a_{ij}, b_{ij}) \quad \square$$

Given an Archimedean *t*-norm *T* with additive generator *f*, and a proximity relation *R* on aset *X*. We can calculate the *T*-transitive closure $\overline{R_T}$ and the *T*-indistinguishability $\underline{R_T}$ based onmax-*T* composition and representation theorem, respectively. Because $\overline{R_T}$ and $\underline{R_T}$ are both *T*-indistinguishabilities, according to proposition 6, one can construct a new T-indistinguishability *E* between $\overline{R_T}$ and $\underline{R_T}$ by using the weighted quasi-arithmetic mean $m_f^{p,1-p}$ with weights *p* and 1-*p* for $\overline{R_T}$ and $\underline{R_T}$. In particular, we have that $E = \overline{R_T}$ as *p* = 1 and $E = \underline{R_T}$ as *p* = 0.

Definition 11. Let *T* be an Archimedean *t*-norm with additive generator *f*, $x \in [0,1]$ and $r \in R^+$ Then $x_T^{((r))} = f^{-1}(r \cdot f(x))$.

Proposition 7. (Garmendiaand Recasens[8]) Let *T* be an Archimedean *t*-norm with additive generator *f*, $E = [e_{ij}]_{n\times n}$ a *T*-indistinguishability on a finite set *X* of cardinality *n* and *r* > 0. Then $E^{((r))} = \left[e_T^{((r))}\right]_{n\times n} = \left[f^{-1}(r \cdot f(e_{ij}))\right]_{n\times n}$ is a *T*-indistinguishability.

Proof. The reflexivity, symmetry and *T*-transitivity of $E^{((r))}$ are verified as follows.

(reflexivity): $f^{-1}(r \cdot f(e_{ii})) = f^{-1}(r \cdot f(1)) = f^{-1}(r \cdot 0) = f^{-1}(0) = 1$

(symmetry): $f^{-1}\left(r \cdot f(e_{ij})\right) = f^{-1}\left(r \cdot f(e_{ji})\right)$ (by $e_{ij} = e_{ji}$)

(*T*-transitivity): $T(f^{-1}(r \cdot f(e_{ik})), f^{-1}(r \cdot f(e_{kj})))$

$= f^{-1}(r \cdot f(e_{ik}) + r \cdot f(e_{kj}))$

$= f^{-1}(r \cdot (f(e_{ik}) + f(e_{kj})))$

$\leq f^{-1}(r \cdot f(e_{ij}))$ (by *f* is decreasing and $f(e_{ij}) \leq f(e_{ik}) + f(e_{kj})$) $\square$

Thanks to this last proposition, by $E^{((r))} \leq E^{((s))}$for $r \geq s$, it allows us to increase or decreasethe values of a *T*-indistinguishability*E*.

3. Obtaining a new t_1-indistinguishability closer to *R* than$\overline{R_{t_1}}$ or $\underline{R_{t_1}}$

In many applications such as fuzzy clustering, knowledge learning reasons and decision making,transitivity of a proximity relation *R* with respect to a *t*-norm *T* is required. In these cases, it isnecessary to replace *R* by a new fuzzy relation *E* also satisfying transitivity; such relations are called*T*-indistinguishabilities. Of course, it is desirable that *E* is as close as possible to *R*. There aredifferent ways to calculate the closeness of two fuzzy relations, many of them related to some metric.

In this paper, the Euclidean distance will be used as a method to compare the closeness of fuzzyrelations.

Definition 12. Let $R = [r_{ij}]_{n\times n}$ and $S = [s_{ij}]_{n\times n}$ be two fuzzy relations on a finite set *X* ofcardinality *n*. The Euclidean distance *D* between *R* and *S* is

$$D(R,S)= \left(\sum_{1\leq i,j\leq n}\left(r_{ij} - s_{ij}\right)^2\right)^{\frac{1}{2}}$$

Corollary 1. Let $R = [r_{ij}]_{n\times n}$ be a proximity relation on a finite set *X* of cardinality *n*, *T* bean Archimedean *t*-norm with additive generator *f*, $\overline{R_T} = [\overline{r_{ij}}]_{n\times n}$ its transitive closure, $\underline{R_T} = [\underline{r_{ij}}]_{n\times n}$ the*T*-indistinguishability obtained from *R* with the Representation Theorem, $p\in[0,1]$and $m_f^{p,(1-p)}(\overline{R_T},\ \underline{R_T})$the *T*-indistinguishability quasi-arithmetic mean of $\overline{R_T}$ and $\underline{R_T}$with weights *p*and 1- *p*. Then

$$D(R,m_f^{p,1-p}(\overline{R_T}\ ,\underline{R_T})) =\left(\sum_{1\leq i,j\leq n}\left(f^{-1}\left(p\cdot f\left(\overline{r_{ij}}\right) + (1-p)\cdot f(\underline{r_{ij}})\right) - r_{ij}\right)^2\right)^{\frac{1}{2}}.$$

Corollary 2. Let $R = [r_{ij}]_{n\times n}$ be a proximity relation on a finite set *X* of cardinality *n*,*T*be anArchimedean *t*-norm with additive generator *f*, $E = [e_{ij}]_{n\times n}$a *T*-indistinguishability on *X*, $p > 0$and $E^{((p))} = \left[e_T^{((p))}\right]_{n\times n} = \left[f^{-1}\left(p\cdot f\left(e_{ij}\right)\right)\right]_{n\times n}$ a T-indistinguishability on *X*. Then

$$D\left(R,E^{((p))}\right) = \left(\sum_{1\leq i,j\leq n}\left(f^{-1}(p\cdot f(e_{ij})) - r_{ij}\right)^2\right)^{\frac{1}{2}}.$$

Given a proximity relation *R* on *X*, there are different t-norms can be chosen to create a *t*-indistinguishability*E* in connection with *R*. For the sake of less distortion and less restriction,the t_1-norm is recommended as the first choice among *t*-norms. This is because we had shownthat $\underline{R_{t_3}} \leq \underline{R_{t_2}} \leq \underline{R_{t_1}} \leq R \leq \overline{R_{t_1}} \leq \overline{R_{t_2}} \leq \overline{R_{t_3}}$ and $R_{t_3}(x) \subseteq R_{t_2}(x) \subseteq R_{t_1}(x)$given former.Under consideration, hence we will focus on the t_1-norm in later discussion. Besides, Garmendia and Recasens[8]also presented methods to find a new t_1-indistinguishability closer to *R* than $\overline{R_{t_1}}$ or $\underline{R_{t_1}}$,which is introduced as follows:

Proposition 8. (Garmendiaand Recasens[8]) Let *T* be the t_1-norm and *R* a proximity relation on a finite set *X* of cardinality *n*. The closest $m_f^{p,(1-p)}(\overline{R_{t_1}}\ ,\underline{R_{t_1}})$ to *R* is attained for

$$p = \frac{\sum_{1\leq i<j\leq n}\ (\overline{r_{ij}} - \underline{r_{ij}})(r_{ij} - \underline{r_{ij}})}{\sum_{1\leq i<j\leq n}\ (\overline{r_{ij}} - \underline{r_{ij}})^2}.$$

Proof. We are looking for the value of *p* that minimize the function$g(p) = D(R,\ m_f^{p,(1-p)}(\overline{R_{t_1}}\ ,\underline{R_{t_1}}))$.

Since *T* be the t_1-norm, $f^{-1}(p\cdot f\left(\overline{r_{ij}}\right) + (1-p)f(\underline{r_{ij}})) = p(\overline{r_{ij}} - \underline{r_{ij}}) + \underline{r_{ij}}$, so

$$g(p) = \left(\sum_{1\leq i,j\leq n}\left(f^{-1}(p\cdot f\left(\overline{r_{ij}}\right) + (1-p)\cdot f(\underline{r_{ij}})) - r_{ij}\right)^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{1\leq i,j\leq n}\left(p(\overline{r_{ij}} - \underline{r_{ij}}) + \underline{r_{ij}} - r_{ij}\right)^2\right)^{\frac{1}{2}}.$$

Due to R, $\overline{R_{t_1}}$ and $\underline{R_{t_1}}$ all have symmetry and reflexivity, it is equivalent to minimize
$g(p) = \sum_{1\le i<j\le n}(p(\overline{r_{ij}} - \underline{r_{ij}}) + \underline{r_{ij}} - r_{ij})^2$.
$g'(p) = \sum_{1\le i<j\le n} 2(p(\overline{r_{ij}} - \underline{r_{ij}}) + \underline{r_{ij}} - r_{ij})(\overline{r_{ij}} - \underline{r_{ij}}) = 0$ implies

$$p = \frac{\sum_{1\le i<j\le n} (\overline{r_{ij}} - \underline{r_{ij}})(r_{ij} - \underline{r_{ij}})}{\sum_{1\le i<j\le n} (\overline{r_{ij}} - \underline{r_{ij}})^2}. \qquad \square$$

Proposition 9. Let Tbe the t_1-norm and R a proximity relation on a finite set X of cardinality n.
If E is a t_1-indistinguishability on X and p> 0, then the closest $E^{((p))}$ to R is attained for

$$p = \frac{\sum_{1\le i<j\le n} (e_{ij} - 1)(r_{ij} - 1)}{\sum_{1\le i<j\le n} (e_{ij} - 1)^2}$$

Proof.We are looking for the value of p that minimize the function $g(p) = D(R,E^{((p))})$. Since T bethe t_1-norm,$f^{-1}\left(p \cdot f(e_{ij})\right) - r_{ij} = p(e_{ij} - 1) + (1 - r_{ij})$, so
$g(p) = \left(\sum_{1\le i,j\le n}(f^{-1}(p \cdot f(e_{ij})) - r_{ij})^2\right)^{\frac{1}{2}}$
$= \left(\sum_{1\le i,j\le n}\left(p(e_{ij} - 1) + (1 - r_{ij})\right)^2\right)^{\frac{1}{2}}$.
Due to R and $E^{((p))}$ both have symmetry and reflexivity, it is equivalent to minimize
$g(p) = \sum_{1\le i<j\le n}\left(p(e_{ij} - 1) + (1 - r_{ij})\right)^2$
$g'(p) = \sum_{1\le i<j\le n} 2(p(e_{ij} - 1) + (1 - r_{ij}))(e_{ij} - 1) = 0$implies

$$p = \frac{\sum_{1\le i<j\le n} (e_{ij} - 1)(r_{ij} - 1)}{\sum_{1\le i<j\le n} (e_{ij} - 1)^2}.\square$$

4. Example and comparisons of closeness using simulative results
In this section, we first illustrate proposition 8 and 9 with a simple example.

Example 4. Let X be a set of cardinality 4 and R be the proximity relation on X given by

$$R = \begin{bmatrix} 1 & 0.9 & 0.3 & 0.4 \\ 0.9 & 1 & 0.7 & 0.2 \\ 0.3 & 0.7 & 1 & 0.6 \\ 0.4 & 0.2 & 0.6 & 1 \end{bmatrix}.$$

Then, for T the t_1-norm, after using the max-T composition and the representation theorem, we get two t_1-indistinguishabilities as follow:

$$\overline{R_{t_1}} = \begin{bmatrix} 1 & 0.9 & 0.6 & 0.4 \\ 0.9 & 1 & 0.7 & 0.3 \\ 0.6 & 0.7 & 1 & 0.6 \\ 0.4 & 0.3 & 0.6 & 1 \end{bmatrix}, \underline{R_{t_1}} = \begin{bmatrix} 1 & 0.6 & 0.3 & 0.3 \\ 0.6 & 1 & 0.4 & 0.2 \\ 0.3 & 0.4 & 1 & 0.5 \\ 0.3 & 0.2 & 0.5 & 1 \end{bmatrix}.$$

By proposition 8, the closest t_1-indistinguishability to R of the type $m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}})$is attained for$p$ = 0.6667. A good t_1-approximation of R is then

$$m_f^{0.6667,0.3333}(\overline{R_{t_1}}, \underline{R_{t_1}}) = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.3667 \\ 0.8 & 1 & 0.6 & 0.2667 \\ 0.5 & 0.6 & 1 & 0.5667 \\ 0.3667 & 0.2667 & 0.5667 & 1 \end{bmatrix}.$$

By proposition 9, $D(R,\overline{R_{t_1}}^{((p))})$ attains its minimum for p = 1.1496 and $D(R,\underline{R_{t_1}}^{((p))})$ for p = 0.8243.
Good t_1-approximations of $R$are therefore

$$\overline{R_{t_1}}^{((1.1496))} = \begin{bmatrix} 1 & 0.8850 & 0.5402 & 0.3102 \\ 0.8850 & 1 & 0.6551 & 0.1953 \\ 0.5402 & 0.6551 & 1 & 0.5402 \\ 0.3102 & 0.1953 & 0.5402 & 1 \end{bmatrix},$$

and

$$\underline{R_{t_1}}^{((0.8243))} = \begin{bmatrix} 1 & 0.6703 & 0.4230 & 0.4230 \\ 0.6703 & 1 & 0.5054 & 0.3406 \\ 0.4230 & 0.5054 & 1 & 0.5879 \\ 0.4230 & 0.3406 & 0.5879 & 1 \end{bmatrix}.$$

The Euclidean distance between R and $\overline{R_{t_1}}$ is 0.4472, between R and $\underline{R_{t_1}}$ is 0.6325, between $R$and $m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}})$with p = 0.6667 is 0.3651, between R and $\overline{R_{t_1}}^{((1.1496))}$is 0.3784 and between $R$and $\underline{R_{t_1}}^{((0.8243))}$ is 0.5024. These results demonstrate the superiority and usefulness of proposition 8or proposition 9 for obtaining good approximations of a proximity relation by t_1-transitive ones.

To achieve a general approach, in the following example, we consider proximity relations R's withdimensions 10, 20, 50, 80, 100 and, for each dimension, 100 proximity relations R's are chosen randomly. With each R, we compute three Euclidean distances: $D(R, m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}}))$,$D(R, \overline{R_{t_1}}^{((p))})$and $D(R, \underline{R_{t_1}}^{((p))})$which are defined as treatment 1, treatment 2 and treatment 3, respectively. Byapplying one-way ANOVA, the null hypothesis that these three treatments are drawn from populations with the same mean values is tested. The results are listed in Table 1 and suggest thattreatment 1 is statistically shortest in all three Euclidean distances with respect to differentdimensions. Therefore, we proposed that the weighted quasi-arithmetic mean $m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}})$isworth an alternative among t_1-indistinguishabilities.

Table 1: ANOVA table for different dimensions

Dimension	Source	SS	DF	MS	F value
n=10	Treatments	4.128	2	2.064	22.145**
	Error	27.347	297	0.092	1<2, 1<3†
	Total	31.475	299		
n=20	Treatments	16.491	2	8.245	70.353**
	Error	34.808	297	0.117	1<3<2†
	Total	51.299	299		
n=50	Treatments	240.455	2	120.228	667.427**
	Error	53.500	297	0.180	1<3<2†
	Total	293.956	299		
n=80	Treatments	616.333	2	308.167	1966.398**
	Error	46.545	297	0.157	1<2, 3<2†
	Total	662.878	299		
n=100	Treatments	961.842	2	480.921	2484.922**
	Error	57.480	297	0.194	1<2, 3<2†
	Total	1019.322	299		

**p-value<0.01

†Sheffe's method

5. Conclusions

Transitivity of a proximity relation R with respect to a t-norm T is required in many applications.

Hence, in case of R without transitivity, it is necessary to replace R by a new fuzzy relation E withtransitivity which is called T-indistinguishability. There are plenty of methods to have such E, forexample, $\underline{R_T}$ can be derived by using the representation theorem or the T-transitive closure $\overline{R_T}$bythe max-T composition. Of course, different t-norm T generates different T-indistinguishability$E$and we expect E to be as close as possible to the original R. Under consideration, the t_1-normis recommend to be the first choice among all t-norms since we have shown$\underline{R_{t_3}} \leq \underline{R_{t_2}} \leq \underline{R_{t_1}} \leq R \leq \overline{R_{t_1}} \leq \overline{R_{t_2}} \leq \overline{R_{t_3}}$. However, Garmendia and Recasens[8] also presented that the followingthree t_1-indistinguishabilities $m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}})$, $\overline{R_{t_1}}^{((p))}$and $\underline{R_{t_1}}^{((p))}$are closer to R than $\overline{R_{t_1}}$ or $\underline{R_{t_1}}$according to their Euclidean distance. Moreover, as we demonstrate in the previous example, theweighted quasi-arithmetic mean$m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}})$with weights p and $1 - p$ for $\overline{R_{t_1}}$ and $\underline{R_{t_1}}$possessthe shortest average Euclidean distance in comparison with those of $\overline{R_{t_1}}^{((p))}$ and $\underline{R_{t_1}}^{((p))}$. Therefore,it suggests that $m_f^{p,(1-p)}(\overline{R_{t_1}}, \underline{R_{t_1}})$would be a better choice of E, statistically.

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