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ANALYTIC FUNCTIONS OF THE THREE-DIMENSIONAL VARIABLE FROM C_3

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Abstract

Annotation. At this scientific work were considered associated-commutative three-dimensional numbers from C_3 .

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This paper is dedicated to the three-dimensional numbers, i.e. associative-commutative hypercomplex numbers as represented below [1]:

$$X = x_1 + jx_2 + kx_3, \quad (1)$$

where the base $(1, j, k)$ is matched by the algebra with the multiplication table of the base units

$$j^2 = k, \quad k^2 = j \quad \text{и} \quad j \cdot k = k \cdot j = 1. \quad (2)$$

In matrix notation they become

$$1 \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad j \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad k \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3)$$

then any number in the form (1) can always be written in matrix form, and conversely

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{pmatrix}. \quad (4)$$

We shall denote the associative-commutative hypercomplex numbers followed the law of multiplication of the base units (2) by C_3 . The algebra in C_3 is an algebra which contains divisors of zero, namely, there are numbers for which reciprocals are not available if the reciprocal is defined as:

$$X \cdot X_{rec} = 1. \quad (5)$$

Hence

$$X_{rec} = \frac{x_1^2 - x_2x_3}{\Delta} + j \frac{x_3^2 - x_1x_2}{\Delta} + k \frac{x_2^2 - x_1x_3}{\Delta}, \quad (6)$$

where

$$\Delta = \Delta(x_1, x_2, x_3) = \det X = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3. \quad (7)$$

For three-dimensional numbers which are not divisors of zero we obtain

$$\frac{1}{X} = \frac{1}{X X^*} = \frac{X^*}{\|X\|^3} = \frac{X^*}{\Delta} = \frac{(x_1^2 - x_2x_3) + (x_3^2 - x_1x_2)j + (x_2^2 - x_1x_3)k}{\Delta}, \quad (8)$$

where $X^* = (x_1^2 - x_2x_3) + (x_3^2 - x_1x_2)j + (x_2^2 - x_1x_3)k$ is an algebraic conjugation from C_3 , and $\|X\| = \sqrt[3]{\Delta}$ is a modulus of X (or norm of X).

Using the properties of the determinants, we expand the determinant of X along a row:

$$\Delta = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = \frac{1}{2}(x_1 + x_2 + x_3)[(x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2]. \quad (9)$$

Then a subset of zero divisors of C_3 consists of following subsets

$$x_1 + x_2 + x_3 = 0 \text{ and } \begin{cases} x_1 - x_2 = 0 \\ x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \tag{10}$$

Therefore, we regard in what follows

$$X = x_1 + x_2 + x_3 \neq 0 \text{ and } X = x_1(1 + j + k) \neq 0 \tag{11}$$

Since the arithmetic average of three numbers is always greater than the geometric average, or equals to zero, we have $\frac{x_1^3 + x_2^3 + x_3^3}{3} \geq \sqrt[3]{x_1^3 x_2^3 x_3^3} = x_1 x_2 x_3$.

Thus a cube of the norm is available, and equals to $\|X\| = \sqrt[3]{\Delta(x_1, x_2, x_3)} \geq 0$. The cube is positive if x_i coordinates $x_i > 0$.

The problem on the characteristic vectors and characteristic values is considered in order to find two other ‘incomplete’ conjunctions. We write a characteristic equation of the matrix (4), and after finding the characteristic values form ‘incomplete’ conjunction of $X \in C_3$ as:

$$\begin{aligned} \bar{X} &= x_1 + j\epsilon x_2 + k\epsilon^2 x_3, \\ \tilde{X} &= x_1 + j\epsilon^2 x_2 + k\epsilon x_3, \end{aligned} \tag{12}$$

where
$$\epsilon = \frac{-1+i\sqrt{3}}{2}, \quad \epsilon^2 = \frac{-1-i\sqrt{3}}{2}, \quad i^2 = -1.$$

Next, we calculate the multiplication using the multiplication table of the base units

$$X\bar{X}\tilde{X} = (x_1 + jx_2 + kx_3)[(x_1^2 - x_2x_3) + j(x_2^2 - x_1x_2) + k(x_2^2 - x_1x_3)] = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = \|X\|^3. \tag{13}$$

The last relation (13) shows that the algebraic conjunction is the result of the multiplication of two ‘incomplete’ conjunctions

$$X^* = \bar{X}\tilde{X} \tag{14}$$

and the norm of the three-dimensional number is defined as

$$\|X\| = \sqrt[3]{X\bar{X}\tilde{X}}. \tag{15}$$

If we arrange a transition from the base $(1, j, k)$ to the base S_1, S_2, S_3 , which objects are associated with basic equations:

$$\begin{aligned} S_1 &= \frac{1}{3}(1 + j + k), \\ S_2 &= \frac{1}{3}(1 + j\epsilon + k\epsilon^2), \\ S_3 &= \frac{1}{3}(1 + j\epsilon^2 + k\epsilon), \end{aligned} \tag{16}$$

then Cayley multiplication table for the base (S_1, S_2, S_3) simply will be

$$\begin{aligned} S_m^2 &= S_m, \\ S_m \cdot S_n &= 0, \quad (m, n = 1, 2, 3 \text{ and } m \neq n). \end{aligned} \tag{17}$$

Inverse relationship of the base units $(1, j, k)$ with zero divisors is expressed by next relations:

$$\begin{aligned} 1 &= S_1 + S_2 + S_3, \\ j &= S_1 + \epsilon^2 S_2 + S_3, \\ k &= S_1 + \epsilon S_2 + \epsilon^2 S_3, \end{aligned} \tag{18}$$

Let us set the base (S_1, S_2, S_3) is absolute and defined in C_3 , where the antisymmetric form defined over the field of complex numbers.

Exponent representation of the C_3 algebra number in the base $(1, j, k)$

Any number in the base $(1, j, k)$ can be represented as:

$$X = x_1 + jx_2 + kx_3$$

If $\exp(j\alpha)$ and $\exp(k\beta)$ are expanded into next equations:

$$\begin{aligned} \exp(j\alpha) &= A(\alpha) + jB(\alpha) + kC(\alpha), \\ \exp(j\beta) &= A(\beta) + jB(\beta) + kC(\beta), \end{aligned}$$

then any number from C_3 which $x_i > 0$ can be represented as

$$X = \|X\| \exp(j\alpha + k\beta) = \|X\| [J_1(\alpha, \beta) + jJ_2(\alpha, \beta) + kJ_3(\alpha, \beta)], \tag{19}$$

where

$$\begin{aligned} J_1(\alpha, \beta) &= A(\alpha)A(\beta) + B(\alpha)B(\beta) + C(\alpha)C(\beta), \\ J_2(\alpha, \beta) &= A(\alpha)C(\beta) + B(\alpha)A(\beta) + C(\alpha)B(\beta), \\ J_3(\alpha, \beta) &= A(\alpha)B(\beta) + B(\alpha)C(\beta) + C(\alpha)A(\beta). \end{aligned} \tag{20}$$

In turn

$$\begin{aligned}
 A(t) &= \frac{1}{3}e^t + \frac{2}{3}e^{-t/2}\cos\frac{\sqrt{3}}{2}t, \\
 B(t) &= \frac{1}{3}e^t + e^{-t/2}\left(-\frac{1}{3}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right), \\
 C(t) &= \frac{1}{3}e^t + e^{-t/2}\left(-\frac{1}{3}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).
 \end{aligned}
 \tag{21}$$

Here t posses value of α or β variables. Real numbers α and β by analogy with complex and double numbers will be called argument of $X \in C_3$.

If we introduce a coordinate system (x_1, x_2, x_3) in the base $(1, j, k)$ in the following manner

$$x_1 = \rho J_1(\alpha, \beta), x_2 = \rho J_2(\alpha, \beta), x_3 = \rho J_3(\alpha, \beta), \tag{22}$$

we obtain two relations

$$x_1 + x_2 + x_3 = \rho[J_1(\alpha, \beta) + J_2(\alpha, \beta) + J_3(\alpha, \beta)] = \rho e^{\alpha+\beta}, \tag{23}$$

$$x_1^3+x_2^3+x_3^3 - 3x_1x_2x_3 = \rho^3[J_1^3(\alpha, \beta) + J_2^3(\alpha, \beta) + J_3^3(\alpha, \beta)] = \rho^3 = \|X\|^3, \tag{24}$$

because of

$$J_1(\alpha, \beta) + J_2(\alpha, \beta) + J_3(\alpha, \beta) = e^{\alpha+\beta}, \tag{25}$$

$$J_1^2(\alpha, \beta) + J_2^2(\alpha, \beta) + J_3^2(\alpha, \beta) - J_1(\alpha, \beta) \cdot J_2(\alpha, \beta) - J_1(\alpha, \beta) \cdot J_3(\alpha, \beta) - J_2(\alpha, \beta) \cdot J_3(\alpha, \beta) = e^{-\alpha-\beta}, \tag{26}$$

$$J_1(\alpha, \beta) = \frac{1}{3}e^{\alpha+\beta} + \frac{2}{3}e^{-\frac{\alpha+\beta}{2}}\cos\frac{\sqrt{3}}{2}(\alpha - \beta), \tag{27}$$

$$J_2(\alpha, \beta) = \frac{1}{3}e^{\alpha+\beta} - \frac{2}{3}e^{-\frac{\alpha+\beta}{2}}\cos\left[\frac{\sqrt{3}}{2}(\alpha - \beta) + \frac{\pi}{3}\right], \tag{28}$$

$$J_3(\alpha, \beta) = \frac{1}{3}e^{\alpha+\beta} - \frac{2}{3}e^{-\frac{\alpha+\beta}{2}}\cos\left[\frac{\sqrt{3}}{2}(\alpha - \beta) - \frac{\pi}{3}\right] \tag{29}$$

and $\rho = \|X\|$ is the norm of $X \in C_3$, then the value of ρ can be naturally called a modulus of the three-dimensional number X .

The difference of ‘nodal’ parameters $(\alpha - \beta)$ defined from the next relation

$$tg\frac{\sqrt{3}}{2}(\alpha - \beta) = \sqrt{3}\frac{x_2-x_3}{x_1-x_2-x_3}, \tag{30}$$

the sum is defined from (23)

$$\alpha + \beta = Ln\frac{x_1+x_2+x_3}{\sqrt{x_1^3+x_2^3+x_3^3-3x_1x_2x_3}}. \tag{31}$$

Hence, if $\rho = \|X\| = 1$, then according to (19) we have $X = J_1(\alpha, \beta) + jJ_2(\alpha, \beta) + kJ_3(\alpha, \beta)$.

The three-dimensional number $J_1(\alpha, \beta) + jJ_2(\alpha, \beta) + kJ_3(\alpha, \beta)$ is labelled as $e^{aj+\beta k}$, in other words, the function $e^{aj+\beta k}$ for any real numbers α and β defined by analogy with Euler formula

$$e^{aj+\beta k} = J_1(\alpha, \beta) + jJ_2(\alpha, \beta) + kJ_3(\alpha, \beta), \tag{32}$$

where the base units j, k satisfy these relations $j^3 = 1, k^3 = 1$. Therefore we shall call them hyperbolic.

The function $e^{j\alpha+k\beta}$ posses usual properties of the exponential function, as if the numbers j, k were real numbers. Note the major properties:

$$e^{j\alpha_1+k\beta_1} \cdot e^{j\alpha_2+k\beta_2} = e^{j(\alpha_1+\alpha_2)+k(\beta_1+\beta_2)}, \tag{33}$$

$$\frac{e^{j\alpha_1+k\beta_1}}{e^{j\alpha_2+k\beta_2}} = e^{j(\alpha_1-\alpha_2)+k(\beta_1-\beta_2)}, \tag{34}$$

$$(e^{j\alpha+k\beta})^n = e^{jn\alpha+kn\beta}, n = 0, \pm 1, \pm 2, \dots \tag{35}$$

Combining (32) and (33) we obtain Moivre formula for the three-dimensional number from C_3 :

$$[J_1(\alpha, \beta) + jJ_2(\alpha, \beta) + kJ_3(\alpha, \beta)]^n = J_1(n\alpha, n\beta) + jJ_2(n\alpha, n\beta) + kJ_3(n\alpha, n\beta), \tag{36}$$

in particular, for $n = 2$ we obtain

$$\begin{aligned}
 J_1(2\alpha, 2\beta) &= J_1^2(\alpha, \beta) + 2J_2(\alpha, \beta)J_3(\alpha, \beta), \\
 J_2(2\alpha, 2\beta) &= J_2^2(\alpha, \beta) + 2J_1(\alpha, \beta)J_2(\alpha, \beta), \\
 J_3(2\alpha, 2\beta) &= J_2^2(\alpha, \beta) + 2J_1(\alpha, \beta)J_3(\alpha, \beta).
 \end{aligned}
 \tag{37}$$

Using equations (33) and (34) we can easily obtain multiplication and division formulas for the three-dimensional numbers from C_3 in the exponential form:

$$X_1 \cdot X_2 = \rho_1 e^{j\alpha_1+k\beta_1} \cdot \rho_2 e^{j\alpha_2+k\beta_2} = \rho_1\rho_2 e^{j(\alpha_1+\alpha_2)+k(\beta_1+\beta_2)} \tag{38}$$

$$\frac{X_1}{X_2} = \frac{\rho_1 e^{j\alpha_1+k\beta_1}}{\rho_2 e^{j\alpha_2+k\beta_2}} = \frac{\rho_1}{\rho_2} e^{j(\alpha_1-\alpha_2)+k(\beta_1-\beta_2)}. \tag{39}$$

It follows from (38) that the modulus of the multiplication of two three-dimensional numbers from C_3 equals to the multiplication of the moduli of these two numbers

$$\|X_1 \cdot X_2\| = \|X_1\| \cdot \|X_2\|, \tag{40}$$

as long as $\|e^{j(\alpha_1-\alpha_2)+k(\beta_1-\beta_2)}\| = J_1^3 + J_2^3 + J_3^3 - 3J_1J_2J_3 = 1$.

We can introduce a concept of a function on the set of three-dimensional numbers from C_3 . Under the exponential function we mean a series:

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots, \tag{41}$$

where $X = x_1 + jx_2 + kx_3$ is an arbitrary three-dimensional number. Then for the exponent of the arbitrary three-dimensional numbers next expression will be correct

$$e^X = e^{x_1}[J_1(x_2, x_3) + jJ_2(x_2, x_3) + kJ_3(x_2, x_3)]. \tag{42}$$

We can associate the function of the three-dimensional variable with a derivative in the direction similarly to their introduction in the algebra of double numbers. C_3 function analyticity means independence of its derivative in the direction $dF = F'dX$ and expressed by analogy of Cauchy-Riemann conditions for the three-dimensional variable [2]:

$$df = FdX \tag{43}$$

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3}, \quad \frac{\partial f_2}{\partial x_1} = \frac{\partial f_3}{\partial x_2} = \frac{\partial f_1}{\partial x_3}, \quad \frac{\partial f_3}{\partial x_1} = \frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_3}, \tag{44}$$

where $f = f_1 + jf_2 + kf_3$, $F = F_1 + jF_2 + kF_3$ and $dX = dx_1 + jdx_2 + kdx_3$. This set of equations represents cross-impact between partial derivatives of function components and argument components. Specifically, the requirement of the function differentiability $f(X) = f_1(x_1, x_2, x_3) + jf_2(x_1, x_2, x_3) + kf_3(x_1, x_2, x_3)$ implies additional conditions on the partial derivatives of f_1, f_2 and f_3 functions.

Theorem 1. The function $f(X) = f_1(x_1, x_2, x_3) + jf_2(x_1, x_2, x_3) + kf_3(x_1, x_2, x_3)$ is differentiable at $X = x_1 + jx_2 + kx_3$ if and only if

- 1) f_1, f_2 and f_3 functions are differentiable at (x_1, x_2, x_3) ;
- 2) Cauchy-Riemann conditions (44) are satisfied at the point (x_1, x_2, x_3) .

Note. If the function is said to be differentiable in a domain it is analytic in the domain.

The Theorem 1 suggests the following definition.

Definition. If a set of functions of the three-dimensional variable satisfies conditions (44) they are called analytic functions in C_3 .

The Theorem 2 is true for C_3 analytic functions.

Theorem 2. Suppose that $f(X) = f_1(x_1, x_2, x_3) + jf_2(x_1, x_2, x_3) + kf_3(x_1, x_2, x_3)$ is differentiable at every point of D domain. Then each component of the function

$$\Phi(X) = (1 - j) \cdot f(X) = (f_1 - f_3) + j(f_2 - f_1) + k(f_3 - f_2)$$

is a harmonic function, i.e

$$\begin{aligned} \Delta[f_1(x_1, x_2, x_3) - f_3(x_1, x_2, x_3)] &= 0, \\ \Delta[f_2(x_1, x_2, x_3) - f_1(x_1, x_2, x_3)] &= 0, \\ \Delta[f_3(x_1, x_2, x_3) - f_2(x_1, x_2, x_3)] &= 0. \end{aligned}$$

Proof of the Theorem 2 follows from the Cauchy-Riemann conditions. The Theorem 2 implies that analytic functions in C_3 contain harmonic functions and possess many properties similar to analytic functions in the complex variable theory.

Cauchy's theorem. Let a function $\Phi(X)$ be differentiable at finite simply connected region D , and let a derivative $\Phi'(X)$ be continuous in S surface. Then an integral taken over a closed curve γ in the S surface equals to zero.

$$\oint_{\gamma} \Phi(X)dX = 0. \tag{45}$$

Proof. If $f = f_1 + jf_2 + kf_3$, then the function $\Phi(X) = (1 - j) \cdot f(X)$ can be represented in the form $\Phi(X) = (f_1 - f_3) + j(f_2 - f_1) + k(f_3 - f_2)$, where $dX = dx_1 + jdx_2 + kdx_3$. Thereafter

$$\oint_{\gamma} \Phi(X)dX = R_1(x_1, x_2, x_3) + jR_2(x_1, x_2, x_3) + kR_3(x_1, x_2, x_3),$$

where

$$\begin{aligned} R_1(x_1, x_2, x_3) &= (f_1 - f_2)dx_1 + (f_3 - f_2)dx_2 + (f_2 - f_1)dx_3, \\ R_2(x_1, x_2, x_3) &= (f_2 - f_1)dx_1 + (f_1 - f_3)dx_2 + (f_3 - f_2)dx_3, \\ R_3(x_1, x_2, x_3) &= (f_3 - f_2)dx_1 + (f_2 - f_1)dx_2 + (f_1 - f_3)dx_3. \end{aligned}$$

Since the function $\Phi(X)$ has a continuous derivative in the S surface, then all first order partial derivatives of f_1, f_2 and f_3 functions are continuous in the S surface, and Cauchy-Riemann conditions are satisfied in this domain.

On account of Stock's theorem

$$R_1(x_1, x_2, x_3) = R_2(x_1, x_2, x_3) = R_3(x_1, x_2, x_3) = 0.$$

Hence $\oint_{\gamma} \Phi(X)dX = 0$, in particular,

$$\begin{aligned} R_1 &= \oint_{\gamma} (f_1 - f_3) dx_1 + (f_3 - f_2) dx_2 + (f_2 - f_1) dx_3 = \\ &= \oint_S \begin{vmatrix} n_1 & n_2 & n_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ (f_1 - f_3) & (f_3 - f_2) & (f_2 - f_1) \end{vmatrix} dS = \\ &= \iint_S \left(\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2} - \frac{\partial f_3}{\partial x_3} + \frac{\partial f_2}{\partial x_3} \right) dx_2 dx_3 + \\ &+ \iint_S \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_1} + \frac{\partial f_1}{\partial x_1} \right) dx_1 dx_3 + \\ &+ \iint_S \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_3}{\partial x_2} \right) dx_1 dx_2 \equiv 0. \end{aligned}$$

References

- [1] *G.I.Garasko*, The three-dimensional numbers which cube of the norm is a non-degenerate form. Hypercomplex numbers in geometry and physics, 1, 2004 (in Russian).
- [2] *M.A. Lavrentiev, B.V.Shabat*, Problems of Hydrodynamics and Mathematical Models, Nauka Publishing House, Moscow, 1977 (in Russian).