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RESEARCH ARTICLE

On Shape Control of a Quartic Trigonometric Bézier Curve with a Shape Parameter

Mridula Dube and Bharti Yadav

Department of mathematics and computer science, R.D. University, Jabalpur, Madhya Pradesh, India.

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Abstract

In this paper, we have constructed a quartic trigonometric Bézier curve with single shape parameter. The shape of the curve can be adjusted as desired, by simply altering the value of shape parameter, without changing the control polygon. The quartic trigonometric Bézier curve can be made close to the quartic Bézier curve or closer to the given control polygon than the quartic Bézier curve. The representation of ellipse and circle are more accurate and act by using quartic trigonometric Bézier curve. Approximation property has been discussed.

Introduction

Curve and surfaces design is important and interesting topic of Computer Aided Geometric Design (CAGD) and Computer Graphics (CG). It has been found that the parametric representation of curve and surfaces specially in polynomial form is more important and suitable for design. Computer Aided Geometric Design (CAGD) generates curve and surfaces with approximate shape with some desired shape features. There are some examples such as car bodies, airplane and wings, ship hulls, bottles and shoe insoles that require free form curves and surfaces. Therefore to fulfill these requirement, parametric representation of curves and surfaces is widely used in the field of CAGD and CG. Trigonometric polynomial is very important in different areas, such as electronics or medicine [9]. In recent years, the trigonometric spline with shape parameters has gained more interest in CAGD, in particular curve design. Han [5] constructed quadratic trigonometric polynomial curve with a shape parameter which are C^1 continuous and similar quadratic B-spine curve. Wu X.et al [13] discussed quadratic trigonometric polynomial curve with multiple shape parameters. Han, X. [6] presented piecewise quadratic trigonometric polynomial curves with C^2 continuity. Bézier technique is one of the methods of analytic representation of curves and surfaces that has won wide acceptance as a valuable tool in CAD/CAM system. In recent years trigonometric polynomial curves like those of Bézier type are considerably in discussion see [1], [3], [4], [7], [8], [10], [11], [12]. Dube, M., et al [2] presented quartic trigonometric Bézier curve with a shape parameter. Motivated by the above ideas we have discussed the quartic trigonometric Bézier curve and its shape preserving properties.

The paper is organized as follows. In section 2, the basic functions of the quartic trigonometric Bézier curve with single shape parameter are established and the properties of the basis function has been described. In section 3, quartic trigonometric Bézier curves and their properties are discussed. In section 4, By using shape parameter, shape control of the curves is studied and explained by using figures. In section 5, the representation of the ellipses and circle are presented. In section 6, Approximation property is discussed.

2. Quartic Trigonometric Bézier Basis Functions

In this section, definition and some properties of quartic trigonometric Bézier basis functions are given:

Definition 2.1: For Selected real values of λ, where $\lambda \epsilon [-1,1]$ the following five functions of t ($t \epsilon [0,1]$) are defined as quartic trigonometric Bézier basis functions with single shape parameter λ:

$$\begin{cases} b_0(t) = (1 - sin\frac{\pi}{2}t)^2(1 - \lambda sin\frac{\pi}{2}t)^2 \\ b_1(t) = sin\frac{\pi}{2}\left(1 - sin\frac{\pi}{2}t\right)\left(1 - \lambda sin\frac{\pi}{2}t\right)(1 + \lambda - \lambda sin\frac{\pi}{2}t) \\ b_2(t) = (1+\lambda)\left(sin\frac{\pi}{2}t + cos\frac{\pi}{2}t - 1\right) \\ b_3(t) = cos\frac{\pi}{2}t\left(1 - cos\frac{\pi}{2}t\right)\left(1 - \lambda cos\frac{\pi}{2}t\right)(1 + \lambda - \lambda\, cos\frac{\pi}{2}t) \\ b_4(t) = (1 - cos\frac{\pi}{2}t)^2(1 - \lambda cos\frac{\pi}{2}t)^2 \end{cases} \quad (2.1)$$

For $\lambda = 0$, the basis functions are quadratic trigonometric polynomial. For $\lambda \neq 0$, the basis functions are quartic trigonometric polynomial.

Theorem 2.1: The basis functions (2.1) have the following properties:
(a) Nonnegative: $b_i(t) \geq 0, i = 0, 1,2,3,4.$
(b) Partition of unity: $\sum_{i=0}^{4} b_i(t) = 1.$
(c) Monotonicity: For a given parameter t, $b_0(t)$and $b_4(t)$ are monotonically decreasing and $b_1(t)$ and $b_3(t)$ are monotonically increasing for the shape parameter λ.
(d) Symmetry: $b_i(t;\lambda) = b_{4-i}(1-t;\lambda)$, for i= 0,1,2,3,4.

Proof: (a) For $t \in [0,1]$ and $\lambda \in [-1,1]$, then $\left(1 - sin\frac{\pi}{2}t\right) \geq 0$, $\left(1 - \lambda\ sin\ \frac{\pi}{2}t\right) \geq 0, \left(1 - cos\frac{\pi}{2}t\right) \geq \left(1 - \lambda\ cos\frac{\pi}{2}t\right) \geq 0$, $(1 - sin\frac{\pi}{2}t)^2 \geq 0, (1 - \lambda sin\frac{\pi}{2}t)^2 \geq 0, sin\frac{\pi}{2}t \geq 0, \cos\frac{\pi}{2}t\ \geq 0, (1+\lambda)\ \geq 0.$ It is obvious that $b_i(t) \geq 0, i = 0, 1, 2, 3, 4.$

(b) $\sum_{i=0}^{4} b_i(t) = (1 - sin\frac{\pi}{2}t)^2\,(1 - \lambda sin\frac{\pi}{2}t)^2 + sin\frac{\pi}{2}t\,\left(1 - sin\frac{\pi}{2}t\right)\left(1 - \lambda sin\frac{\pi}{2}t\right)\left(1 + \lambda - \lambda sin\frac{\pi}{2}t\right) + (1+\lambda)\left(sin\frac{\pi}{2}t + cos\frac{\pi}{2}t - 1\right) + cos\frac{\pi}{2}t\,\left(1 - cos\frac{\pi}{2}t\right)\left(1 - \lambda cos\frac{\pi}{2}t\right)\left(1 + \lambda - \lambda cos\frac{\pi}{2}t\right) + (1 - cos\frac{\pi}{2}t)^2(1 - \lambda cos\frac{\pi}{2}t)^2 = 1$

The remaining cases follow abviously.
Fig.1. shows the curves of the quartic trigonometric basis function for $\lambda = 1$ (red solid) and $\lambda = -1$ (blue dashed).

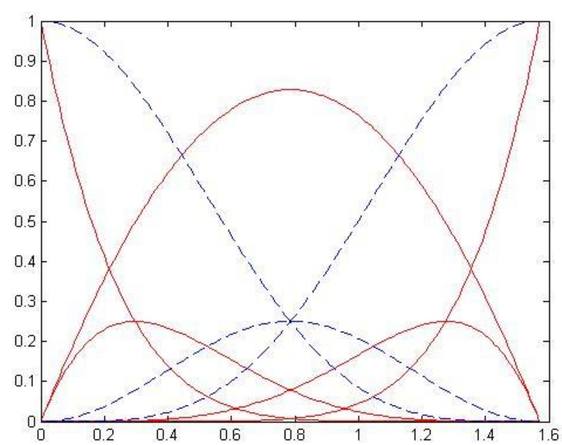

Figure 1: The quartic trigonometric basis function.

3. Quartic trigonometric Bézier curve

We construct the quartic trigonometric Bézier curve with shape parameter as follows:
Definition 3.1: Given the control points $P_i(i = 0,1,2,3,4)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$C(t) = \sum_{i=0}^{4} P_i b_i\,(t) \quad (3.1)$$

$t\ \in [0,1], \lambda \in [-1, 1]$ is called a quartic trigonometric Bézier curve with shape parameter.

The curve defined by (3.1) possesses some properties which can be obtained easily from the properties of the basis function.

Theorem 3.1: The Quartic trigonometric Bézier curve (3.1) have the following properties:

(a) End point properties:

$C(0) = P_0, C(1) = P_4$

$C'(0) = \frac{\pi}{2}(1+\lambda)[(P_1+P_2) - 2P_0]$

$C'(1) = \frac{\pi}{2}(1+\lambda)[2P_4 - (P_2+P_3)]$

$C''(0) = \left(\frac{\pi}{2}\right)^2 [(1-\lambda)P_1 - (1+\lambda)P_2 - 2((1+\lambda)^2 + 2\lambda)P_3 + 2((1+\lambda)^2 + 2\lambda)P_4]$

$C''(1) = \left(\frac{\pi}{2}\right)^2 [(1-\lambda)P_3 - (1+\lambda)P_2 - 2((1+\lambda)^2 + 2\lambda)P_1 + 2((1+\lambda)^2 + 2\lambda)P_0]$

(b) Symmetry: P_0, P_1, P_2, P_3, P_4 *and* P_4, P_3, P_2, P_1, P_0 define the same curve in different parametrizations, that is $C(t; \lambda; P_0, P_1, P_2, P_3, P_4) = C(1-t; \lambda; P_4, P_3, P_2, P_1, P_0)$,
$t \in [0,1], \lambda \in [-1,1]$.

(c) Geometric invariance: The shape of the curve (3.1) is independent of the choice of coordinates, i.e., it satisfies the following two equation:

$C(t; \lambda;\ P_0+q, P_1+q, P_2+q, P_3+q, P_4+q) = C(t; \lambda; P_0, P_1, P_2, P_3, P_4) + q$

$C(t; \lambda;\ P_0 * T, P_1 * T, P_2 * T, P_3 * T, P_4 * T) = C(t; \lambda; P_0, P_1, P_2, P_3, P_4) * T$

where q is an arbitrary vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ and T is an arbitrary $d \times d$ matrix, d = 1 or 3.

(d) Convex hull property: From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points.

4. Shape control of the quartic Trigonometric Bézier curve

The parameter λ controls the shape of the curve (3.1). In figure 2, The quartic trigonometric Bézier curve $C(t)$ gets closer to the control polygon as the value of the parameter λ increases. In figure 2, the curves are generated by setting the values of λ as $\lambda = -1$ (blue solid), $\lambda = -0.5$ (red dashed), $\lambda = 0$ (green dashed), $\lambda = 0.5$ (black dashed), $\lambda = 1$ (red soild).

In order to construct a closed quartic trigonometric Bézier curves, we can set $P_n = P_0$. The closed quartic trigonometric Bézier curves for $\lambda = -0.5, \lambda = 0, \lambda = 0.5, \lambda = 1$ are shows in figure 3: (a), (b), (c) and (d) respectively.

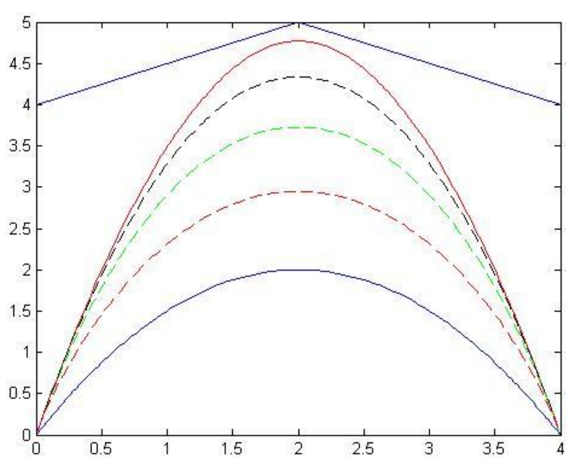

Figure 2: The effect on the shape of quartic trigonometric Bézier curves of altering the value of λ

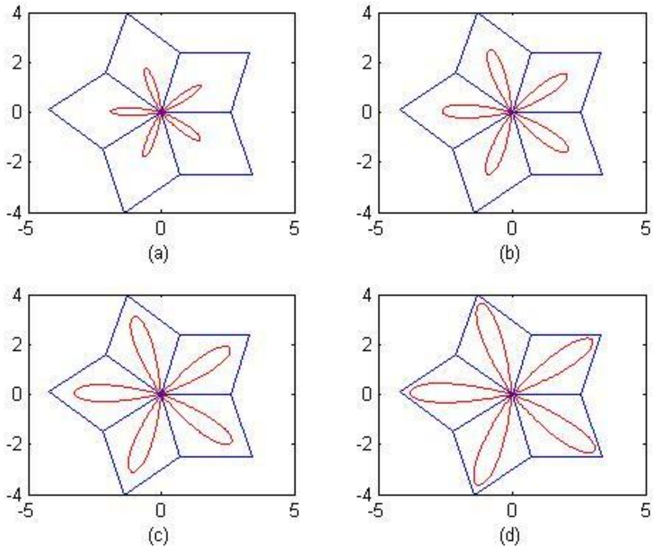

Figure 3: The quartic trigonometric Bézier curves with different values of shape parameter λ (flower)

5. The representation of Ellipse

Theorem 5.1: Let P_0, P_1, P_2, P_3 and P_4 be five control points on an ellipse with semi axes $\sqrt{2}$ and $\frac{3}{\sqrt{2}}$. By the proper selection of coordinates, their coordinates can be written in the form

$$P_0 = \begin{pmatrix} -a \\ 0 \end{pmatrix}, P_1 = \begin{pmatrix} -a \\ \frac{b}{2} \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}, P_3 = \begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix}, P_4 = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

Then the corresponding Quartic Bézier curve with the shape parameter $\lambda = 0$ and local domain $t \in [0,4]$ represents arc of an ellipse with

$$\begin{cases} x(t) = a\left(sin\frac{\pi}{2}(t) - cos\frac{\pi}{2}(t)\right) \\ y(t) = \frac{3}{2}b(sin\frac{\pi}{2}(t) + cos\frac{\pi}{2}(t) - 1) \end{cases} \tag{5.1}$$

Proof: If we take $\lambda = 0$ and

$$P_0 = \begin{pmatrix} -a \\ 0 \end{pmatrix}, P_1 = \begin{pmatrix} -a \\ \frac{b}{2} \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}, P_3 = \begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix}, P_4 = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

into (3.1), then the coordinates of quartic trigonometric Bézier curve are

$$\begin{cases} x(t) = a\left(sin\frac{\pi}{2}(t) - cos\frac{\pi}{2}(t)\right), \\ y(t) = \frac{3}{2}b(sin\frac{\pi}{2}(t) + cos\frac{\pi}{2}(t) - 1) \end{cases}$$

This gives the intrinsic equation

$$\left(\frac{x}{\sqrt{2}a}\right)^2 + \left(\frac{y + \frac{3}{2}b}{\frac{3}{\sqrt{2}b}}\right)^2 = 1.$$

It is an equation of an ellipse. Fig. 4 shows the Ellipse.

Corollary 5.2: According to theorem (5.1), if, $a = \frac{1}{\sqrt{2}}, b = \frac{\sqrt{2}}{3}$ then the corresponding Quratic trigonometric Bézier curve with the shape parameter $\lambda = 0$ and local domain $t \in [0, 4]$ represents arc of an circle. Fig. 5 shows the Circle.

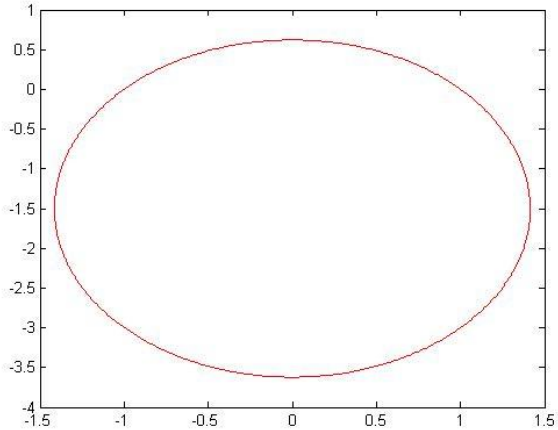

Figure 4: The representation of ellipse.

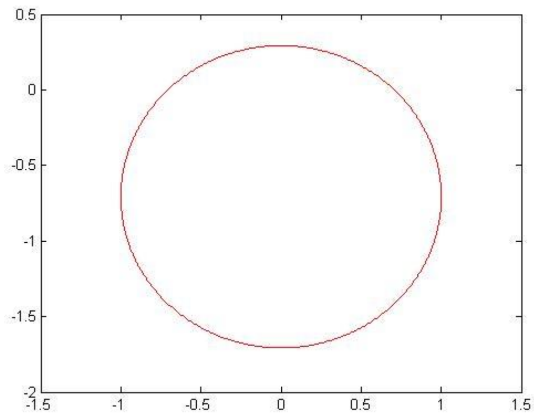

Figure 5: The representation of circle.

6. Approximability

Control polygons plays an important role in geometric modeling. It is an advantage that, if the curve being modeled tends to preserve the shape of its control polygon. Now we have shown the relation of the quartic trigonometric Bézier curves and quartic Bézier curves with same control points.

Theorem 6.1: Suppose P_0, P_1, P_2, P_3 and P_4 are not collinear; the relationship between quartic trigonometer Bézier curve $C(t)$ (3.1) and quartic Bézier curve $B(t) = \sum_{i=0}^{4} P_i \binom{4}{i} (1-t)^{4-i} (t)^i$ with the same control points $P_i (i = 0,1,2,3,4)$ are as follows:

$C\,(0) = B(0), \qquad C(1) = B(1)$

$$C\left(\frac{1}{2}\right) - P^* = B\left(\frac{1}{2}\right) - P^* \tag{6.1}$$

where $P^* = \frac{1}{8}(P_2)$ and $P_2 = \frac{1}{2}(P_1 + P_3)$ and $\lambda = \frac{\sqrt{2}-1}{2}$.

Proof: By simple computation, then

$C\,(0) = P_0 = B(0),\ C(1) = P_4 = B(1)$ and

$$B\left(\frac{1}{2}\right) = \frac{1}{16}(P_0 + 4P_1 + 6P_2 + 4P_3 + P_4)$$

$$B\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{P_1+P_3}{2} - P_2\right) = \frac{1}{16}(P_0 + 3P_1 + 8P_2 + 3P_3 + P_4) \tag{6.2}$$

According to (6.2), we have

$$C\left(\frac{1}{2}\right) = \frac{1}{4}\left[(\sqrt{2}-1)(\sqrt{2}-\lambda)\right]^2 (P_0 + P_4) + \frac{1}{4}\left[(\sqrt{2}-1)(\sqrt{2}-\lambda)\left(\sqrt{2}+\lambda\sqrt{2}-\lambda\right)\right](P_1 + P_3) + [(1+\lambda)(\sqrt{2} - 1)]\, P_2$$

If $\lambda = \frac{\sqrt{2}-1}{2}$ then,

$$C\left(\frac{1}{2}\right) = \frac{1}{16}(P_0 + 3P_1 + 8P_2 + 3P_3 + P_4)$$
$$C\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{P_1 + P_3}{2} - P_2\right)$$
$$C\left(\frac{1}{2}\right) - \frac{1}{8}(P_2) = B\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{P_1 + P_3}{2}\right)$$
$$C\left(\frac{1}{2}\right) - P^* = B\left(\frac{1}{2}\right) - P^*.$$

Then (6.1) holds.

Fig.6 shows the relationship between the quartic trigonometric Bézier curve and quartic Bézier curve. The quartic trigonometric Bézier curve is closer to quartic Bézier curve. The quartic trigonometric Bézier curve (blue solid) with parameter $\lambda = \frac{\sqrt{2}-1}{2}$ is analogous to ordinary quartic Bézier curve (black solid).

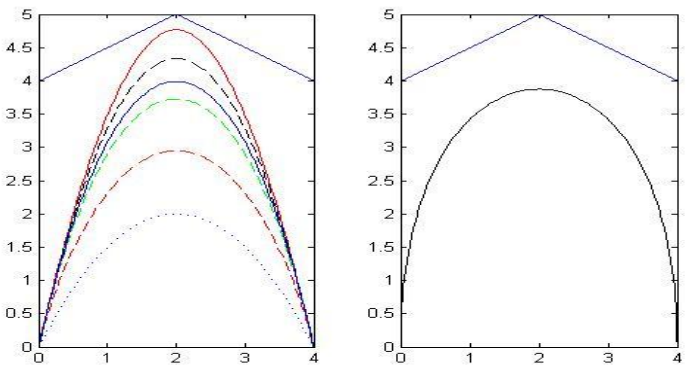

Figure 6: The relationship between the quartic trigonometric Bézier curve and quartic Bézier curve.

7. Conclusion

In this paper, we have presented the quartic trigonometric Bézier curve with a shape parameter and analysis of quartic trigonometric Bézier curve are similar to the ordinary quartic Bézier curve. Each section of the curve only refers to the five control points. We can design different shape curves by changing parameter. The curve can represent ellipse and circle by adjusting the control points and parameter value.

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