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RESEARCH ARTICLE

CONVERGENCE OF DEFICIENT C^2 QUARTIC SPLINE INTERPOLATION

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Abstract

This paper attempts in obtaining the existence, uniqueness and error bounds of quartic spline interpolation

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1. INTRODUCTION

Piecewise lower degree interpolation are widely used in the method of piecewise polynomial approximation to represent a non analytic function. Although in piecewise linear interpolation the maximum error between a function and its interpolant can be controlled by mesh spacing but such functions have corners at the joint of two linear pieces and therefore we usually require more data than higher order method to get desired accuracy. Thus, for a smooth and more efficient approximation higher degree splines are useful. (See De'Boor [1]). An asympatically precise estimate for the error of quadratic spline interpolating at the end points between the successive mesh point has been obtained by Rana [8] (see also Rana [9]). In the direction of more higher degree splines Jianzhong and Hung [10] have obtained optimal error bounds for quartic and quintic interpolatory splines (see also Howell and Verma [3], Dubey and Shukla [11]). Error inequality in polynomial interpolation and their application given by Agrawal and Wong [7], Gemeling-Meyling [6] and Hallet, Mund and Munert [5]). In the present paper we shall obtain the existence, uniqueness and error bounds for deficient quartic spline interpolation matching the given function at mesh points and second derivative at interior points with appropriate boundary conditions.

2. EXISTENCE AND UNIQUENESS

Consider a mesh P of [0, 1] given by $0=x_0<x_1<......<x_n=1$ such that $x_{i+1}-x_i=h_i$, *i=0, 1,.....n-1*. For a positive integer m, let π_m [0, 1] denote the set all algebraic polynomials of degree not greater than m. For a function defined over [0, 1], we denote by s_i the restriction of s over $[x_i, x_{i+1}]$. The class of all deficient quartic spline functions over [0, 1] with mesh P is defined by

$$S(4,P)=\{s: s\in C^2[0,1], s_i\in\pi_4[0,1] \;\; i=1,2,.....n-1\}$$

where in S_1 (4, P) denotes the class of all deficient quartic splines S (4, P) which satisfy the boundary condition

$$s'(x_0)=f'(x_0),$$

$s'(x_n) = f'(x_n)$. (2.1)

Introducing the following interpolatory conditions

$s(x_i) = f(x_i)$ i = 0, 1,.....n (2.2)

$s''(\alpha_i) = f''(\alpha_i)$ i = 1, 2,....n (2.3)

where $\alpha_i = x_i + \theta h_i$ $0 < \theta < 1.$ $\theta \neq \frac{1}{2}$

We shall prove the following

THEOREM 2.1 : Let *f'*, f'' exist, then there exists a unique deficient quartic spline in $S_1(4, P)$ which satisfies the interpolating conditions (2.2) - (2.3) and boundary conditions (2.1) if (i) $\frac{1}{4} < \theta < \frac{1}{2}$ and $h_{i-1} > h_i$ or (ii) $\frac{1}{2} < \theta < \frac{3}{4}$ and $h_{i-1} < h_i$.

PROOF : Considering a quartic polynomial P(z) on [0, 1], we can easily verify that

$$P(z) = P(0)Q_1(z) + P(1)Q_2(z) + P''(\theta)Q_3(z)$$

$$+ P'(0)Q_4(z) + P'(1)Q_5(z), \tag{2.4}$$

where

$$Q_1(z) = [1 + \{(24\theta - 36\theta^2)z^2 + (-8 + 24\theta^2)z^3$$

$$+ (6 - 12\theta)z^4\}/(2 - 12\theta + 12\theta^2)],$$

$$Q_2(z) = \{24\theta + 36\theta^2)z^2 + (8 - 24\theta^2)z^3$$

$$-(6 - 12\theta)z^4\}/(2 - 12\theta + 12\theta^2),$$

$$Q_3(z) = \left[z^2 - 2z^3 + z^4\right]/(2 - 12\theta + 12\theta^2),$$

$$Q_4(z) = [z + \{18\theta - 24\theta^2)z^2 + (-6 + 12\theta^2)z^3$$

$$+ (4 - 6\theta)z^4\}](2 - 12\theta + 12\theta^2),$$

and

$$Q_5(z) = [(6\theta - 12\theta^2)z^2 + (-2 + 12\theta^2)z^3$$

$$+ (2 - 6\theta)z^4]/(2 - 12\theta + 12\theta^2).$$

Now writing $t = (x - x_i)/h_i, 0 \leq t \leq 1,$ (2.4) may be expressed in terms of the restriction s_i of *s* as follows:

$$s_i(x) = f(x_i)Q_1(t) + f(x_{i+1})Q_2(t) + h_i^2\ f''(\alpha_i)$$

$$Q_3(t) + h_i f'(x_i)Q_4(t) + h_i f'(x_{i+1})Q_5(t) \qquad (2.5)$$

which clearly satisfies the conditions (2.1) - (2.3) and $s_i(x)$ is quartic in $[x_i, x_{i+1}]$ for i=0,......n-1. Since $s \in C^2[0,1]$, therefore applying continuity condition of second derivative of s in (2.5), we have

$$(-36\theta + 24\theta^2 + 12)h_i\, s_{i-1}^{'} + [-60\theta + 48\theta^2 + 12)$$

$$h_i - h_{i-1}(36\theta - 48\theta^2)]s_i^{'} + h_{i-1}(24\theta^2 - 12\theta)s_{i+1}^{'}$$

$$= \{(48\theta - 72\theta^2)\frac{h_{i-1}}{h_i} + (24 + 72\theta^2 - 96\theta)\frac{h_i}{h_{i-1}}\} f_i$$

$$-(24 + 96\theta + 72\theta^2)\frac{h_i}{h_{i-1}} f_{i-1} + 2h_i h_{i-1}$$

$$\{f''(\alpha_i) - f''(\alpha_{i-1})\} + (-48\theta + 72\theta^2)\frac{h_{i-1}}{h_i} f_{i+1}. \qquad (2.6)$$

It may be verified that in (2.6) the coefficient of $s_{i-1}^{'}$ is non negative for $0 \le \theta < \frac{1}{2}$ and non positive for $\frac{1}{2} < \theta \le 1$ whereas coefficient of $s_{i+1}^{'}$ is non positive for $0 \le \theta < \frac{1}{2}$ and non negative for $\frac{1}{2} < \theta \le 1.$ The coefficient of $s_i^{'}$ is non positive for $\frac{1}{4} \le \theta \le \frac{3}{4}.$ Thus considering the case (i) of the Theorem 2.1 we see that excess of the positive value of the coefficient of $s_i^{'}$ over the sum of the absolute values of coefficient of $s_{i-1}^{'}$ and s_{i+1} is $d_i(\theta) = 24\left[(-1 + 4\theta - 3\theta^2)h_i + (\theta - \theta^2)h_{i-1}\right]$ which is positive under the condition (i) of the Theorem 2.1 For case (ii) Theorem (2.1) the excess of the positive value of the coefficient of s_i^1 over the sum of the absolute values of the coefficient of $s_{i-1}^{'}$ and $s_{i+1}^{'}$ is given by

$$d_i^*(\theta) = 24\theta[(1-\theta)h_i + (2 - 3\theta)h_{i-1}]$$

which is positive under the condition (ii) of the Theorem 2.1. Thus, the coefficient matrix of the system of equation (2.6) is diagonally dominant and hence invertible. This complete the proof of Theorem 2.1.

3. ERROR BOUNDS

Following the methods of Hall and Meyer's (see also Hall & Verma [3]), in this section we shall obtain the bounds of error function $e^r(x) = f^r(x) - s^r(x)$, r = 0, 1, for the spline interpolant of theorem 2.1 which are best possible. Let $s(x)$ be twice continuously differentiable function satisfying the conditions of Theorem 2.1. Now consider $f \in C^5[0,1]$ and writing $L_i[f,x]$ for the unique quartic which agree with $f^r(x_i), f^r(x_{i+1}), r=0,1$ and $f''(\alpha_i)$ we see that for $x \in [x_i, x_{i+1}]$ we have

$$|f(x)-s(x)| \le |f(x)-s_i(x)| \le |f(x)-$$

$$L_i[f,x]| + |L_i[f,x]-s_i(x)|. \qquad (3.1)$$

In order to obtain the bounds of *e(x)*, we proceed to get pointwise bounds of both the terms on the right hand side of (3.1). The estimate of the first term can be obtained by following a well-known Cauchy remainder theorem for polynomial interpolation (see Devis [4]) i.e.

$$|f(x)-L_i[f,x]| \le \frac{h^5}{5!}|t^2(1-t)^2(\frac{1}{2}-t)|F, \qquad (3.2)$$

where $t=(x-x_i)/h_i, and\ F=\max_{\theta \le x \le 1}|f^{(5)}(x)|$ and h = $\max_i h_i$

To get the bounds of $|L_i[f,x]-s_i(x)|$, we have from (2.4).

$$L_i[f,x]-s_i(x)=h_i[f'(x_i)-s'(x_i)]Q_4(t)$$

$$+h_i[f'(x_{i+1})-s'(x_{i+1})]Q_5(t) \qquad (3.3)$$

Thus

$$|L_i[f,x]-s_1(x)| \le h_i|e'(x_i)||Q_4(t)|+h_i|e'(x_{i+1})||Q_5(t)| \qquad (3.4)$$

Let $K_1(t)=|t+\{(18\theta-24\theta^2)t^2+(-6+12\theta^2)$

$t^3+(2-6\theta)t^4\}/(2-12\theta+12\theta^2)|,$

and $K_2(t)=|(6\theta-12\theta^2)t^2+(-2+12\theta^2)t^3+(2-6\theta)t^4\}/(2-12\theta+12\theta^2)|.$

Let $K(t)=\max_{0<t<1}\{K_1(t), K_2(t)\}$, then (3.5)

$$|L_i(f,x)-s_i(x)| \le h_i \max\{e'(x_i), e'(x_{i+1})K(t) \qquad (3.6)$$

Let the $\max|e'(x_i)|$ exist for i=j, then equation (3.6) may be written as

$$|L_i[f,x]-s_1(x)| \le h|e'(x_j)|K(t) \qquad (3.7)$$

Now we proceed to obtain $|e'(x_j)|$, Replacing $s'(x_j)$ by $e'(x_j)$ in equation (2.6) we have,

$$h_i(-36\theta+24\theta^2+12)e'_{j-1}+[(-60\theta+48\theta^2+12)$$

$$h_j-h_{j-1}(36\theta-48\theta^2)]e'_j+h_{j-1}(24\theta^2-12\theta)e'_{j+1}=E(f) \tag{3.8}$$

where $E(f)=\left\{(48\theta-72\theta^2)\frac{h_{j-1}}{h_j}+(24+72\theta^2-96\theta)\frac{h_j}{h_{j-1}}\right\}$

$$f_j-(24+96\theta+72\theta^2)\frac{h_j}{h_{j-1}}f_{j-1}+2h_jh_{j-1}(f''(\alpha_j)-f''(\alpha_{j-1})\}$$

$$+(-48\theta+72\theta^2)\frac{h_{j-1}}{h_j}f_{j+1}-(-36\theta+24\theta^2+12)$$

$$h_j\ f'_{j-1}-[(-60\theta+48\theta^2+12)h_j-h_{j-1}(36\theta-48\theta^2)\,f'_j-h_{j-1}(24\theta^2-12\theta)\,f'_{j+1}$$

In view of that *E(f)* is a linear functional which is zero for polynomial of degree 4, or less, we can apply the Peano theorem (see Davis [4]) to obtain

$$E(f)=\int_{n_{j-1}}^{n_{j+1}}\frac{f^5(y)}{4!}E[(x-y)_+^4]dy. \tag{3.9}$$

Thus, from (3.9) we have

$$|E(f)|\le\frac{1}{4!}f\int_{n_{j-1}}^{n_{j+1}}[E(x-y)_+^4]dy. \tag{3.10}$$

Further, it can be observed from (3.9) that $x_{j-1}\le y\le x_{j+1}$

$$E[(x-y)_+^4]=[(48\theta-72\theta^2)\frac{h_{j-1}}{h_j}+\frac{h_j}{h_{j-1}}$$

$$(24+72\theta^2-96\theta^2)(x_j-y)_+^4+2h_jh_{j-1}$$

$$\{12(\alpha_j-y)_+^2-12(\alpha_{j-i}-y)_+^2\}+(-48\theta+72\theta^2)$$

$$\frac{h_{j-1}}{h_j}(x_{j+1}-y)_+^4-4[(-60\theta+48\theta^2+12)h_j-h_{j-1}$$

$$(36\theta-48\theta^2)(x_j-y)_+^3-h_{j-1}(24\theta^2-12\theta)(x_{j+1}-y)^3].$$

Rewriting the above expression in the following symmetric form, we get

$$[E(x-y)_+^4]=$$

$$\frac{h_{j-1}}{h_j}[(-48\theta+72\theta^2)(x_j-y)-24\theta^2 h_j]$$

$$[(x_j-y)+h_j]^3, \qquad \alpha_j<y<x_{j+1}$$

$$\frac{h_{j-1}}{h_j}[(-48+72\theta^2)(x_j-y)^2+\{-144\theta+192\theta^2\}(x_j-y)h_j$$

$$+[(-144\theta+144\theta^2)+24)h_j^2](x_j-y)^2, \qquad x_j<y<\alpha_j$$

$$-\frac{h_j}{h_{j-1}}[(-48\theta+72\theta^2)(x_j-y)^2-(-144\theta+192\theta^2)(x_j-y)h_{j-1}$$

$$+(-144\theta+144\theta^2+24)h_{j-1}^2]\,(x_j-y)^2, \qquad \alpha_{j-1}<y<x_j$$

$$-\frac{h_j}{h_{j-1}}[(-48\theta+72\theta^2)(x_j-y)+24\theta^2 h_{j-1}](x_j-y-h_{j-1})^3$$

$$x_{j-1}<y<\alpha_{j-1}.$$

Thus, it is clear from the above expression that $[E(x-y)_+^4]$ is non negative for $x_{j-1}\le y\le x_{j+1}$.

Therefore, it follows that

$$\int_{x_{j-1}}^{x_{j+1}}|E(x-y)_+^4|dy=\frac{4h_j h_{j-1}[h_j^3+h_{j-1}^3]}{5}(3+18\theta-20\theta^2+120\theta^3). \qquad (3.11)$$

Combining (3.10) with (3.11), we have

$$|E(f)|\le\frac{F\,h_j h_{j-1}(h_j^3+h_{j-1}^3)K^*(\theta)}{5!}$$

where $K^*(\theta)=4(3+18\theta-20\theta^2+120\theta^3)$. (3.12)

Thus, from (3.8) and (3.10), (3.12) it follows that

$$|e'(x_j)|=|e_j'|\le\frac{F\,h_j h_{j-1}(h_j^3+h_j^3)K^*(\theta)}{5!\{(3\theta^2-4\theta+1)h_j+(3\theta^2-2\theta)h_{j-1}\}} \qquad (3.13)$$

Now using (3.2) (3.7) along with (3.13) in (3.1), we have

$$|e(x)| \le \frac{h^5}{5!} |t^2(1-t)^2(t-\theta)| F + h_j |e_j^{'}| |K(t)| \tag{3.14}$$

$$\le \frac{h^5}{5!} F |c(t)|$$

where $c(t) = \left[t^2(1-t)^2 |t-\theta| + \frac{2K^*(\theta)K(t)}{(6\theta^2 - 6\theta + 1)} \right].$ (3.15)

Thus, we prove the following.

Theorem 3.1 : Suppose *s(x)* is the quartic spline of Theorem 2.1 interpolating a function *f(x)* and $f \in C^5[0,1]$ then

$$|e(x)| \le K \frac{h^5}{5!} \max_x |f^5(x)|, \tag{3.16}$$

where $K = \max_{0 \le t \le 1} |c(t)| = t^2(1-t)^2 |(t-\theta)| + \frac{2|K(t)K^*(\theta)|}{(6\theta^2 - 6\theta + 1)}.$

Also we have

$$|e_j^{'}| = |(e^{'}(x_j)| \le \frac{h^4}{5!} \frac{2K^*(\theta)}{(6\theta^2 - 6\theta + 1)} \max_x |f^5(x)| \tag{3.17}$$

Further more K in (3.16) can not be improved for an equally spaced partition. Inequality (3.17) is the best possible.

Also, we have

$$|e'(x)| \le K_1 \frac{h^4}{5!} |f^5(x)| \tag{3.18}$$

where K_1 is some positive constant Equation (3.15) proves (3.16). Inequality (3.17) is a direct consequence of (3.13).

However, we shall show that inequality (3.16) is best possible in the limit. Considering $f(x) = \frac{x^5}{5!}$ and using Cauchy formula Davis [4] we have

$$\frac{x^5}{5!} - L_i[\frac{t^5}{5!}, x] = \frac{h^5}{5!} (1-t)^2 t^2 (t-\theta) \tag{3.19}$$

Moreover, for the function under consideration (3.8) gives the following for equally spaced knots

$$E\left(\frac{x^5}{5}\right)=\frac{2h^4K*(\theta)}{5!}$$

$$=(-36\theta+24\theta^2+12)e'_{j-1}+$$

$$[-96\theta+96\theta^2+12]e'_j+(24\theta^2-12\theta)e'_{j+1} \qquad (3.20)$$

Consider for a moment,

$$e'(x_j)=e'(x_{j-1})=e'(x_{j+1})=\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+1)5!} \qquad (3.21)$$

We have from (3.4)

$$L_i[f,x]-s(x)=\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+1)5!}\left(Q_4(t)+Q_5(t)\right) \qquad (3.22)$$

$$=\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+1)}[(2+12\theta^2-12\theta)t$$

$$+t^2(24\theta-36\theta^2)+(-8+24\theta^2)t^3+(6-12\theta)t^4]/(2-12\theta+12\theta^2). \qquad (3.23)$$

From (3.23) it is clearly observed that (3.15) is the best possible provided we could prove that

$$e'(x_{j-1})=e'(x_j)=e'(x_{j+1})=\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+1)5!} \qquad (3.24)$$

In fact (3.24) is attained only in the limit. The difficulty will take place in the case of boundary condition i.e. $e'(x_0)=e'(x_n)=0$. However it can be shown that as we move many subintervals away from the boundaries $e'(x_j)\rightarrow\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+1)5!}$.

For that we shall apply (3.20) inductively to move away from the end conditions $e'(x_0)=e'(x_n)=0$. The first step in this direction is to show that $e'(x_j)\geq 0$ for j=0,1,.....n which can be proved by a contradiction assumption.

Let $e'(x_j)<0$ for some i, i=1,...,n-1. Now making use of (3.17), we get

$$\frac{h^4K*(\theta)}{5!(6\theta^2-6\theta+1)12}\geq\max|e'(x_j)|>(-36\theta+24\theta^2+12)e'_{j-1}$$

$$+(96\theta^2-96\theta+12)e'_j+(24\theta^2-12\theta)e'_{j+1}$$

$$=\frac{2h^4K*(\theta)}{5!},$$

i.e. $1>12(6\theta^2-6\theta+1)$

This is a contradiction. Thus, $e'(x_j)\geq 0$ for j=0,1,.....n. Now from (3.20)

$$(-96\theta+96\theta^2+12)e_j^{'}=\frac{2h^4K*(\theta)}{5!}-$$

$$(-36\theta+24\theta^2+12)e_{j-1}^{'}-(24\theta^2-12\theta)e_{j+1}^{'} \qquad \because e_j^{'}\geq 0$$

$$e_j^1\leq\frac{2h^4K*(\theta)}{(-96\theta+96\theta^2+12)5!} \text{ for j=1,2,......n-1} \qquad (3.25)$$

Now again using (3.25) in (3.20) we have

$$e_j^{'1}\leq\frac{2h^4K*(\theta)}{5!((-96\theta+96\theta^2+12)}\left[1+(\frac{48\theta^2+48\theta-12}{-96\theta+96\theta^2+12})\right]$$

$$=\frac{2h^4K*(\theta)}{5(-96\theta+96\theta^2+12)}$$

Repeated use of (3.25), we have

$$=\frac{2h^4K*(\theta)}{5(-96\theta+96\theta^2+12)}\left[1+\frac{(-48\theta^2+48\theta-12)}{(96\theta^2-96\theta+12}+(\frac{-48\theta^2+48\theta-12}{96\theta^2-96\theta+12})^2+......\right]$$

Now, it can be seen easily that the right hand side of (3.26) tend to

$$\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+7)5!}$$

Hence $$e'(x_j)\leq\frac{h^4K*(\theta)}{12(6\theta^2-6\theta+1)(5!)} \qquad (3.27)$$

Thus, equation (3.27) verifies the proof of (3.17).

REFERENCES :

1. Deboor, C.A. Practical Guide to Splines, Applied Mathematical Science, Vol. 27 Spoinger, Varlag, New York 1979.

2. Hall, C.A. and Meyer, W.W., J. Approximation Theory 16 (1976), pp 105-122.

3. Howell, G. and Verma, A.K. Best Error Bound of Quartic Spline Interpolation, J. Approx. Theory 58 (1989), 58-67.

4. Davis, P.J. Interpolation and approximation, New York, 1969.

5. Hallet P., Mund, E,H. & Munert, J.P. Jour. of Comp. App. Maths 1 (1975), pp. 279-288.

6 Gemlling, R.H.J. and Meyling, G. in Interpolation by Bivartate Quintic Splines of Class Construction of Theory of function 87 (ed) Sendor et al (1987) 152-61.

7 Agrawal, R.P. and Wong, P.J.Y. Error Inqualities in Polynomial Interpolation and their application, Kumar Academic Publisher's (1993).

8 Rana, S.S. Convergence of a class of deficient interpolatory splines, Rocky Mont. Jour. of Math., 18 (1988), 825-831.

9 Rana, S.S. Quadratic spline interpolation, Jour. of Approximation Theory, 57 (1989), 300-305.

10. Jienzhong, W. and Hung, D. Scientia Sinica 24 (1982) pp. 1130-1141.

11. Dubey, Y.P. and Shukla, A. The deficient C^1 quartic spline interpolation, international general of Engineering & Science, Vol. 2, 9April 2013, pp. 24-30.