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DISCRETE QUARTIC SPLINE INTERPOLATION OVER UNIFORM MESH

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Abstract

In this paper, we have studied existence, uniqueness and error bounds for deficient discrete quartic spline over uniform mesh.

Introduction

Discrete splines have been introduced by Mangasarian and Schumaker [4] in connection with certain studies of minimization problems involving differences. Deficient splines are more useful because they require less smooth data. Deficient discrete quartic spline which interpolates given functional values at one intermediate points of a uniform mesh has been studied in [10]. These results were generalized by Dubey, Rana and Dubey [9] for non uniform meshes. Rana and Dubey [11] have obtained asymptotic precise estimate of the difference between discrete cubic spline interpolant and the function interpolated, which is some time used to smooth a histogram. Sharma and Tzimbalario [2] have present a unified treatment of the asymptotic error comparison both for even and for odd degree interpolatory splines. In the direction for some constructive aspect of discrete splines we refer to Astor and Duris [1], Jia [6] and Schumaker [3]. The object of present paper, is to obtained existence, uniqueness and convergence properties of deficient discrete quartic spline interpolation which agree with the given function at two interior points of an interval and first difference at the mid point.

Let us consider a mesh P on [0, 1], which is defined by

$P{:}0=x_0<x_1<.......<x_n=1$. Let P be the length of the mesh interval $[x_{i-1},x_i]$ such that $x_{\_}-x_{i-1}=P,$ for i = 1,2,....n. Throughout, h will represent a given positive real number. Consider a real continuous function $s(x,h)$ defined over [0, 1] which is such that its restriction s_i on $[x_{i-1},x_i]$ is a Polynomial of degree 3 or less for i = 2,.....,n. Then $s(x,h)$ defined a deficient discrete quartic spline if

$$D_h^{(j)}\, s_i(x_i,h)=D_h^{(j)}\, s_{i+1}(x_i,h) \quad j=0,1 \tag{1.1}$$

Where the difference operator D_n are defined as

$$D_n^{(0)} f(x) = f(x),\ D_n^{(1)}\ f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

The class of all discrete deficient quartic splines is denoted by R (4, 1, P, h) denotes the class of all discrete quartic splines of deficiency one which satisfy the boundary condition

$$s(x_0, h) = f(x_0, h)$$

$$s(x_n, h) = f(x_n, h) \qquad (1.2)$$

For a given function f, we introduced the following interpolatory conditions

$$s(\alpha_i) = f(\alpha_i) \qquad (1.3)$$

$$s(\beta_i) = f(\beta_i) \qquad (1.4)$$

$$D_h^{(1)} s(\gamma_i) = D_h^{(1)} f(\gamma_i) \qquad (1.5)$$

Where $\alpha_i = x_{i-1} + \frac{1}{3} p$

$$\beta_i = x_{i-1} + \frac{1}{2} p = \gamma_i$$

and pose the following.

PROBLEM 1.1 : Given h > 0, for what restriction on P does there exist a unique $s(x,h) \in R(4,1,P,h)$ which satisfy the boundary conditions (1.2) and interpolatory codition (1.3)-(1.5).

Let Q(t) be a discrete quartic polynomial on $[0,1]$, then we can show that

$$Q(t) = Q\left(\frac{1}{3}\right) P_1(t) + Q\left(\frac{1}{2}\right) P_2(t) + D_n^{(1)}\ Q\left(\frac{1}{2}\right) P_3(t) + Q(0)\, P_4(t) + Q(1) P_5(t) \qquad (1.6)$$

Proof : Let $t = \frac{(x - x_i)}{p}, 0 \le t \le 1$. We can write (1.6) in the form of the restriction $s_i(x,h)$ of the deficient discrete quartic spline s(x,h) on $[x_i\ x_{i+1}]$ as follows :-

$$s_i(x) = f(\alpha_i) P_1(t) + f(\beta_i) P_2(t) + pD_h^{(1)}\ f(\gamma_i)$$

$$P_3(t) + s(x_i)\, P_4(t) + s(x_{i+1}) P_5(t) \qquad (1.7)$$

Where

$$P_1(t) = \frac{\left[H\left(\frac{-9}{8}, \frac{9}{2}\right) t + H\left(\frac{45}{8}, \frac{-45}{2}\right) t^2 + H(-9, 36) t^3 + H\left(\frac{9}{2}, -18\right) t^4 \right]}{H\left(\frac{-1}{36}, \frac{1}{9}\right)}$$

$$P_2(t)=\frac{\left[H\left(\frac{8}{9},\frac{-32}{9}\right)t+H\left(\frac{-44}{9},\frac{176}{9}\right)t^2+H(8,-32)t^3+H(-4,16)t^4\right]}{H\left(\frac{-1}{36},\frac{1}{9}\right)}$$

$$P_3(t)=\left[H\left(\frac{-1}{9},0\right)t+H\left(\frac{2}{3},0\right)t^2+\left(\frac{-11}{9},0\right)t^3+H\left(\frac{2}{3},0\right)t^4\right]/H\left(\frac{-1}{36},\frac{1}{9}\right)$$

$$P_4(t)=\frac{\left[1+H\left(\frac{2}{9},\frac{-4}{3}\right)t+H\left(\frac{-23}{36},\frac{47}{9}\right)t^2+H\left(\frac{7}{9},-8\right)t^3+H\left(\frac{-1}{3},4\right)t^4\right]}{H\left(\frac{-1}{36},\frac{1}{9}\right)}$$

$$P_5(t)=\frac{\left[H\left(\frac{1}{72},\frac{7}{18}\right)t-H\left(\frac{-7}{72},\frac{41}{18}\right)t^2+H\left(\frac{2}{9},4\right)t^3-H\left(\frac{1}{6},2\right)t^4\right]}{H\left(\frac{-1}{36},\frac{1}{9}\right)}$$

Where $H(a,b)=a+bh^2$, a and b are real number.

Observing that (1.7) it may easily seen that $s_i(x,h)$ is a quartic on $[x_i,x_{i+1}]$ i=0,1.....n-1 satisfying (1.2)-(1.5).

2. **Main results :**

We are set to answer the problem 1.1 in the following.

Theorem 2.1 : For any h > 0 and then there exist a unique deficient discrete quartic spline $s(x,h)\in R(4,1,P,h)$ which satisfies the condition (1.2)-(1.5).

Proof : Now applying continuity of the first difference of $s(x,h)$ at x_i, we get the following system of equations

$$\left\{H\left(\frac{-1}{18},\frac{-10}{9}\right)P^2+H\left(\frac{89}{9},-8\right)h^2\right\}s_{i-1}+s_i\left[H\left(\frac{13}{72},\frac{-7}{6}\right)P^2+H\left(\frac{11}{9},-4\right)h^2\right]$$

$$+s_{i+1}\left[H\left(\frac{1}{18},\frac{7}{18}\right)P^2+H\left(\frac{2}{9},4\right)h^2\right]$$

$$=F_i,\ \text{i = 1,2,.....n-1} \qquad (2.1)$$

Where

$$F_i=\left[\left\{H\left(\frac{9}{8},\frac{-9}{2}\right)P^2+H(9,-36)h^2\right\}f(\alpha_{i-1})\right.$$

$$+\left\{H\left(\frac{-8}{9},\frac{32}{9}\right)P^2-H(8,32)h^2\right\}f(\beta_{i-1})$$

$$+P\left\{P^2H\left(\frac{2}{9},0\right)+H\left(\frac{13}{9},0\right)h^2\right\}D_h^{(1)}\ f(\gamma_{i-1})\Bigg]$$

$$-\Bigg[\left\{H\left(\frac{8}{9},\frac{-32}{9}\right)P^2+H\left(\frac{172}{9},-32\right)h^2\right\}f(\beta_i)$$

$$-\left\{H\left(\frac{9}{2},\frac{9}{2}\right)P^2+4(-9,36)h^2\right\}f(\alpha_i)$$

$$+P\,D_h^{(1)}\ f(\gamma_i)\left\{P^2H\left(\frac{1}{9},0\right)+H\left(\frac{11}{9},0\right)h^2\right\}\Bigg]$$

We can easily see that excess of the absolute value of the coefficient of $s(x_i)=m_i$ (Say) dominant for the sum of the absolute values of the coefficient of m_{i-1} and m_{i+1} in (2.1) under the conditions of Theorem 2.1 and is given by

$$J_i(h)=\left[H\left(\frac{5}{72},\frac{4}{9}\right)P^2+H\left(\frac{80}{9},0\right)h^2\right]$$

Therefore the coefficient matrix of the system of equation (2.1) is diagonally dominant and hence invertible. Thus, the system of equation has unique solution this complete the proof of theorem (2.1).

3. ERROR BOUNDS :

It may be observed that system of equation (2.1) may be written as

$$A(h)M(h)=F \qquad (3.1)$$

Where A(h) is coefficient matrix and $M(h)=s(x_i,h)$, we note that row max norm $\|A^{-1}(h)\|$ satisfies following inequality

$$\|A^{-1}(h)\|\le y(h) \qquad (3.2)$$

Where $y(h)=\max\{J_i(h)\}^{-1}$

We introduced the set $R_h=\{jh: j\,is\,an\,\text{int}\,eger's\}$ and define a discrete intervals as follows $[0,1]_h=[0,1]\cap R_h$. For a function f and two distinct points x_1 and x_2 in the domain, the first divided difference are define by

$$[x_1,x_2]=\frac{f(x_1)-f(x_2)}{x_1-x_2}$$

For convenience, we write $f^{(1)}$ for $D_h^{(1)}$ and $w(f,P)$ is the modulus of continuity of f, the discrete norm of the function f over the intervals $[0,1]_h$ is defined by $\|f\|=\max_{x\in[0,1]}|f(x)|$

Without assuming any smoothness condition on the data f, we shall obtain error bounds for the function over discrete interval $[0,1]_h$.

Theorem 3.1 : Suppose $s(x,h)$ is the discrete quartic spline interpolant of Theorem 2.1. Then

$$\| e_i^{(1)}(x) \| \leq K(P,h) w(f,p) \tag{3.3}$$

$$\| e_i(x) \| \leq K_1(P,h) w(f,P) \tag{3.4}$$

and

$$\| e(x) \| \leq P\, K^*(P,h) w(f,P) \tag{3.5}$$

Where $K(P,h), K_1(P,h) and\, K^*(P,h)$ are positive constant of P and h.

Proof : We replace $m_i(h)$ by error function $e(x_i) = D_n^{(i)} s(x,h) - f_i = L_i$ in (3.1) and to find error bounds (3.3) - (3.5) we need following Lemma due to Lyche [4.5].

Lemma 3.1 : Let $\{a_i\}_{i=1}^n$ and $\{b_j\}_{j=1}^m$ be a given sequence of non-negative real number's such that $\sum a_i = \sum b_j$ then for any real valued function f defined on a discrete interval [0,1], we have

$$\left| \sum_{i=1}^n a_i \left[x_{i_0}, x_{i_1} .... x_{i_k}\right]_f - \sum_{j=1}^m b_j \left[y_{j_0}, y_{j_1} ... y_{j_k}\right]_f \right| \leq w\left(f^{(1)}, |1-P|\right) \sum \frac{a_i}{K!}$$

Where $x_{1_k}, y_{j_k} \in [0,1]_h$ for relevant values of i, j and k.

Now equation (3.1) can be written as

$A(h).e(x) = F_i(h) - A(h) f_i = L_i$ by using **Lemma 3.1 w**e get

$$|(L_i)| \leq w\left(f^{(1)}, 1-P\right) \sum_{i=1}^5 a_i = \sum_{j=1}^5 b_j \tag{3.6}$$

Where

$$\left[P^3 H\left(\frac{1}{432}, \frac{-13}{36}\right) + P\, H\left(\frac{3}{2}, -6\right) h^2 + P\, H\left(\frac{-40}{27}, \frac{8}{3}\right) h^2 \right]$$

$$= \sum_{i=1}^5 a_i = \sum_{j=1}^5 b_j$$

Where

$$a_1 = P\left[H\left(\frac{-17}{144}, \frac{17}{36}\right) P^2 - h^2 H\left(\frac{1}{2}, -2\right) \right]$$

$$a_2 = P\left[H\left(\frac{1}{18}, \frac{-10}{9}\right) P^2 + H\left(\frac{5}{9}, -8\right) h^2 \right]$$

$$a_3 = P\left[H\left(\frac{-4}{27},\frac{16}{27}\right)P^2 + H\left(\frac{-4}{3},\frac{16}{3}\right)h^2\right]$$

$$a_4 = P\left[H\left(\frac{-1}{108},\frac{-7}{27}\right)P^2 - H\left(\frac{4}{27},\frac{8}{3}\right)h^2\right]$$

$$a_5 = P\,H\left(\frac{2}{9},\frac{13}{9}\right)$$

$$b_1 = P\left\{H\left(\frac{3}{16},\frac{-3}{4}\right)P^2 + H\left(\frac{3}{2},-6\right)h^2\right\}$$

$$b_2 = P^3\,H\left(\frac{1}{36},\frac{1}{9}\right)$$

$$b_3 = P^3\,H\left(\frac{1}{12},\frac{1}{9}\right)$$

$$b_4 = \frac{-P^3}{9}$$

$$b_5 = -P\frac{11}{9}h^2$$

and $x_{1_0} = \beta_{i-1} = y_{1_1}, x_{1_1} = x_i = x_{2_1} = y_{2_1} = y_{4_0}$

$x_{2_0} = x_{i-1}, x_{3_0} = \gamma_{i-1} - h, x_{3_1} = \gamma_{i-1} + h$

$y_{1_0} = \alpha_{i-1}, y_{2_0} = \alpha_i, y_{3_0} = \gamma_{i-1} - h$

$y_{3_1} = \gamma_{i-1} + h, x_{4_0} = \alpha_i, x_{4_1} = \beta_i, x_{5_0} = \alpha_i, x_{5_1} = x_{i+1}$ thus applying Lemma 3.1 in (3.6), we get

$$|L_i| \le N(P,h)w\left(f^{(1)},|1-p\right) \tag{3.7}$$

Now using (3.2), (3.7) and (3.6) we get

$$|e(x_i)| < K(P,h)w\left(f^{(1)},P\right) \tag{3.8}$$

Where $K(P,h)$ is some positive function of P and h, this complete the proof of inequality (3.4).

Now from (1.5) equation

$$e(x) = e(x_i)Q_4(t) + e(x_{i+1})Q_5(t) + M_i(f) \tag{3.9}$$

Where

$$M_i(f) = f(\alpha_i)Q_1(t) + f(\beta_i)Q_2(t) + P\,f^{(1)}(\gamma_i)$$

$Q_3(t) + f_{i-1}Q_4(t) + f_iQ_5(t) - f(x)$, we write $M_i(f)$ is of the form of divided difference and by using Lemma 3.1. We get

$$|M_i(f)| \le \left| \sum_{i=1}^{3} a_i \left[x_{1_0}, x_{i_1}\right]_f - \sum_{j=1}^{2} b_j \left[y_{j_0}, y_{j_1}\right]_f \right| \tag{3.10}$$

Where

$$a_1 = P\left[tH\left(\frac{3}{16}, \frac{3}{4}\right) + t^2 H\left(\frac{-15}{16}, \frac{15}{4}\right) + H\left(\frac{3}{2}, -6\right)t^3 + H\left(\frac{-3}{4}, 3\right)t^4 \right]$$

$$a_2 = P\left[H\left(\frac{-1}{9}, \frac{2}{3}\right)t + H\left(\frac{23}{72}, \frac{-47}{18}\right)t^2 + H\left(\frac{-7}{18}, 4\right)t^3 + H\left(\frac{4}{3}, -2\right)t^4 \right]$$

$$a_3 = P\left[H\left(\frac{1}{9}, 0\right)t + H\left(\frac{2}{3}, 0\right)t^2 + H\left(\frac{-11}{9}, 0\right)t^3 + H\left(\frac{2}{3}, 0\right)t^4 \right]$$

$$b_1 = P\left[tH\left(\frac{-1}{144}, \frac{17}{36}\right) + H\left(\frac{7}{144}, \frac{79}{36}\right)t^2 - H\left(\frac{1}{9}, 2\right)t^3 + H\left(\frac{1}{12}, 1\right)t^4 \right]$$

and $b_2 = P\,t\,H\left(\frac{-1}{36}, \frac{1}{9}\right)$

Clearly $\sum a_i = \sum b_j = P\left[-H\left(\frac{5}{144}, \frac{1}{12}\right)t + t^2 H\left(\frac{7}{144}, \frac{41}{36}\right) - H\left(\frac{2}{9}, 2\right)t^3 + H\left(\frac{1}{4}, 1\right)t^4 \right]$

and $x_{1_0} = \alpha_i, x_{1_1} = \beta_i = x_{2_1}, x_{2_0} = x_{i-1}, x_{3_0} =$

$\gamma_1 - h, x_{3_1} = \gamma_1 + h, y_{1_0} = \beta_i, y_{1_1} = x_i,$

$\gamma_{2_0} = x_{i-1}, y_{2_1} = x$ Therefore

$$|M_i(f)| < N^*(P,h)\,w\left(f^{(1)}, P\right) \tag{3.11}$$

From (3.8), (3.9) and (3.1) we get inequality (3.5).

Next from (1.5) equation, we get

$$P e^{(1)}(x) = e_{i-1} Q_4^{(1)}(t) + e_i Q_5^{(1)}(t) + U_i(f) \qquad \text{(Say)} \tag{3.12}$$

Where

$$U_i(f) = f(\alpha_i) Q_1^{(1)}(t) + f(\beta_i) Q_2^{(1)}(t) + P\, f^{(1)}(\gamma_i) Q_3^{(1)}(t) + f_{i-1} Q_4^{(1)}(t)$$

$$+ f_i Q_5^{(1)}(t) - P.f^{(1)}(x) \tag{3.13}$$

We write $U_i(f)$ in the form of divided difference as follows -

$$|U_i(f)| = \left| \sum_{i=1}^{3} a_i \left[x_{1_0}, x_{1_1}\right]_f - \sum_{j=1}^{2} b_j \left[y_{1_0}, y_{j_1}\right]_f \right|$$

Where $a_1 = P\left[H\left(\frac{-1}{9}, \frac{2}{3}\right) + 2H\left(\frac{23}{36}, \frac{-47}{9}\right)t + \left(3t^2 + h^2\right)H \right.$

$$\left(\frac{-59}{6},4\right)+4t\left(t^2+h^2\right)H\left(\frac{1}{6},-2\right)\Bigg] \quad (3.14)$$

$$a_2=P\left[H\left(\frac{1}{144},\frac{7}{36}\right)+2tH\left(\frac{-7}{72},\frac{-59}{18}\right)+\left(3t^2+h^2\right)H\left(\frac{1}{9},2\right)-\right.$$

$$\left.4H\left(\frac{1}{12},1\right)t\left(t^2+h^2\right)\right]$$

$$a_3=P\left[\frac{-1}{9}+\frac{11}{3}t-\frac{11}{9}\left(3t^2+h^2\right)t\frac{8}{3}t\left(t^2+h^2\right)\right]$$

$$b_1=P\left[-H\left(\frac{3}{16},\frac{3}{4}\right)+2tH\left(\frac{15}{8},\frac{-15}{2}\right)+H\left(\frac{-3}{2},6\right)\right.$$

$$\left.\left(3t^2+h^2\right)+4H\left(\frac{3}{4},-3\right)t\left(t^2+h^2\right)\right]$$

$$b_2=PH\left(\frac{-1}{36},\frac{1}{9}\right).$$

and $y_{1_0}=\alpha_i, y_{1_1}=\beta_i, y_{2_0}=x+h, y_{2_1}=x-h$

$x_{1_0}=\beta_i, x_{1_1}=x_i, x_{3_0}=\gamma_i-h, x_{2_0}=\beta_i$

$x_{2_1}=x_{i+1}, x_{3_1}=\gamma_i+h$

Since $\sum_{i=1}^{3}a_i=\sum_{j=1}^{2}b_j=P\left[H\left(\frac{-105}{8},\frac{31}{36}\right)+2tH\left(\frac{15}{8},\frac{-15}{2}\right)+\left(3t^2+h^2\right)\right.$

$$\left.H\left(\frac{-3}{2},6\right)+4t\left(t^2+h^2\right)H\left(\frac{3}{4},-3\right)\right]$$

Again applying Lemma 3.1 in $U_i(f)$. We get

$$\left|U_i(f)\right|\leq N(p,h)\,w\left(f^{(1)},P\right) \quad (3.15)$$

Thus using (3.8) and (3.15) in (3.12). We get the following

$$\| e^{(1)}(x)\|\leq K(P,h)\,w\left(f^{(1)},P\right) \quad (3.16)$$

This complete proof of (3.3) inequality.

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