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RESEARCH ARTICLE

A Study on a Class of Rational Cubic Trigonometric Bézier Curves with Shape parameters

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Abstract

In this paper a rational cubic trigonometric Bézier curve based on trigonometric basis functions with shape parameters is presented. The basis function share the properties with Bernstein basis functions, so the generated curves inherit many properties of traditional Bézier curves. In this paper the shape parameters provides a local control on shape of the curve without changing the control polygon, by selecting suitable values of these parameters, which enables the designer to control the curve more than ordinary Bézier curve.

Introduction

Designing of curves and surfaces is very important topic of Computer Aided Geometric Design and computer graphics. In recent years parametric representation of curves and surfaces with shape parameters has received a lot attention. Trigonometric splines and polynomials have attracted wide spread interest within curve design; see [1, 3, 4, 5, 6, 7]. Han [5] presented a class of quadratic trigonometric polynomial curves with a shape parameter. The shape of the curve was easily controlled by altering the values of shape parameter than the ordinary quadratic B-spline curves. Bézier technique is one of the techniques of analytic representation of curves and surfaces that has received wide acceptance as a important tool in CAD/CAM system. They are used to produce curves which appear reasonably smooth at all scales. The quadratic trigonometric Bézier curve with single shape parameter was presented by Bashir et al (see [2]). Qian et al (see [8]) described trigonometric Bézier curves and its applications to draw some shapes. Shang and Benyue [9] presented Quasi cubic trigonometric bases and surfaces by their tensor product.

The Present work is organized as follows: In section 2, the basis function of the cubic trigonometric Bézier curve with two shape parameters are established and the properties of the basis function have been described. In section 3, rational cubic trigonometric Bézier curves and their properties are discussed. Section 4 is about shape control of the curves by shape parameter. Section 5 includes the representation of ellipse and circle. In section 6, the approximation of the rational cubic trigonometric Bézier curve to the ordinary rational cubic Bézier curve is presented. In section 7, conclusion of the paper.

2. Cubic Trigonometric Bézier Basis Functions

In this section, definition and some properties of Cubic Trigonometric Bézier Basis Functions with two shape parameters are given:

Definition 2.1 Let $\alpha, \beta \in [-2,1]$, for $t \in [0,1]$ the following four functions are defined the cubic trigonometric Bézier basis functions, with two shape parameters α and β :

$R_0(t) = (1-\sin\frac{\pi}{2}t)^2(1-\alpha\sin\frac{\pi}{2}t)$,
$R_1(t) = \sin\frac{\pi}{2}t\,(1-\sin\frac{\pi}{2}t)(2+\alpha(1-\sin\frac{\pi}{2}t))$,

$R_2(t) = \cos\frac{\pi}{2}t\,(1-\cos\frac{\pi}{2}t)\,(2+\beta(1-\cos\frac{\pi}{2}t))$,

$R_3(t) = (1-\cos\frac{\pi}{2}t)^2\,(1-\beta\cos\frac{\pi}{2}t)$ (1)

If α = β = 0, then the basis functions reduce to quadratic trigonometric polynomials.

Theorem 2.1 The basis functions (1) have the following properties:
(i) Nonnegativity: $R_i(t) \geq 0$, i=0,1,2,3
(ii)Partition of unity: $\sum_{i=0}^{3} R_i(t) = 1$
(iii) Monotonicity: For the given values of parameters α and β, $R_0(t)$ is monotonically decreasing and $R_3(t)$ is monotonically increasing.
(iv) Symmetry: $R_i(t;\alpha,\beta) = R_{3-i}(1-t;\alpha,\beta)$, i = 0,1,2,3

Proof:
(i) For $t \in [0,1]$ and $\alpha, \beta \in [-2,1]$,
$\sin\frac{\pi}{2}t \geq 0$, $(1-\sin\frac{\pi}{2}t) \geq 0$, $(1-\alpha\sin\frac{\pi}{2}t) \geq 0$, $(2+\alpha(1-\sin\frac{\pi}{2}t)) \geq 0$, $\cos\frac{\pi}{2}t \geq 0$, $(1-\cos\frac{\pi}{2}t) \geq 0$,
$(1-\beta\cos\frac{\pi}{2}t) \geq 0$, $(2+\beta(1-\cos\frac{\pi}{2}t)) \geq 0$. It is obvious that $R_i(t) \geq 0$, i=0,1,2,3
(ii) $\sum_{i=0}^{3} R_i(t) = (1-\sin\frac{\pi}{2}t)^2(1-\alpha\sin\frac{\pi}{2}t) + \sin\frac{\pi}{2}t\,(1-\sin\frac{\pi}{2}t)(2+\alpha(1-\sin\frac{\pi}{2}t))$
$+ \cos\frac{\pi}{2}t\,(1-\cos\frac{\pi}{2}t)(2+\beta(1-\cos\frac{\pi}{2}t)) + (1-\cos\frac{\pi}{2}t)^2(1-\beta\cos\frac{\pi}{2}t) = 1$.
(iii) Monotonicity of the functions is seen in Fig.1, the curves of the cubic trigonometric basis functions for α = β = −2 (blue solid lines) and for α = β = 1 (red dashed lines).

3. Rational Cubic Trigonometric Bézier Curve

We construct the Rational Cubic Trigonometric Bézier curve with two shape parameters as follows:

Definition 3.1 Given the control points P_i(i = 0,1,2,3) in R^2 or R^3 $t \in [0,1]$ and $\alpha, \beta \in [-2,1]$, We define the Rational Cubic Trigonometric Bézier Curve with two shape parameters as:

$$T(t) = \frac{\sum_{i=0}^{3} m_i P_i R_i(t)}{\sum_{i=0}^{3} m_i R_i(t)} \quad (2)$$

where $R_i(t)$, (i = 0,1,2,3) are the basis functions defined in (1) and m_i is scalar, called weight function. We assume that $m_i \geq 0$. If $m_i = 1$, we get non-rational trigonometric Bézier curve, since the denominator is identically equal to one. The curve defined by (2) possesses some geometric properties which can be obtained easily from the properties of basis functions.
Theorem 3.1 The Rational cubic trigonometric Bézier curve has the following properties:
(i) End point properties
$T(0) = P_0$, $T(1) = P_3$. $T'(0) = \frac{m_1}{m_0}\frac{\pi}{2}(2+\alpha)[P_1 - P_0]$, $T'(1) = \frac{m_2}{m_3}\frac{\pi}{2}(2+\beta)[P_3 - P_2]$
If $m_0 = m_3 = \pi$ and $m_1 = m_2 = 2$, then $T'(0) = (2+\alpha)[P_1 - P_0]$, $T'(1) = (2+\beta)[P_3 - P_2]$
(ii) Symmetry: P_0, P_1, P_2, P_3 and P_3, P_2, P_1, P_0 define the same curve in different parameterizations, that is
$T(t;\alpha,\beta;P_0,P_1,P_2,P_3) = T(1-t;\alpha,\beta;P_3,P_2,P_1,P_0)$ $t \in [0,1]$ and $\alpha, \beta \in [-2,1]$
(iii) Geometric Invariance: The shape of the curve (2) is independent of the choice of coordinates, i. e. it satisfies the following two equations:
$T(t;\alpha,\beta;P_0+q,P_1+q,P_2+q,P_3+q) = T(t;\alpha,\beta;P_0,P_1,P_2,P_3)+q$
$T(t;\alpha,\beta;P_0*q,P_1*q,P_2*q,P_3*q) = T(t;\alpha,\beta;P_0,P_1,P_2,P_3)*q$, $t \in [0,1]$ and $\alpha, \beta \in [-2,1]$
(iv) Convex hull property: From the non negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by control points.

4. Shape Control of rational cubic trigonometric Bézier Curve

The parameters α and β controls the shape of the curve (2). In Fig.2, The rational cubic rigonometric Bézier Curve T(t) gets closer to the control polygon as the values of the parameters α and β increases.In Fig.2, The curves are generated by setting the values of α, β as α = β = -2(blue dashed lines), α = β = -1(green dashed lines), α = β = 0.5(red dashed lines), α = β = 1(black solid lines),

In Fig.3 the curves are generated by setting α = -2(blue dashed lines), α = -1(green dashed lines), α = 0.5 (red dashed lines), α = 1(black solid lines) and β = -2.

In Fig.4 the curves are generated by setting β = -2(blue dashed lines), β = -1(green dashed lines), β = 0.5 (red dashed lines), β = 1(black solid lines) and α = -1.

In Fig.5 the curves are generated by setting α = β = -0.5 (red dashed lines), α = β = -1 (blue solid lines), α =β = 0.5 (green dashed lines).

In order to construct a closed rational cubic trigonometric Bézier curve, we set $P_3 = P_0$, $m_1 = m_2 = 2$ and $m_3 = m_4 = 3$. In Fig 6(a, b, c, d) , The closed rational cubic trigonometric Bézier curve of altering the values of α and β at the same time.

5. The representation of ellipse

Theorem 5.1 The basis function (2) have the following properties:
Let P_0, P_1, P_2, P_3 are the four control points on the ellipse with semiaxes $(a\sqrt{2})$ and $(b/\sqrt{2})$, by proper selection of coordinates, their coordinates can be written in the form
$P_0 = \begin{pmatrix} -a \\ 0 \end{pmatrix}$, $P_1 = \begin{pmatrix} -a \\ \frac{b}{2} \end{pmatrix}$, $P_2 = \begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix}$, $P_3 = \begin{pmatrix} a \\ 0 \end{pmatrix}$
then corresponding rational cubic trigonometric Bézier curve with the shape parameters α = β = 0 with $m_1 = m_2 = m_3 = m_4 = \frac{1}{4}$ and local domain $t \in [0,4]$ represent arc of an ellipse with

$$x(t) = a(\sin\frac{\pi}{2}t - \cos\frac{\pi}{2}t)$$
$$y(t) = \frac{b}{2}(\sin\frac{\pi}{2}t + \cos\frac{\pi}{2}t - 1) \quad (3)$$

Proof: If we take α = β = 0 and $m_1 = m_2 = m_3 = m_4 = \frac{1}{4}$

$P_0 = \begin{pmatrix} -a \\ 0 \end{pmatrix}$, $P_1 = \begin{pmatrix} -a \\ \frac{b}{2} \end{pmatrix}$, $P_2 = \begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix}$, $P_3 = \begin{pmatrix} a \\ 0 \end{pmatrix}$ into (2), then the coordinates of rational cubic trigonometric Bézier Curve are

$$x(t) = a(\sin\frac{\pi}{2}t - \cos\frac{\pi}{2}t)$$
$$y(t) = \frac{b}{2}(\sin\frac{\pi}{2}t + \cos\frac{\pi}{2}t - 1)$$

This gives the intrinsic equation $$\left(\frac{x(t)}{(\sqrt{2})a}\right)^2 + \left(\frac{y(t)+1}{\frac{b}{(\sqrt{2})}}\right)^2 = 1 \quad (4)$$

It is an equation of ellipse. Fig.7(a) shows the ellipse.

Corollary 5.2: According to theorem(5.1), if a = 7.5, b = 12 then the corresponding rational cubic trigonometric Bézier curve with the shape parameters α = β = 0 and local domain $t \in [0,4]$, represents arc of an circle. Fig.7 (b) shows the circle.

6. Approximability

Control polygons provide an important tool in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show some relations of the rational cubic trigonometric Bézier curves and the rational cubic Bézier curves corresponding to their control polygons.

Theorem 6.1 Suppose P_0, P_1, P_2, P_3 are not collinear; The relationships between rational cubic trigonometric Bézier curve T(t) (2) and rational cubic Bézier curve $B(t) = \frac{\sum_{i=0}^{3} m_i P_i B_i(t)}{\sum_{i=0}^{3} m_i B_i(t)}$ (5)

where $B_i(t) = \sum_{i=0}^{3} \binom{3}{i} (1-t)^{3-i} t^i$ (6)

$t \in [0,1]$, with the same control points P_i (i = 0,1,2,3) and m_i is scalar, called weight function, are as follows:
T(0) = B(0), T(1) = B(1),

$$T\left(\frac{1}{2}\right) - P^* = \frac{7\left(2-\sqrt{2}-\alpha\left(\sqrt{2}-1\right)\right)}{\left(2+3\sqrt{2}+\alpha\left(\sqrt{2}-1\right)\right)}\left(B\left(\frac{1}{2}\right) - P^*\right) \tag{7}$$

where $P^* = \frac{(P_1+P_2)}{2}$ with assumption α = β.

Proof : Let $m_0 = m_3 = 1$, $m_1 = m_2 = 2$
Then rational cubic Bézier curve (5) will be of the form

$$B(t) = \frac{(1-t)^3 P_0 + 6(1-t)^2 t P_0 + 6(1-t) t^2 P_0 + t^3 P_0}{(1-t)^3 + 6(1-t)^2 t + 6(1-t) t^2 + t^3}$$

by simple computation, we get

T(0) $= P_0 =$ B(0), T(1) $= P_3 =$ B(1) and

$$B\left(\frac{1}{2}\right) = \frac{1}{14}\left(P_0 + 6P_1 + 6P_2 + P_3\right)$$

$$B\left(\frac{1}{2}\right) - P^* = \frac{1}{14}\left(P_0 - P_1 - P_2 + P_3\right) \tag{8}$$

For α = β and $m_0 = m_3 = 1$, $m_1 = m_2 = 2$

$$T\left(\frac{1}{2}\right) = \frac{\left(2-\sqrt{2}-\alpha\left(\sqrt{2}-1\right)\right)(P_0+P_3) + 2\left(2\sqrt{2}+\alpha\left(\sqrt{2}-1\right)\right)(P_1+P_2)}{2\left(2+3\sqrt{2}+\alpha\left(\sqrt{2}-1\right)\right)}$$

$$T\left(\frac{1}{2}\right) - P^* = \frac{\left(2-\sqrt{2}-\alpha\left(\sqrt{2}-1\right)\right)(P_0-P_1-P_2+P_3)}{2\left(2+3\sqrt{2}+\alpha\left(\sqrt{2}-1\right)\right)}$$

$$T\left(\frac{1}{2}\right) - P^* = \frac{7\left(2-\sqrt{2}-\alpha\left(\sqrt{2}-1\right)\right)}{\left(2+3\sqrt{2}+\alpha\left(\sqrt{2}-1\right)\right)}\left(B\left(\frac{1}{2}\right) - P^*\right)$$

Corollary 6.1 When $\alpha = \beta = \frac{6-5\sqrt{2}}{4(\sqrt{2}-1)}$, then rational cubic trigonometric Bézier curve is close to rational cubic Bézier curve, i.e. $T\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right)$.

Fig.8 shows the relationship between the rational cubic trigonometric Bézier curve and rational cubic Bézier curve. The rational cubic trigonometric Bézier curve (blue dashed) with shape parameter $\alpha = \beta = \frac{6-5\sqrt{2}}{4(\sqrt{2}-1)}$ is analogous to the rational cubic Bézier curve (black solid).

7. Conclusion

In this paper, Rational cubic trigonometric Bézier curve with two shape parameters is presented for which all geometric properties are similar to rational cubic Bézier curve. Due to shape parameters it is more useful to design different shapes as compared to rational cubic Bézier curve. The curve exactly represents the arc of an ellipse and circle under certain conditions.

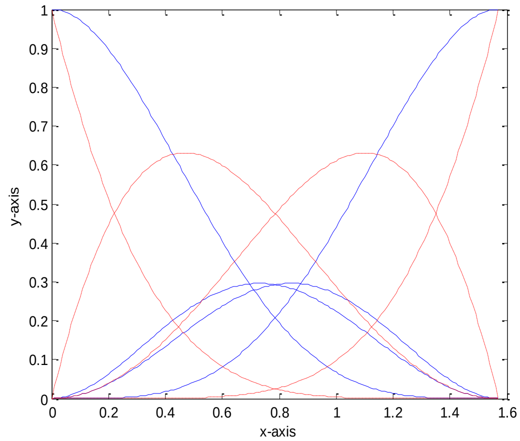

Fig.1 : The cubic trigonometric basis functions

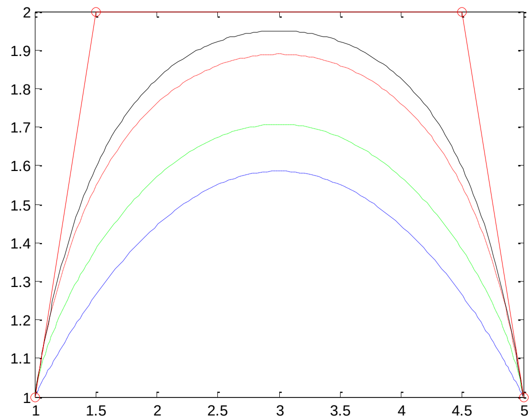

Fig.2 : The effect on the shape of rational cubic trigonometric Bézier curves with altering the values of α and β simultaneously

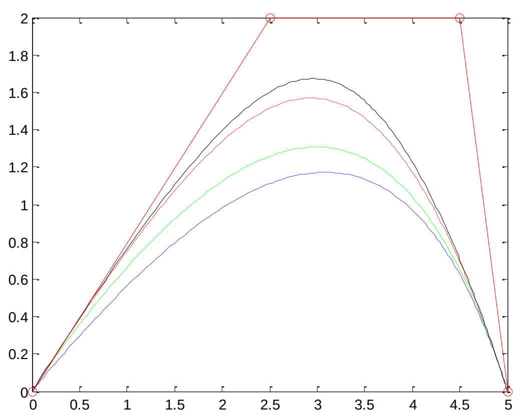

Fig.3 : The effect on the shape of rational cubic trigonometric Bézier curves with altering the values of α and β

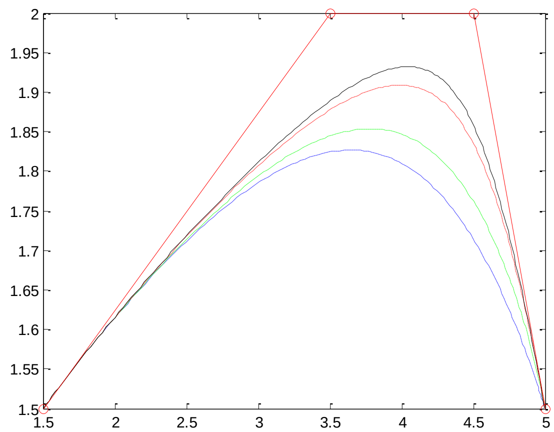

Fig.4 : The effect on the shape of rational cubic trigonometric Bézier curves with altering the values of α and β

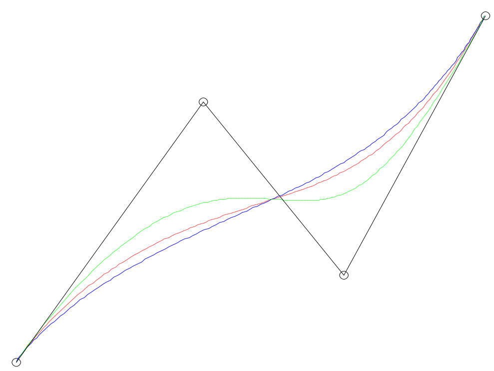

Fig.5 : Open rational cubic trigonometric Bézier curves with altering the values of α and β

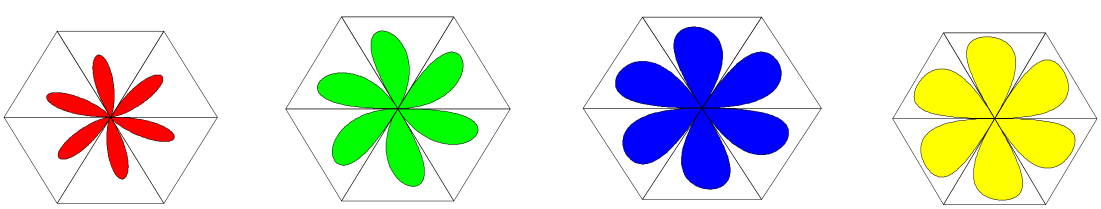

Fig.6(a) α = -1.5, β = -0.5 **Fig.6(b) α = -0.5, β = 0.5** **Fig.6(c) α = 0.5, β = 0.5** **Fig.6(d) α = 1, β = 1**

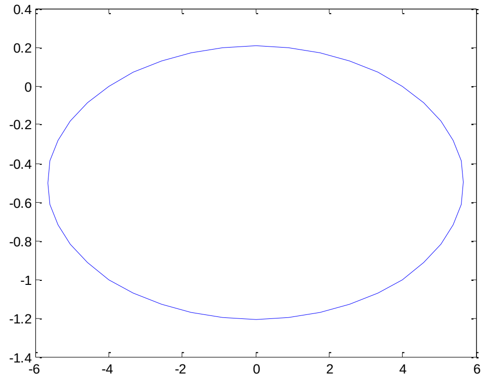

Fig.7(a) The representation of ellipse

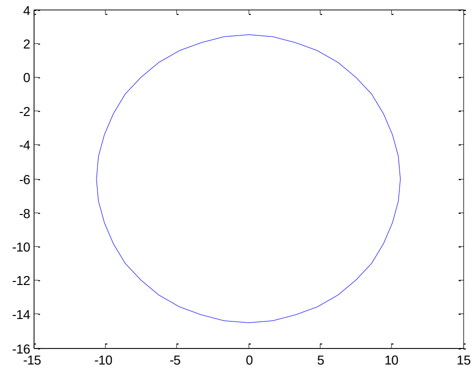

Fig.7(b) The representation of circle

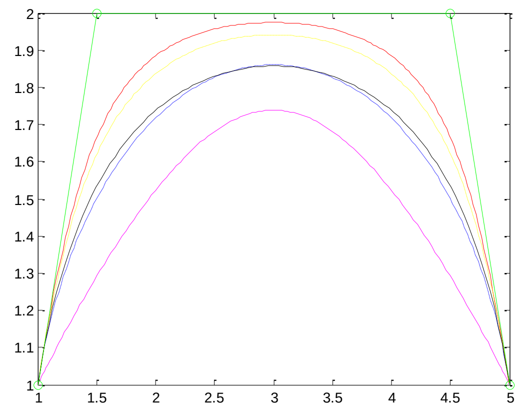

Fig.8 The relationship between the rational cubic trigonometric Bézier curve and rational cubic Bézier curve

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