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RESEARCH ARTICLE

Markov Chain Monte Carlo and Lindley's Approximation for Estimation Weibull Distribution Based on Right Censored Data

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Abstract

This paper contains the Bayesian Estimator using Markov Chain Monte Carlo and Lindley's approximationestimation of the Weibull distribution based on Type I censored data. The Maximum likelihood estimation is used, where the shape parameter estimation is not available in closed forms, although it can be solved by numerical methods. Moreover, the Bayesian estimates of the parameters, the survival and hazard functions cannot be solved analytically. Hence Markov Chain Monte Carlo and Lindley's approximationmethod are used, where the full conditional distribution for the parameters of Weibull distribution are obtained via Gibbs sampling and Metropolis-Hastings algorithm followed by the survival and hazard functions estimates. The methods are compared to Maximum likelihood counterparts and the comparisons are made with respect to the Mean Square Error (MSE) and absolute bias to determine the better method in parameters, the survival and hazard functions.

Introduction

One of the most appealing classical statistical techniques used for fitting statistical models to data as well as providing estimates for the parameters and survival and hazard functions of a model is the Bayesian approach and maximum likelihood estimation (MLE).Sinha (1986) used Lindley's approximation technique to estimate the survival and hazard functions of Weibull distribution with Jeffreys prior information. Smith and Naylor(1987) developed the maximum likelihood and Bayesian estimators and compared them using the parameter Weibull distribution. Berger and Sun (1993) considered the Bayesian using Gibbs sampler for estimating the Poly Weibull distribution using informative priors. Sun (1997), estimated the two parameters of Weibull distribution where he compared Jeffreys prior with that of the reference prior under Bayesian methods. Hossain and Zimmer (2003) estimated the scale and shape parameters of Weibull distribution based on censored data by maximum likelihood estimator and least squares method. Nassar and Eissa (2005) consider Bayesian approach for type II censored data to estimate the two shape parameters and the reliability function of the Exponentiated Weibull distribution. Singh *et .al.* (2005) obtained Bayesian and Maximum likelihood estimation for the parameter of Exponentiated Weibull distribution with censored data. Soliman *et al*. (2006) estimated Weibull distribution by using maximum likelihood estimator and Bayesian method following by estimated the hazard and reliability functions.Mudholkar *et al*. (1996) described the application of the Weibull distribution in modeling and analyzing survival data. Gupta *et al*. (2008) estimated Weibull extension model by Bayesian approach using Markov Chain Monte Carlo (MCMC). where the Gibbs sampler and the Metropolis-Hasting algorithms are used to simulate sample from the posterior. Kundu and Howlader. (2010) obtained Bayesian inference and prediction of the inverse Weibull distribution with Type-II censored data. Pandey *et al*. (2011) compared between Bayesian and maximum likelihood estimation of the scale parameter in Weibull distribution with known shape under Linex loss function. .Joarder *et al*. (2011) considered the parameters of a Weibull distribution under Type-I censored data where the Bayesian approach using Gibbs sampling technique.

The objective of this paper is to estimate the parameters, survival and hazard functions of the Weibull distribution based on type-I censored data by using Bayesian approach with help of theMarkov Chain Mote Carlo and Lindley's approximationand compared to maximum likelihood estimator by using mean square error (MSE) and absolute bias to determine the best estimator under several conditions.

2. Methodology

2.1. Maximum Likelihood Estimation (MLE)

The likelihood function as given in Klein and Moeschberger, (2003) is

$$L(\theta, p\,|\,t)=\prod_{i=1}^{n}[\,f(t_i;\theta,p)]^{\delta_i}\;[S(t_i;\theta,p)]^{1-\delta_i} \tag{1}$$

The pdf of Weibull distribution is,

$$f(t;\theta,p)=\frac{p}{\theta}t^{\,p-1}\exp(-\frac{t^p}{\theta})$$

The logarithm of the likelihood function is given below,

$$\ln L(\theta\,|\,p,t)=\sum_{i=1}^{n}\left[\delta_i\left(\ln p-\ln\theta+(p-1)\ln t_i\right)-\frac{t_i^p}{\theta}\right] \tag{2}$$

To obtain the equations for the unknown parameters, we differentiate Eq. (2) partially with respect to the scale and shape parameters and equal it to zero. The resulting equations are given respectively as,

$$U_i(\theta)=\frac{\partial L(\theta\,|\,p,t)}{\partial\theta_i}=-\frac{\sum_{i=1}^{n}\delta_i}{\theta}+\frac{\sum_{i=1}^{n}t_i^p}{\theta^2}$$

$$U_i(p)=\frac{\partial L(\theta\,|\,p,t)}{\partial p_i}=\frac{\sum_{i=1}^{n}\delta_i}{p}+\sum_{i=1}^{n}\delta_i\ln(t_i)-\frac{\sum_{i=1}^{n}t_i^p\ln(t_i)}{\theta} \tag{3}$$

The MLE for the scale parameter of Weibull distribution is

$$\hat{\theta}_M=\frac{\sum_{i=1}^{n}t_i^p}{\sum_{i=1}^{n}\delta_i} \tag{4}$$

The shape parameter can't be solved analytically, and for that we used Newton Raphson method to find the numerical solution following (Hossain and Zimmer, 2003)

The survival and hazard functions of Weibull distribution are given as follows

$$\hat{S}_M(t)=\exp\left(-\frac{t^{\hat{p}_M}}{\hat{\theta}_M}\right) \tag{5}$$

$$\hat{h}_M(t)=\frac{\hat{p}_M}{\hat{\theta}_M}t^{\hat{p}_M-1} \tag{6}$$

Where $\hat{\theta}_M$ is the scale parameter estimated by MLE and the $\hat{p}_M$ is the shape parameter estimated by MLE.

2.2 Bayesian using non informative prior (Jeffreys prior)

The Jeffreys prior is the square root of the determinant of the Fisher information matrix parameters per observation as,

$$g_1(\theta, p) \propto \sqrt{(I(\theta, p)}$$

The Fisher information matrix of scale and shape parameters is

$$I(\theta, p) = -E\begin{vmatrix} \dfrac{\partial^2 \ln f(t\,|\,\theta, p)}{\partial\theta^2} & \dfrac{\partial^2 \ln f(t\,|\,\theta, p)}{\partial\theta\,\partial p} \\ \dfrac{\partial^2 \ln f(t\,|\,\theta, p)}{\partial\theta\,\partial p} & \dfrac{\partial^2 \ln f(t\,|\,\theta, p)}{\partial p^2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{\theta^2} & \dfrac{\ln\theta + \Gamma'(2)}{\theta p} \\ \dfrac{\ln\theta + \Gamma'(2)}{\theta p} & \dfrac{1 + (\ln\theta)^2 + \Gamma''(2) + 2\ln\theta\,\Gamma'(2)}{p^2} \end{vmatrix} = \frac{d}{\theta^2 p^2}$$

where,

$$\Gamma'(2) = \int_0^\infty u \ln(u) \exp(-u)\, du$$

$$\Gamma''(2) = \int_0^\infty u \ln^2(u) \exp(-u)\, du \quad , d = 0.6449$$

Then,

$$g_1(\theta, p) = k\,\frac{\sqrt{d}}{\theta p} \tag{7}$$

See Sinha (1986)

The posterior density function is derived as see Al Omari *et al.* (2012)

$$\prod{}_1(\theta, p\,|\,t) = \frac{\prod_{i=1}^{n}[f(t_i;\theta, p)]^{\delta_i}\,[S(t_i;\theta, p)]^{1-\delta_i}\, g_1(\theta, p)}{\int_0^\infty\int_0^\infty \prod_{i=1}^{n}[f(t_i;\theta, p)]^{\delta_i}\,[S(t_i;\theta, p)]^{1-\delta_i}\, g_1(\theta, p)\, d\theta dp}$$

$$= \frac{\dfrac{1}{\theta^{\sum_{i=1}^{n}\delta_i + 1}}\, p^{\sum_{i=1}^{n}\delta_i - 1} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\dfrac{\sum_{i=1}^{n} t_i^p}{\theta}\right)}{(\sum_{i=1}^{n}\delta_i - 1)!\int_0^\infty \dfrac{p^{\sum_{i=1}^{n}\delta_i - 1}}{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i}} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} dp} \tag{8}$$

The Bayes estimates for the scale and shape parameters under squared error loss function are given as:

$$\hat{\theta}_{BJ} = \frac{\int_0^\infty p^{\sum_{i=1}^n \delta_i - 1} \frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n \delta_i - 1\right)\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n \delta_i - 1}} \, dp}{\int_0^\infty p^{\sum_{i=1}^n \delta_i - 1} \frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n \delta_i}} \, dp} \tag{9}$$

$$\hat{p}_{BJ} = \frac{\int_0^\infty p^{\sum_{i=1}^n \delta_i} \frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n \delta_i}} \, dp}{\int_0^\infty p^{\sum_{i=1}^n \delta_i - 1} \frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n \delta_i}} \, dp} \tag{10}$$

The integration of the scale and shape parameter we can't solve it analytical for that Markov Chain Monte Carlo and Lindley's approximation are used to solve the problem.

The survival function of the Weibull distribution as shown below

$$\hat{S}_{BJ}(t/\theta, p) = \frac{\int_0^\infty\int_0^\infty \exp\left(-\frac{t^p}{\theta}\right) L(t;\theta, p, \delta) g_1(\theta, p)\, d\theta dp}{\int_0^\infty\int_0^\infty L(t;\theta, p, \delta) g_1(\theta, p)\, d\theta dp}$$

$$= \frac{\int_0^\infty p^{\sum_{i=1}^n \delta_i - 1} \frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n t_i^p + t^p\right)^{\sum_{i=1}^n \delta_i}} \, dp}{\int_0^\infty p^{\sum_{i=1}^n \delta_i - 1} \frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n \delta_i}} \, dp} \tag{11}$$

The hazard function is given below,

$$\hat{h}_{BJ}(t/\theta,p)=\frac{\int_0^\infty\int_0^\infty \frac{p}{\theta}t_i^{p-1}L(t;\theta,p,\delta)g_1(\theta,p)\,d\theta dp}{\int_0^\infty\int_0^\infty L(t;\theta,p,\delta)g_1(\theta,p)\,d\theta dp}$$

$$=\frac{\sum_{i=1}^{n}\delta_i\int_0^\infty p^{\sum_{i=1}^{n}\delta_i}\,t_i^{p-1}\,\frac{\prod_{i=1}^{n}t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^{n}t_i^p\right)^{\sum_{i=1}^{n}\delta_i+1}}\,dp}{\int_0^\infty p^{\sum_{i=1}^{n}\delta_i-1}\,\frac{\prod_{i=1}^{n}t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^{n}t_i^p\right)^{\sum_{i=1}^{n}\delta_i}}\,dp} \tag{12}$$

The equation of the survival and hazard functions we can't solve it analytical for that we used Markov Chain Monte Carlo and Lindley's approximation.

2.2.1 Gibbs Sampling for Jeffreys Prior

The posterior probability density function of θ and p given in Equation (8) for Bayesian using Jeffreys prior can we rewrite as follows

$$\Pi(\theta,p\,|\,t)=\frac{\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+1}}\,p^{\sum_{i=1}^{n}\delta_i-1}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^p}{\theta}\right)}{\int_0^\infty\int_0^\infty\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+1}}\,p^{\sum_{i=1}^{n}\delta_i-1}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^p}{\theta}\right)d\theta\,dp} \tag{13}$$

We propose to use the Gibbs sampling technique to estimate the scale parameter.
Therefore, the full conditional of the posterior density function of scale and shape given the data are combining the Jeffreys prior with likelihood as given below

$$\Pi_1(\theta,p\,|\,t)=\frac{1}{J_1}L(\theta,p\,|\,t)*g_1(\theta,p)$$

$$=\frac{1}{J_1\theta^{\sum_{i=1}^{n}\delta_i+1}}\,p^{\sum_{i=1}^{n}\delta_i-1}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^p}{\theta}\right) \tag{14}$$

Where,

$$J_1 = \int_0^\infty \int_0^\infty \frac{1}{\theta^{\sum_{i=1}^{n}\delta_i + 1}} p^{\sum_{i=1}^{n}\delta_i - 1} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right) d\theta \, dp$$

From equation (14) we can get the conditional posterior of the scale parameter θ as given below

$$\Pi_1(\theta \mid p,t) \propto \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i}}{\Gamma\left(\sum_{i=1}^{n}\delta_i\right)} \theta^{-\sum_{i=1}^{n}\delta_i - 1} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right) \tag{15}$$

The conditional posterior of the scale parameter is follow inverse gamma density function with scale and shape parameters $\sum_{i=1}^{n} \delta_i, \sum_{i=1}^{n} t_i^p$ respectively, we propose to use Gibbs sampling technique as shown in Algorithm 1to get the estimation of scale parametric, see for example in Gibbs sampling Smith and Roberts (1993).

Algorithm 1:

1. Start with initial value θ^0
2. Generate the scale parameter θ from inverse gamma $\left(\sum_{i=1}^{n} \delta_i, \sum_{i=1}^{n} t_i^p\right)$
3. Generate $\theta_1, \ldots, \theta_n$
4. The Bayesian estimation of the scale parameter is
   $\hat{E}_1(\theta \mid p,t) = \frac{1}{n}\sum_{i=1}^{n} \theta_i$
5. Obtain the posterior variance of scale parameter as

$$\hat{V}_1(\theta \mid p,t) = \frac{1}{n}\sum_{i=1}^{n} \left(\theta_i - \hat{E}(\theta \mid p,t)\right)^2$$

2.2.2 Metropolis-Hastings Algorithm

The Metropolis- Hastings algorithm is a very general Markov Chain Mote Carlo method, it can be used to obtain random samples from any arbitrarily complicated target distribution of any dimension that is known up to a normalizing constant. In fact, Metropolis algorithm is an alternative to Gibbs sampler that does not require availability of full conditionals see Hastings (1970) and Soliman *et al*, (2011).
The conditional posterior of the shape parametric *p* is given below

$$\Pi_1(p \mid \theta,t) \propto p^{\sum_{i=1}^{n}\delta_i - 1} \prod_{i=1}^{n} t_i^{\delta_i (p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta} \right) \tag{16}$$

As show in the conditional posterior of the shape parameter *p* it's not follow any close distribution for that we suggest to use the Metropolis-Hastings algorithm to generate MCMC sample,
Algorithm 2 are used to estimate the shape parametric *p* by Metropolis-Hastings sample technique as follows

Algorithm 2:

1. Start with initial value p_0

2. The current value p_i and generate the candidate value p^* from arbitrary distribution Uniform (0, 1)

3. Taken the ratio at the candidate value p^* and current value p_i

$$\alpha = \min\left\{1,\ \frac{\Pi(p^* \mid \theta, t)}{\Pi(p_i \mid \theta, t)}\right\}$$

4. The next value for the p_i is given below as

$$p_{i+1} = \begin{cases} p^* \text{with probability } \alpha \\ p_i \text{with probability } 1\text{-}\alpha \end{cases}$$

5. Generate *u* from Uniform (0, 1)

6. Accept p^* with probability α if $p^* < \alpha$ and return to step 2, otherwise accept p_i and return to step 2.

7. The Bayesian estimation of the shape parameter *p* under the squared error loss function is given as

$$\hat{E}_1(p \mid \theta, t) = \frac{1}{n}\sum_{i=1}^{n} p_i$$

8. Obtain the posterior variance of shape parameter for Bayesian using Jeffreys prior as

$$\hat{V}_1(\theta \mid p, t) = \frac{1}{n}\sum_{i=1}^{n}\left(\theta_i - \hat{E}(\theta \mid p, t)\right)^2$$

In survival and hazard functions we can solve it according to the estimation scale parameter θ_i from Algorithm 1 and shape parameter p_i from Algorithm 2, which has density as follows,

$$S_1^*(t) = \int_0^\infty\int_0^\infty S(t)\,\Pi_1(\theta, p \mid t)\,d\theta dp$$

$$\approx \frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{t_j^{p_i}}{\theta_i}\right) \quad (17)$$

$$h_1^*(t) = \int_0^\infty\int_0^\infty h(t)\,\Pi_1(\theta, p \mid t)\,d\theta dp$$

$$\approx \frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{\theta_i} t_j^{p_i - 1} \quad (18)$$

Where the θ_i is the Bayesian estimation using Jeffreys prior of the scale parameter from Gibbs sampling (Algorithm 1) and p_i is the shape parameter estimation by Metropolis-Hastings sample technique as shown in Algorithm 2.

2.3 Bayesian using Extension of Jeffreys Prior Estimation

The extension of Jeffreys prior is, $g_2(\theta, p) \propto [(I(\theta, p)]^c, c \in R^+$

$$\text{Then, } g_2(\theta, p) = k\,\frac{d^c}{\theta^{2c} p^{2c}} \quad (19)$$

The posterior probability density function of extension of Jeffreys prior of θ and *p* given as follows

$$\prod{}_2(\theta, p\,|\,t) = \frac{\dfrac{1}{\theta^{\sum_{i=1}^{n}\delta_i+2c}}\, p^{\sum_{i=1}^{n}\delta_i-2c} \prod_{i=1}^{n} t_i^{\delta_i(p-1)} \exp\left(-\dfrac{\sum_{i=1}^{n} t_i^p}{\theta}\right)}{(\sum_{i=1}^{n}\delta_i + 2c-2)! \int_0^\infty \dfrac{p^{\sum_{i=1}^{n}\delta_i-2c}}{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i}} \prod_{i=1}^{n} t_i^{\delta_i(p-1)} dp}$$

The Bayesian estimation using extension of Jeffreys prior of the scale parameter under squared error loss function is given as:

$$\hat{\theta}_{BE} = \int_0^\infty p^{\sum_{i=1}^{n}\delta_i-2c} \frac{\prod_{i=1}^{n} t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^{n}\delta_i + 2c-2\right)\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i+2c-2}} \quad dp \tag{20}$$

And the shape parameter under squared error loss function by Bayesian using extension of Jeffreys prior is given as:

$$\hat{p}_{BE} = \int_0^\infty p^{\sum_{i=1}^{n}\delta_i-2c+1} \frac{\prod_{i=1}^{n} t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i+2c-1}} \quad dp \tag{21}$$

$$where \quad J_1 = \int_0^\infty p^{\sum_{i=1}^{n}\delta_i-2c} \frac{\prod_{i=1}^{n} t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i+2c-1}} \quad dp$$

In Bayesian estimation using extension of Jeffreys prior, the survival function of the Weibull distribution are taking the integration for the scale and shape parameters for the survival function of the Weibull distribution combining with the posterior as shown below

$$\hat{S}_{BE}(t/\theta, p) = \int_0^\infty p^{\sum_{i=1}^{n}\delta_i-2c} \frac{\prod_{i=1}^{n} t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^{n} t_i^p + t_i^p\right)^{\sum_{i=1}^{n}\delta_i+2c-1}} dp \tag{22}$$

In Bayesian estimation using extension of Jeffreys prior, the hazard function of the Weibull distribution are taking the integration for the scale and shape parameters for the hazard function of the Weibull distribution combining with the posterior as shown below

$$\hat{h}_{BE}(t/\theta,p)=(\sum_{i=1}^{n}\delta_i+2c-1)\int_0^{\infty}p^{\sum_{i=1}^{n}\delta_i-2c+1}\,t_i^{p-1}\frac{\prod_{i=1}^{n}t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^{n}t_i^{p}\right)^{\sum_{i=1}^{n}\delta_i+2c}}dp \tag{23}$$

The posterior probability density function of θ and p given the data with extension of Jeffreys prior is combining equation (19) with likelihood function and using Bayesian theorem, the joint posterior distribution is derived as see Al Omari& Ibrahim. (2010)

$$\Pi_2(\theta,p\,|\,t)=\frac{\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+2c}}\,p^{\sum_{i=1}^{n}\delta_i-2c}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^{p}}{\theta}\right)}{\int_0^{\infty}\int_0^{\infty}\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+2c}}\,p^{\sum_{i=1}^{n}\delta_i-2c}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^{p}}{\theta}\right)d\theta\,dp} \tag{24}$$

Also, the posterior using extension of Jeffreys prior in equation (24) cannot be computed explicitly in this case, we propose to use the Gibbs sampling technique to generate MCMC samples for estimation the scale parametric.
Therefore, the full conditional of the posterior density function using extension of Jeffreys prior of θ and p given the data are combining the extension of Jeffreys prior with likelihood as given below

$$\Pi_2(\theta,p\,|\,t)=\frac{1}{J_2}L(\theta,p\,|\,t)*g_2(\theta,p)$$

$$=\frac{1}{J_2\theta^{\sum_{i=1}^{n}\delta_i+2c}}\,p^{\sum_{i=1}^{n}\delta_i-2c}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^{p}}{\theta}\right) \tag{25}$$

Where,

$$J_2=\int_0^{\infty}\int_0^{\infty}\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+2c}}\,p^{\sum_{i=1}^{n}\delta_i-2c}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^{p}}{\theta}\right)d\theta dp$$

From equation (24) we can get the conditional posterior using extension of Jeffreys prior of the scale parametric θ as follows

$$\Pi_2(\theta\,|\,p,t)\propto\frac{\left(\sum_{i=1}^{n}t_i^{p}\right)^{\sum_{i=1}^{n}\delta_i+2c-1}}{\Gamma\left(\sum_{i=1}^{n}\delta_i+2c-1\right)}\theta^{-\sum_{i=1}^{n}\delta_i-2c}\exp\left(-\frac{\sum_{i=1}^{n}t_i^{p}}{\theta}\right) \tag{26}$$

Now we saw that, the conditional posterior with extension of Jeffreys prior of the scale parametric θ is follow inverse gamma density function with scale and shape parameters $\left(\sum_{i=1}^{n}\delta_i + 2c - 1\right), \sum_{i=1}^{n} t_i^p$ respectively, we propose to use Gibbs sampling technique to generate MCMC sample

Also from equation (24) we can get the conditional posterior using extension of Jeffreys prior of the shape parameter *p* as given below,

$$\Pi_2(p\,|\,\theta,t) \propto p^{\sum_{i=1}^{n}\delta_i - 2c} \prod_{i=1}^{n} t_i^{\delta_i(p-1)} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right) \tag{27}$$

As show in the conditional posterior for extension of Jeffreys of the shape parameter *p* it's not follow any close distribution for that we suggest to use the Metropolis-Hastings algorithm to generate MCMC sample

The survival and hazard functions as follow,

$$S_2^*(t) = \int_0^\infty\int_0^\infty S(t)\,\Pi_2(\theta,p\,|\,t)\,d\theta dp$$

$$\approx \frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{t_j^{p_{Ei}}}{\theta_{Ei}}\right) \tag{28}$$

$$h_2^*(t) = \int_0^\infty\int_0^\infty h(t)\,\Pi_2(\theta,p\,|\,t)\,d\theta dp$$

$$\approx \frac{1}{n}\sum_{i=1}^{n}\frac{p_{Ei}}{\theta_{Ei}}\,t_j^{p_{Ei}-1} \tag{29}$$

Where the θ_{Ei} is the estimation of the scale parameter from Gibbs sampling and p_{Ei} the estimation of the by Metropolis-Hastings.

2.4 Lindley's Approximation

As shown the integrals involved in (9) to (12) and (20) to (23) cannot be solved analytically and for that we obtained Lindley's Expansion to solve the parameters approximation.

According to Abdel-Wahid (1987), Lindley proposed a ratio of integral of the form

$$\int w(\theta)\exp\{L(\theta)\}\,d\theta \Big/ \int \upsilon(\theta)\exp\{L(\theta)\}\,d\theta$$

Where $L(\theta)$ is the log-likelihood and $w(\theta),\upsilon(\theta)$ are arbitrary functions of θ In applying this procedure, it is assumed that $\upsilon(\theta)$ is the prior distribution for θ and $w(\theta) = u(\theta).\upsilon(\theta)$ with $u(\theta)$ being some function of interest.

The posterior expectation according to Sinha (1986) is

$$E(u(\theta)\,|\,t) = \int \upsilon(\theta)\exp\{L(\theta)+\rho(\theta)\}\,d\theta \Big/ \int \exp\{L(\theta)+\rho(\theta)\}\,d\theta$$

Lindley expansion is therefore approximated asymptotically by

$$E(u(\theta)\,|\,t) = u + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + u_1\rho_1\sigma_{11} + u_2\rho_2\sigma_{22} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2 + L_{03}u_2\sigma_{22}^2)$$

where *L* is the log-likelihood equation in (2). Taking the scale parameter θ estimation, where

$$u=\theta, u_1=\frac{\partial u}{\partial \theta}=1$$

$$u_2=u_{11}=u_{22}=0$$

For the shape parameter p

$$u=p, u_2=\frac{\partial u}{\partial p}=1$$

$$u_1=u_{11}=u_{22}=0$$

For the survival function

$$u=\exp\left(-\frac{t^p}{\theta}\right)$$

$$u_1=\frac{\partial u}{\partial \theta}=\frac{t^p}{\theta^2}u, \quad u_2=\frac{\partial u}{\partial p}=-\frac{t^p \ln t}{\theta}u$$

$$u_{11}=\frac{\partial^2 u}{\partial \theta^2}=\frac{t^p}{\theta^3}u\left(\frac{t^p}{\theta}-2\right), \quad u_{22}=\frac{\partial^2 u}{\partial p^2}=\frac{t^p(\ln t)^2}{\theta}u\left(\frac{t^p}{\theta}-1\right)$$

For the Hazard function

$$u=\frac{p}{\theta}t^{p-1} \quad u_1=\frac{\partial u}{\partial \theta}=\frac{-u}{\theta}, \quad u_2=\frac{\partial u}{\partial p}=-\frac{t^{p-1}+pt^{p-1}\ln t}{\theta}$$

$$u_{11}=\frac{\partial^2 u}{\partial \theta^2}=\frac{2u}{\theta^2}, \quad u_{22}=\frac{\partial^2 u}{\partial p^2}=\frac{t^{p-1}\ln t\left(2+pt^{p-1}(\ln t)^2\right)}{\theta}$$

$$\rho_1=\frac{\partial \rho}{\partial \theta}=-\frac{1}{\theta}, \quad \rho_2=\frac{\partial \rho}{\partial p}=-\frac{1}{p}$$

$$\sigma_{11}=(-L_{20})^{-1}, \quad \sigma_{22}=(-L_{02})^{-1}$$

$$L_{20}=\frac{\partial^2 L}{\partial \theta^2}=\frac{\sum_{i=1}^{n}\delta_i-2n}{\theta^2}$$

$$L_{02}=\frac{\partial^2 L}{\partial p^2}=-\frac{\sum_{i=1}^{n}\delta_i}{p^2}-\frac{\sum_{i=1}^{n}t_i^p(\ln t_i)^2}{\theta}$$

$$L_{30}=\frac{\partial^3 L}{\partial \theta^3}=\frac{4n-2\sum_{i=1}^{n}\delta_i}{\theta^3}$$

$$L_{03}=\frac{\partial^3 L}{\partial p^3}=\frac{2\sum_{i=1}^{n}\delta_i}{p^3}-\frac{\sum_{i=1}^{n}t_i^p(\ln t_i)^3}{\theta}$$

For extension of Jeffreys prior estimator we substitute $\rho_1=\frac{\partial \rho}{\partial \theta}=-\frac{1}{\theta^{2c}}, \quad \rho_2=\frac{\partial \rho}{\partial p}=-\frac{1}{p^{2c}}$

The Bayesian with extension of Jeffreys prior estimation for the scale and shape parameters, the survival function and the hazard function estimator can be obtained in a similar manner.

3. Simulation Study

To assess the performance of the Maximum likelihood and Bayesian using Jeffreys prior and extension of Jeffreys prior with help of the Lindley's Approximation and Markov Chain Monte Carlo, where the Gibbs sampling are used to estimate the scale parameter and Metropolis-Hastings Algorithm to estimate the shape parameters, follow by estimated the survival and hazard functions , the estimates and the mean squared errors (MSE) and bias for each method were calculated using 10,000 replications for sample size *n*=25, 50 and 100 of Weibull distribution with censored data for different value of parameters were the scale parametric θ = 1 and 2, shape parametric p = 0.5 and 1.5 and the two values of Jeffreys extension were c=1.5 and 3, the considered values of θ, *p* and *c* are meant for illustration only and other values can also be taken for generating the samples from Weibull distribution.

Table 1: Estimated scale parameter

Size	Estimators	θ =1		θ =2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	1.1251	1.1273	1.7258	2.2306
	BJ	1.1556	1.1685	1.7029	2.2621
	BE(*c*=1.5)	1.0283	1.0369	1.7058	2.2579
	BE(*c*=3)	1.0342	1.0253	1.6902	2.2477
	BG	1.0511	0.9623	1.8235	1.8026
	BM(*c*=1.5)	0.9950	0.9932	1.8866	1.9038
	BM(*c*=3)	0.9893	0.9855	1.8354	1.8576
50	MLE	1.1091	1.1086	1.7512	2.1634
	BJ	1.1866	1.1786	1.6457	2.1752
	BE(*c*=1.5)	1.0721	1.0620	1.6451	2.2271
	BE(*c*=3)	1.2063	1.2097	1.5992	2.2152
	BG	1.0798	0.9596	1.8164	1.8504
	BM(*c*=1.5)	1.0531	0.9720	1.8901	1.9353
	BM(*c*=3)	1.0638	0.9630	1.8690	1.9153
100	MLE	1.0106	1.0054	1.8238	2.1117
	BJ	1.0585	1.0376	1.7671	2.1570
	BE(*c*=1.5)	1.0102	1.0024	1.7661	2.1815
	BE(*c*=3)	1.1083	1.1040	1.7435	2.1558
	BG	1.0272	0.9769	1.8962	1.9042
	BM(*c*=1.5)	0.9976	0.9922	1.9324	1.9634
	BM(*c*=3)	0.9892	0.9896	1.9164	1.9369

Table 2: Estimated shape parameter

Size	Estimators	θ =1		θ =2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.9266	1.1992	1.8244	2.2873
	BJ	0.9218	1.2183	1.8213	2.2987
	BE(*c*=1.5)	0.9246	1.2096	1.7973	2.3192
	BE(*c*=3)	0.9181	1.2328	1.7606	2.3477
	BG	0.9147	0.8676	1.7546	1.7261
	BM(*c*=1.5)	0.9351	0.8747	1.8525	1.7459
	BM(*c*=3)	0.9122	0.8761	1.8693	1.7845
50	MLE	0.9341	1.1501	1.9111	2.1592
	BJ	0.9302	1.1685	1.9096	2.1604
	BE(*c*=1.5)	0.9313	1.1691	1.8973	2.1606
	BE(*c*=3)	0.9243	1.1701	1.8789	2.1699
	BG	0.9264	0.8930	1.8529	1.8481
	BM(*c*=1.5)	0.9463	0.8896	1.9365	1.8551
	BM(*c*=3)	0.9266	0.8718	1.9211	1.8796
100	MLE	0.9594	1.0802	1.9366	2.0221
	BJ	0.9394	1.1241	1.9158	2.0396

	BE(c=1.5)	0.9425	1.1242	1.9096	2.0697
	BE(c=3)	0.9420	1.1147	1.8904	2.1143
	BG	0.9409	0.9238	1.8722	1.8760
	BM(c=1.5)	0.9605	0.9126	1.9440	1.8953
	BM(c=3)	0.9155	0.9111	1.9318	1.9135

Table 3: MSE of scale parameter

Size	Estimators	θ=1		θ=2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.0603	0.0696	0.2045	0.5162
	BJ	0.1057	0.1177	0.2610	0.5541
	BE(c=1.5)	0.0583	0.0707	0.2192	0.5251
	BE(c=3)	0.0966	0.0759	0.2341	0.5232
	BG	0.0661	0.0550	0.2015	0.5076
	BM(c=1.5)	0.0501	0.0381	0.1662	0.2227
	BM(c=3)	0.0601	0.0413	0.1947	0.4129
50	MLE	0.0288	0.0258	0.1753	0.4481
	BJ	0.0439	0.0442	0.2032	0.5005
	BE(c=1.5)	0.0275	0.0264	0.1892	0.4498
	BE(c=3)	0.0376	0.0370	0.1907	0.4985
	BG	0.0286	0.0224	0.1696	0.3870
	BM(c=1.5)	0.0252	0.0193	0.1272	0.3386
	BM(c=3)	0.0274	0.0232	0.1594	0.3673
100	MLE	0.0127	0.0151	0.1445	0.3203
	BJ	0.0154	0.0188	0.1811	0.3539
	BE(c=1.5)	0.0123	0.0156	0.1528	0.3302
	BE(c=3)	0.0143	0.0171	0.1463	0.3390
	BG	0.0155	0.0127	0.1378	0.3131
	BM(c=1.5)	0.0116	0.0105	0.0926	0.2501
	BM(c=3)	0.0125	0.0113	0.1076	0.2947

Table 4: MSE of shape parameter

Size	Estimators	θ=1		θ=2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.01078	0.10941	0.01081	0.09022
	BJ	0.01087	0.12021	0.01098	0.09125
	BE(c=1.5)	0.01079	0.12590	0.01164	0.09041
	BE(c=3)	0.01257	0.11717	0.01261	0.09170
	BG	0.01195	0.13837	0.01235	0.09167
	BM(c=1.5)	0.00914	0.11410	0.00945	0.09233
	BM(c=3)	0.01169	0.18931	0.00914	0.09255
50	MLE	0.00432	0.03937	0.00427	0.04113
	BJ	0.00433	0.04114	0.00438	0.04251
	BE(c=1.5)	0.00440	0.04215	0.00432	0.04203
	BE(c=3)	0.00509	0.04052	0.00496	0.04273
	BG	0.00441	0.04182	0.00439	0.04358
	BM(c=1.5)	0.00382	0.03957	0.00401	0.04297
	BM(c=3)	0.00436	0.04126	0.00393	0.04335
100	MLE	0.00211	0.01950	0.00205	0.01843
	BJ	0.00212	0.01943	0.00209	0.01890
	BE(c=1.5)	0.00214	0.02017	0.00208	0.01883
	BE(c=3)	0.00233	0.02174	0.00217	0.01851
	BG	0.00235	0.02163	0.00206	0.01869
	BM(c=1.5)	0.00203	0.01976	0.00178	0.01872

	BM(c=3)	0.00228	0.02196	0.00183	0.01928

Table 5: MSE of survival function

Size	Estimators	θ =1		θ =2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.00563	0.00517	0.01362	0.01426
	BJ	0.00775	0.00566	0.01551	0.01570
	BE(c=1.5)	0.00628	0.00538	0.01533	0.01682
	BE(c=3)	0.00562	0.00573	0.01361	0.01757
	BG	0.00619	0.00557	0.01419	0.01496
	BM(c=1.5)	0.00532	0.00542	0.01182	0.01449
	BM(c=3)	0.00551	0.00532	0.01202	0.01445
50	MLE	0.00251	0.00266	0.01110	0.01124
	BJ	0.00307	0.00322	0.01346	0.01378
	BE(c=1.5)	0.00323	0.00331	0.01356	0.01401
	BE(c=3)	0.00244	0.00339	0.00993	0.01423
	BG	0.00303	0.00270	0.01218	0.01231
	BM(c=1.5)	0.00216	0.00272	0.00955	0.01295
	BM(c=3)	0.00238	0.00284	0.00984	0.01275
100	MLE	0.00124	0.00120	0.01019	0.01006
	BJ	0.00137	0.00130	0.01173	0.01279
	BE(c=1.5)	0.00141	0.00132	0.01171	0.01327
	BE(c=3)	0.00123	0.00134	0.00966	0.01293
	BG	0.00132	0.00121	0.01172	0.01125
	BM(c=1.5)	0.00113	0.00122	0.00915	0.01137
	BM(c=3)	0.00119	0.00123	0.00934	0.01176

Table 6: MSE of hazard function

Size	Estimators	θ =1		θ =2	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	2.5547	0.2558	2.7446	0.1058
	BJ	2.6206	0.3473	2.8625	0.1408
	BE(c=1.5)	2.4481	0.3249	2.8208	0.1363
	BE(c=3)	2.6462	0.3419	2.8816	0.1224
	BG	2.5711	0.3204	2.7632	0.1359
	BM(c=1.5)	2.3154	0.3217	2.6616	0.1174
	BM(c=3)	2.3554	0.3020	2.7114	0.1196
50	MLE	1.9296	0.0949	1.8737	0.0847
	BJ	2.1553	0.0995	1.9119	0.1228
	BE(c=1.5)	1.8913	0.0972	1.9103	0.1299
	BE(c=3)	2.1214	0.1110	1.9148	0.1285
	BG	2.1244	0.1011	1.9188	0.1011
	BM(c=1.5)	1.7541	0.0995	1.8719	0.1186
	BM(c=3)	1.8366	0.1281	1.8350	0.1207
100	MLE	1.5137	0.0447	1.1829	0.0773
	BJ	1.6887	0.0497	1.2718	0.0952
	BE(c=1.5)	1.4677	0.0484	1.2482	0.0989
	BE(c=3)	1.5266	0.0472	1.2926	0.0982
	BG	1.5256	0.0458	1.1853	0.0885
	BM(c=1.5)	1.3399	0.0450	1.1665	0.0861
	BM(c=3)	1.4567	0.0463	1.1552	0.0848

Table 7: Bias of scale parameter

Size	Estimators	$\theta=1$		$\theta=2$	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.1868	0.1891	0.2739	0.4552
	BJ	0.2649	0.2877	0.3199	0.4881
	BE(*c*=1.5)	0.1641	0.2569	0.2877	0.4604
	BE(*c*=3)	0.2413	0.2647	0.2957	0.4793
	BG	0.1987	0.1622	0.2696	0.4402
	BM(*c*=1.5)	0.1799	0.1405	0.2446	0.4135
	BM(*c*=3)	0.1859	0.1501	0.2563	0.4347
50	MLE	0.1342	0.1322	0.2501	0.3944
	BJ	0.1581	0.1609	0.2854	0.4205
	BE(*c*=1.5)	0.1277	0.1499	0.2691	0.4021
	BE(*c*=3)	0.1508	0.1515	0.2778	0.4118
	BG	0.1454	0.1221	0.2426	0.3799
	BM(*c*=1.5)	0.1164	0.1183	0.2138	0.2776
	BM(*c*=3)	0.1215	0.1318	0.2351	0.3832
100	MLE	0.0868	0.0932	0.1769	0.3573
	BJ	0.0952	0.1017	0.2046	0.3868
	BE(*c*=1.5)	0.0967	0.1015	0.1856	0.3605
	BE(*c*=3)	0.0985	0.0994	0.1981	0.3755
	BG	0.0957	0.0901	0.1729	03478
	BM(*c*=1.5)	0.0816	0.0852	0.1516	0.2489
	BM(*c*=3)	0.0864	0.0877	0.1633	0.3115

Table 8: Absolute bias of shape parameter

Size	Estimators	$\theta=1$		$\theta=2$	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.08122	0.23616	0.07588	0.23583
	BJ	0.08413	0.24572	0.07664	0.23609
	BE(*c*=1.5)	0.08333	0.25161	0.07895	0.23730
	BE(*c*=3)	0.09368	0.24527	0.08099	0.23902
	BG	0.09243	0.28175	0.07860	0.23713
	BM(*c*=1.5)	0.08055	0.29543	0.07397	0.24103
	BM(*c*=3)	0.08539	0.31422	0.07239	0.25772
50	MLE	0.05058	0.15828	0.05520	0.16226
	BJ	0.05151	0.16144	0.05658	0.16313
	BE(*c*=1.5)	0.05198	0.16343	0.05836	0.16424
	BE(*c*=3)	0.05707	0.16078	0.05980	0.16510
	BG	0.05367	0.21627	0.05729	0.16421
	BM(*c*=1.5)	0.05012	0.16009	0.05351	0.16521
	BM(*c*=3)	0.05278	0.22733	0.05147	0.17055
100	MLE	0.03573	0.10687	0.03557	0.10091
	BJ	0.03670	0.10778	0.03639	0.10164
	BE(*c*=1.5)	0.03727	0.10846	0.03894	0.10191
	BE(*c*=3)	0.03836	0.10781	0.04055	0.10218
	BG	0.04081	0.12575	0.03717	0.10385
	BM(*c*=1.5)	0.03444	0.12726	0.03579	0.10503
	BM(*c*=3)	0.03699	0.12852	0.03540	0.10653

Table 9: Absolute bias of survival function

Size	Estimators	$\theta=1$		$\theta=2$	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	0.0562	0.0563	0.0985	0.0968
	BJ	0.0640	0.0639	0.1031	0.1047
	BE(*c*=1.5)	0.0588	0.0575	0.1077	0.1131
	BE(*c*=3)	0.0583	0.0593	0.0969	0.1174
	BG	0.0572	0.0601	0.0994	0.0993
	BM(*c*=1.5)	0.0559	0.0545	0.0921	0.0988
	BM(*c*=3)	0.0576	0.0556	0.0942	0.0972
50	MLE	0.0393	0.0376	0.0908	0.0864
	BJ	0.0419	0.0401	0.0964	0.0902
	BE(*c*=1.5)	0.0401	0.0395	0.0969	0.0911
	BE(*c*=3)	0.0395	0.0380	0.0892	0.0921
	BG	0.0395	0.0387	0.0911	0.0866
	BM(*c*=1.5)	0.0343	0.0368	0.0862	0.0871
	BM(*c*=3)	0.0364	0.1366	0.0873	0.0889
100	MLE	0.0281	0.0254	0.0858	0.0821
	BJ	0.0295	0.0265	0.0881	0.0867
	BE(*c*=1.5)	0.0299	0.0266	0.0892	0.0876
	BE(*c*=3)	0.0305	0.0268	0.0841	0.0871
	BG	0.0324	0.0271	0.0867	0.0832
	BM(*c*=1.5)	0.0313	0.0234	0.0829	0.0841
	BM(*c*=3)	0.0330	0.0245	0.0833	0.0859

Table 10: Absolute bias of hazard function

Size	Estimators	$\theta=1$		$\theta=2$	
		p=0.5	*p*=1.5	*p=0.5*	*p=1.5*
25	MLE	1.2374	0.3411	1.3594	0.2708
	BJ	1.2801	0.3699	1.3806	0.3065
	BE(*c*=1.5)	1.2363	0.3639	1.3758	0.2976
	BE(*c*=3)	1.3116	0.3522	1.3901	0.2811
	BG	1.2436	0.3773	1.3924	0.2917
	BM(*c*=1.5)	1.1184	0.3676	1.3023	0.3090
	BM(*c*=3)	1.1741	0.3653	1.3285	0.3114
50	MLE	1.2185	0.2170	1.2512	0.2499
	BJ	1.2656	0.2213	1.2688	0.2897
	BE(*c*=1.5)	1.1826	0.2282	1.2831	0.2794
	BE(*c*=3)	1.2444	0.2389	1.3022	0.2759
	BG	1.2238	0.2271	1.2707	0.2513
	BM(*c*=1.5)	1.1594	0.2364	1.2143	0.2541
	BM(*c*=3)	1.1429	0.2361	1.2371	0.2671
100	MLE	0.9281	0.1502	0.9399	0.2180
	BJ	0.9406	0.1522	0.9567	0.2287
	BE(*c*=1.5)	0.8934	0.1528	0.9645	0.2336
	BE(*c*=3)	0.9362	0.1572	0.9455	0.2318
	BG	0.9379	0.1532	0.9409	0.2319
	BM(*c*=1.5)	0.8718	0.1535	0.9199	0.2298
	BM(*c*=3)	0.8721	0.1565	0.9208	0.2217

4. Discussion.

As shown in Table (1), the estimate of the scale parameter of Weibull distribution is obtained by Maximum likelihood (MLE), Bayesian using Lindley's approximation with Jeffreys prior (BJ) and extension of Jeffreys prior (BE) and Bayesian using Gibbs sampling Jeffreys prior (BG) and extension of Jeffreys prior (BM). Table (2) we estimated the shape parameter of Weibull distribution by employing the Maximum likelihood (MLE), Bayesian using Lindley's approximation with Jeffreys prior (BJ) and extension of Jeffreys prior (BE) and Bayesian using Metropolis-Hastings Algorithm Jeffreys prior (BG) and extension of Jeffreys prior (BM).

As shown in Table (3), the estimate of the scale parameter of Weibull distribution with type – I censored data was compared by mean squared error (MSE). We observed that the Bayesian using Gibbs sampling with extension of Jeffreys prior (BM) is the best compare to the others. Moreover, Bayesian using extension of Jeffreys with help from Lindley's approximation is better than MLE estimators when θ =1 and p=0.5. When the number of sample size increases the mean squared error (MSE) decreases in all cases.
As shown in Table (4), the shape parameter estimates of Weibull distribution with type-I censored data was compared by mean square error (MSE), we observed that the maximum likelihood (MLE) is better compare to others, but, Bayesian using Gibbs sampling with extension of Jeffreys is better than the MLE estimators when shape parameter p=0.5.
In Table (5), when we compare the mean squared error (MSE) of the survival function of Weibull distribution for type-I censored data, we found thatBayesian using Markov Chain Monte Carlo (MCMC) with extension of Jeffreys prior (BM) is better compare to the others when p=0.5, and maximum likelihood estimation is better when shape parametric p=1.5. When the number of sample size increases the mean squared error (MSE) decreases in all cases.

In Tables (6), when we compared the hazard estimates of Weibull distribution with type-I censored data by mean squared error (MSE), we found that Bayesian using Markov Chain Monte Carlo (MCMC) with extension of Jeffreys prior (BM) is better compare to the others when p=0.5, and maximum likelihood estimation is better when shape parametric p=1.5.
In Table (7), when we compared the Absolute bias of the scale parameter of Weibull distribution,we observed that the Bayesian using Gibbs sampling with extension of Jeffreys prior (BM) is the best compare to the others.
In Table (8), when we compare the Absolute bias of the shape parameter of Weibull distribution, we observed that the maximum likelihood (MLE) is better compare to others, but, Bayesian using Gibbs sampling with extension of Jeffreys is better than the MLE estimators when shape parametric p=0.5.
In Table (9), when we compare the Absolute bias of the shape parameter of Weibull distribution, we found thatBayesian using Markov Chain Monte Carlo (MCMC) with extension of Jeffreys prior (BM) is better compare to the others when p=0.5, and maximum likelihood estimation is better when shape parametric p=1.5. When the number of sample size increases the mean squared error (MSE) decreases in all cases.
In Tables (10), when we compared the hazard estimates of Weibull distribution we found that Bayesian using Markov Chain Monte Carlo (MCMC) with extension of Jeffreys prior (BM) is better compare to the others when p=0.5, and maximum likelihood estimation is better when shape parametric p=1.5. When the number of sample size increases the mean squared error (MSE) decreases in all cases.

5. Conclusion

The Bayesian using Gibbs sampling with extension of Jefferys prior for estimate the scale parameter is the best compare to others for all cases. On the other hand the Maximum Likelihood estimate of shape parameter is more efficient than their Bayesian using the Metropolis-Hastings algorithm and Lindley's approximation. However the Bayesian using Metropolis-Hastings algorithm with extension of Jeffreys is better than MLE for certain conditions. In the survival and hazard function, the Maximum Likelihood (MLE) is better than the Bayesian usingMarkov Chain Monte Carlo and Lindley's approximation when the shape parameter p=1.5, on the other hand, the Bayesian using Markov Chain Monte Carlo with extension of Jeffreys prior will be better when the shape parameter p=0.5. When the number of sample size increases the mean squared error (MSE) and absolute bias decreases in all cases.

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