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RESEARCH ARTICLE

ON THE STABILITY OF MODIFIED SP- ITERATION AND PICARD S-ITERATION PROCESSES DEALING WITH CONTACTION - LIKE OPERATORS

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§1 INTRODUCTION AND PRELIMINARIES:

Let X be a normed linear space, C be a non-empty convex subset of X , and $T: C \rightarrow C$ be a self-map.

There are many iterative approximation of fixed points appeared over years by many authors, like modified SP – iteration [4], defined by:

$$\begin{aligned} x_0 &\in C \\ x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)z_n + \alpha_n Tz_n \\ z_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned} \quad (1.1)$$

Also, Picard S – iteration [2] defined by:

$$\begin{aligned} x_0 &\in C \\ x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)Tx_n + \alpha_n Tz_n \\ z_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned} \quad (1.2)$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$.

We abbreviate (1.1) and (1.2) by:

$$x_{n+1} = f(T, x_n); n = 0, 1, 2, \dots$$

where $x_0 \in C$ is the initial approximation and f is a function defined by (1.1) or (1.2).

When the sequence $\{x_n\}$ calculated by (1.1) or (1.2) some errors occurs and we obtain an approximate sequence $\{y_n\}$. In fact,

$$x_2 \cong y_1 = f(T, x_1) \quad (1.3)$$

Also, to calculate x_3 , we use y_1 , so we get:

$$x_3 \cong y_2 = f(T, y_1)$$

and so on, we get an approximate sequence $\{y_n\}$ that we hope to be closed to $\{x_n\}$.

Recall that a given fixed point iteration method is stable with respect to T or T -stable if for any arbitrary sequence $\{y_n\}$ in X , and let

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\| \tag{1.4}$$

then,

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} y_n = p$$

where p is a fixed point of T .

Many authors proved several stability results dealing with different contractive types. Imoru and Olatinwo [3] proved the stability of Picard and Mann iteration processes under the following contractive definition:

There exists $\delta \in (0,1)$ and a monotone increasing function $\varphi: [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ such that for each $x, y \in C$,

$$\|Tx - Ty\| \leq \varphi(\|x - Tx\|) + \delta\|x - y\| \tag{1.5}$$

which is called contractive – like operator.

In this paper, we consider the modified SP – iterative (1.1) and Picard S – iterative (1.2) processes to establish some stability results under contraction – like operators.

\$2 Main Results:

In this section we discuss the stability results of Picard S- iterative and modified SP – iterative processes, but first we need the following lemma.

Lemma 2.1. [1]:

If q is a real number such that $0 \leq q \leq 1$, and $\{b_n\}_{n=1}^\infty$ is a sequence of positive numbers such that $\lim_{n \rightarrow \infty} b_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=1}^\infty$ satisfying:

$$u_{n+1} \leq qu_n + b_n; \quad n = 0,1, \dots \tag{2.1}$$

we have:

$$\lim_{n \rightarrow \infty} u_n = 0.$$

Now, we will prove the stability of modified SP- iterative process.

Theorem 2.2:

Let X be a Banach space and $T: X \rightarrow X$ be a contractive – like operator. Suppose T has a fixed point p , let x_0 be arbitrary in X and $\{x_n\}$ be a modified SP- iteration (1.1). if there is a λ suht that $0 < \lambda < \alpha_n, \beta_n < 1$, then modified SP- iterative process is T - stable.

Proof:

Let $\{y_n\}_{n=1}^\infty$ be an arbitrary sequence in X , and defined $\{\varepsilon_n\}$ by

$$\begin{aligned} \varepsilon_n &= \|y_{n+1} - Tq_n\| \quad n \geq 0 \\ q_n &= (1 - \alpha_n)w_n + \alpha_n Tw_n \\ w_n &= (1 - \beta_n)y_n + \beta_n Ty_n \end{aligned}$$

Assume $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We want to prove $\lim_{n \rightarrow \infty} y_n = p$:

$$\|y_{n+1} - p\| \leq \|y_{n+1} - Tq_n\| + \|Tq_n - Tp\| = \varepsilon_n + \delta\|q_n - p\| \tag{2.1}$$

Now,

$$\begin{aligned} \|q_n - p\| &= \|(1 - \alpha_n)w_n + \alpha_n Tw_n - p\| \leq (1 - \alpha_n)\|w_n - p\| + \alpha_n\|Tw_n - p\| \\ &\leq (1 - \alpha_n(1 - \delta))\|w_n - p\| \end{aligned} \tag{2.2}$$

Also,

$$\begin{aligned} \|w_n - p\| &= \|(1 - \beta_n)y_n + \beta_n Ty_n - p\| \leq (1 - \beta_n)\|y_n - p\| + \beta_n\|Ty_n - p\| \\ &\leq (1 - \beta_n(1 - \delta))\|y_n - p\| \end{aligned} \tag{2.3}$$

Comparing (2.1), (2.2) and (2.3) we get

$$\|y_{n+1} - p\| \leq \varepsilon_n + \delta(1 - \alpha_n(1 - \delta))(1 - \beta_n(1 - \delta))\|y_n - p\| = \varepsilon_n + \delta(1 - \lambda(1 - \delta))^2\|y_n - p\|$$

Since, $0 < \delta(1 - \lambda(1 - \delta))^2 < 1$, then by lemma (2.1), $\lim_{n \rightarrow \infty} \|y_n - p\| = 0$.

Conversely, suppose $\lim_{n \rightarrow \infty} y_n = p$, thus by (2.2) and (2.3) we get:

$$\begin{aligned} \varepsilon_n &= \|y_{n+1} - Tq_n\| \leq \|y_{n+1} - p\| + \|Tq_n - p\| \leq \|y_{n+1} - p\| + \delta\|q_n - p\| \\ &= \|y_{n+1} - p\| + \delta(1 - \alpha_n(1 - \delta))(1 - \beta_n(1 - \delta))\|y_n - p\| \\ &= \|y_{n+1} - p\| + \delta(1 - \lambda(1 - \delta))^2\|y_n - p\| \end{aligned}$$

Since, $\lim_{n \rightarrow \infty} y_n = p$, then $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. □

The following is the stability of Picard S-iterative process.

Theorem 2.3:

Let X be a Banach space and $T: X \rightarrow X$ be a contractive – like operator. Suppose T has a fixed point p , let x_0 be arbitrary in X and $\{x_n\}$ be a Picard S- iteration (1.2). if there is a λ such that $0 < \lambda < \alpha_n, \beta_n < 1$, then Picard S- iterative process is T - stable.

Proof:

Let $\{y_n\}_{n=1}^{\infty}$ be an arbitrary sequence in X , and defined $\{\varepsilon_n\}$ by

$$\begin{aligned}\varepsilon_n &= \|y_{n+1} - Tq_n\| \quad n \geq 0 \\ q_n &= (1 - \alpha_n)Ty_n + \alpha_n Ts_n \\ s_n &= (1 - \beta_n)y_n + \beta_n Ty_n\end{aligned}$$

Assume $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Claim that $\lim_{n \rightarrow \infty} y_n = p$:

$$\|y_{n+1} - p\| \leq \|y_{n+1} - Tq_n\| + \|Tq_n - p\| = \varepsilon_n + \delta \|q_n - p\| \quad (2.4)$$

Now,

$$\begin{aligned}\|q_n - p\| &= \|(1 - \alpha_n)Ty_n + \alpha_n Ts_n - p\| \leq (1 - \alpha_n)\|Ty_n - p\| + \alpha_n \|Ts_n - p\| \\ &\leq \delta(1 - \alpha_n)\|y_n - p\| + \alpha_n \delta \|s_n - p\| \quad (2.5)\end{aligned}$$

Also,

$$\begin{aligned}\|s_n - p\| &= \|(1 - \beta_n)y_n + \beta_n Ty_n - p\| \leq (1 - \beta_n)\|y_n - p\| + \beta_n \|Ty_n - p\| \\ &\leq (1 - \beta_n(1 - \delta))\|y_n - p\| \quad (2.6)\end{aligned}$$

Comparing (2.4), (2.5) and (2.6) we get

$$\begin{aligned}\|y_{n+1} - p\| &\leq \varepsilon_n + \delta[\delta(1 - \alpha_n)\|y_n - p\| + \alpha_n \delta(1 - \beta_n(1 - \delta))\|y_n - p\|] \\ &\leq \varepsilon_n + [\delta^2(1 - \lambda^2(1 - \delta))]\|y_n - p\|\end{aligned}$$

Since, $0 < \delta^2(1 - \lambda^2(1 - \delta)) < 1$, then by lemma (2.1), $\lim_{n \rightarrow \infty} \|y_n - p\| = 0$.

Conversely, suppose $\lim_{n \rightarrow \infty} y_n = p$, hence by (2.5) and (2.6), we get:

$$\begin{aligned}\varepsilon_n = \|y_{n+1} - Tq_n\| &\leq \|y_{n+1} - p\| + \|Tq_n - p\| \leq \|y_{n+1} - p\| + \delta \|q_n - p\| \\ &\leq \|y_{n+1} - p\| + \delta(1 - \alpha_n \beta_n(1 - \delta))\|y_n - p\| \leq \|y_{n+1} - p\| + \delta(1 - \lambda^2(1 - \delta))\|y_n - p\|\end{aligned}$$

Since, $\lim_{n \rightarrow \infty} y_n = p$, then $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. \square

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