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ON TRANSFORMATION GROUPS

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Abstract

INTRODUCTION

Transformation groups are one of the most important groups which is defined as a collection of transformation that forms a group with composition as operation but we should note that not every collection will form a group because groups are sets with binary operations and should satisfy four axioms. In transformation groups the operation is composition and there are certain properties that any collection of transformations must have to be a group. These properties are listed here together with definition of transformation and transformation group and examples of transformation groups.

2. some preliminaries.

2.1 Definition of transformation :

A transformation is a mapping from a set to itself i.e. given a non-empty set X , Then

$F : X \longrightarrow X$

In other words a transformation is a general term of four specific ways to manipulate the shape of a point , a line , or shape . The original shape of the object is called the pre-image and the final shape and position of the object is the image under the transformation.

Definition 2.2

A collection of transformations which forms a group with composition as the operation. A dynamical system or, more generally , a topological group G together with a topological space X where each g in G gives rise to a homeomorphism of X in a continuous manner with respect to the algebraic structure of G.

2.3 Lie group

A Lie group is a group that is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. Lie groups are named after Sophus Lie, who laid the foundations of the theory of continuous transformation groups.

Definition 2 . 4

A **real Lie group** is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$$\mu : G \times G \to G \quad \mu(x,y) = xy$$

means that μ is a smooth mapping of the product manifold $G\times G$ into G. These two requirements can be combined to the single requirement that the mapping

$$(x,y)\mapsto x^{-1}y$$

be a smooth mapping of the product manifold into G.

2.5 Lie algebra

Lie algebras are closely related to Lie groups which are groups that are also smooth manifolds, with the property that the group operations of multiplication and inversion are smooth maps. Any Lie group gives rise to a Lie algebra. Conversely, to any finite-dimensional Lie algebra over real or complex numbers, there is a corresponding connected Lie group. This correspondence between Lie groups and Lie algebras allows one to study Lie groups in terms of Lie algebras.

A composition of transformations means that two or more transformations will be performed on one object . For instance , We could perform a reflection and then a translation on the same point .

A collection of transformations can form a group called the transformation group .It is not true that every collection of transformations will form a group ,these collection of transformations should have some properties which is stated later on .

3. **properties of a transformation group :**

First composition :

The first one is that is closed under composition, if T is one transformation in the group , and U is another , then we could first perform T , and then perform U . The result is that we will have performed the composition of T followed by U , and this is often denoted $U\,O\,T$. Suppose that T is the transformation that translates a point a one unit to the right .In terms of a coordinate system , T will translate the point $a = (a_1,a_2)$ to the point $Ta = (a_1 + 1 , a_2)$.

Suppose also that U is another transformation that reflects a point across the diagonal line y = x . Then $U(a_1,a_2) = (a_2 , a_1)$. The composition, $U°T$ will first move a one unit right , then reflect it across the diagonal line y = x , so that

$$(U\,O\,T)(a_1, a_2) = U(a_1 + 1, a_2) = (a_2, a_1 + 1).$$

Second : the identity element.

The transformation that does nothing is called the identity transformation and we denote it by I . Thus for any point a in the plane

$Ia = a$

So we can say that when the identity transformation is composed with any other transformation, the other transformation is all that results, that is to each transformation T ,

$$I\,O\,T = T \;,\quad\text{and}\quad T\,O\,I = T.$$

Third : inverses

For each transformation T there is an inverse transformation T^{-1} which undoes what T does. For instance, if T translates the plane one unit to the right , then T^{-1} translates it back to the left one unit. If T is a 45 –rotation clockwise about a fixed point , thenT^{-1} rotates the plane 45 counterclockwise about the same point .

The composition of a transformation and its inverse is the trivial transformation that does nothing .The inverse T^{-1} of a transformation T is characterized by the two equations:

$$T^{1}O\,T = I \quad\text{and}\quad T\,O\,T^{-1} = I$$

Forth : associativity:

Associativity is obvious for transformation groups . If you have three transformations T , U and V, then the triple composition V O UO T can be found in either of two ways in terms of ordinary composition, either

(1) compose V with the result of composing U with T or ,

(2) compose V ° U (which is the result of composing V with U) with T .

In other words ,composition satisfies the associative id

$$V O\ (U\,O\,T) = (V O\,U)\,O T.$$

Composition is always an associative property.

Fifth : COMMUTIVITY :

Usually transformation groups aren’t commutative, that is don’t expect that

$$T\ O\ U = U\ O\ T$$

For instance, with the example transformations T and U above, where T is the translation to right by one unit, Ta = $(a_1 + 1 , a_2)$, and U is the reflection across the diagonal line y = x so that

U $(a_1, a_2) = (a_2 , a_1)$,

we found that the composition U O T was given by the formula

(U O T)(a_1 , a_2)= (a_2, a_1 + 1),

but we can show that the reverse composition T O U is given by the formula

(T O U)(a_1, a_2) = (a_2+1, a_1).

These aren't equal, so U O T does not equal T O U .

So transformation groups aren't usually commutative .

Examples

4. Isometrics (as an example of transformation group)

There are several kinds of isometrics of the plane , there are

1-translations

2- rotations

3- reflections

4-glide reflections.

And as an example we talk here about one of the rotation groups which is the special orthogonal group So (n),

4.1 special orthogonal group

The special orthogonal group denoted by So(n) is an important subgroup of the orthogonal group O(n) which is the group of $n \times n$ orthogonal matrices, where the group operation is given by matrix multiplication, and an orthogonal matrix is a real matrix whose inverse equals its transpose.) . The determinant of an orthogonal matrix being either 1 or −1.When the determinant is 1,this is called the special orthogonal group. This group is also called the **rotation group**, because, in dimensions 2 and 3, its elements are the usual rotations around a point (in dimension 2) or a line (in dimension 3). In low dimension, these groups have been widely studied,

4.2 special unitary group

The **special unitary group** of degree n, denoted SU(n), is the group of $n \times n$ unitary matrices with determinant 1. The group operation is that of matrix multiplication. The special unitary group is a subgroup of the unitary group U(n), consisting of all $n \times n$ unitary matrices, which is itself a subgroup of the general linear group GL(n, **C**).

The SU(n) groups find wide application in the Standard Model of particle physics, especially SU(2) in the electroweak interaction and SU(3) in QCD.

The simplest case, SU(1), is the trivial group, having only a single element.

4.3 the relation between SU(2) and SO(3)

The group SU(2) is isomorphic to the group of quaternions of absolute value 1, and is thus diffeomorphic to the 3-sphere. Since unit quaternions can be used to represent rotations in 3-dimensional space (up to sign), we have a subjective homomorphism **from SU(2) to the** rotation group **SO(3) whose** kernel **is { +** I**, −** I**}.**

4.4 Lie algebra representation

The representations of the group are found by considering representations of **SU(2)**, the Lie algebra of SU(2). In principle this is the 'infinitesimal version' of SU(2); Lie algebras consist of infinitesimal transformations, and their Lie groups to 'integrated' transformations. In what follows, we shall consider the complex Lie algebra (i.e. the complexification of the Lie algebra), which doesn't affect the representation theory.

The Lie algebra is spanned by three elements e, f and h with the Lie brackets

$$[h, e] = e$$
$$[h, f] = -f$$
$$[e, f] = h$$

(These elements may be expressed in terms of matrices I_1, I_2 and I_3 which are related to the Pauli matrices by multiplication by a factor of $-i$. $e = I_1 + i I_2$, $f = I_1 - i I_2$, and $h = I_3$.)

Since **SU(2)** is semi simple, the representation $\rho(h)$ is always diagonalizable (for complex number scalars). Its eigenvalues are called the weights. Its eigenvectors can be taken as a basis for the vector space the group acts upon. The dimension of the representation can be determined by counting the number of these eigenvectors. Suppose x is an eigenvector of weight α. Then,

$$h[x] = \alpha x$$
$$h[e[x]] = (\alpha + 1)e[x]$$
$$h[f[x]] = (\alpha - 1)f[x]$$

In other words, e raises the weight by one and f reduces the weight by one. e and f are referred to as ladder operators, taking us between eigenvectors or to 0. A consequence is that

$$h^2 + ef + fe$$

is a Casmir invariant and commutes with the generators of the algebra. By Schur's lemma, its action is proportional to the identity map, for irreducible representations. It is convenient to write the constant of proportionality as $\lambda(\lambda + 1)$. (The expression $h^2 + ef + fe$ is equal to I^2 defined as $I_1^2 + I_2^2 + I_3^2$, which is related to the magnitude of angular momentum operator in quantum physics.)

Finite-dimensional representations only have finitely many weights, and have a greatest and least weight. (They are both highest weight representations and lowest weight representations.)

Let α_1 be a weight which is greater than all the other weights. Let x be an h-eigenvector of eigenvalue α_1. Then $e(x) = 0$. If the representation is irreducible, using the commutation relations we can calculate that $(h^2 + ef + fe)x = (\alpha_1^2 + \alpha_1)x = \lambda(\lambda + 1)x$. Since x is nonzero, α_1 is either λ or $-\lambda - 1$.

Likewise, let α_2 be a weight which is lower than all the other weights. Let x be an eigenvector of α_2, so $f(x) = 0$. If the representation is irreducible, using the commutation relations $(\alpha_2^2 - \alpha_2)x = \lambda(\lambda + 1)x$, and so α_2 is either $\lambda + 1$ or $-\lambda$.

For an irreducible finite-dimensional representation, the highest weight can't be less than the lowest weight. In addition, the difference between them has to be an integer because if the difference isn't an integer, there will always be a weight which is one more or one less than any given weight, contradicting the assumption of finite-dimensionality.

Since $\lambda < \lambda + 1$ and $-\lambda - 1 < -\lambda$, without any loss of generality we can assume the highest weight is λ (if it's $-\lambda - 1$, just redefine a new λ' as $-\lambda - 1$) and the lowest weight would then have to be $-\lambda$. This means λ has to be an integer or half-integer. Every weight is a number between λ and $-\lambda$ which differs from them by an integer.

Furthermore, each weight has multiplicity one. If this were not the case, we could define a proper sub representation generated by an eigenvector of λ and f applied to it any number of times, contradicting the assumption of irreducibility.

This construction also shows for any given nonnegative integer multiple of half λ, all finite-dimensional irreps with λ as its highest weight are equivalent (just make an identification of a highest weight eigenvector of one with one of the other).

4.5 Application in physics:

As stated above, representations of SU(2) describe non-relativistic spin due to double covering of the rotation group of Euclidean 3-space. Relativistic spin is described with representation, a super group of SU(2), which in the similar way covers $SO^+(1;3)$, the relativistic version of rotation group. SU(2) symmetry also supports concepts of isobaric spin and weak isospin, collectively known as *isospin*.

$\lambda = 1/2$ gives the **2** representation, the fundamental representation of SU(2). When an element of SU(2) is written as a complex 2×2 matrix, it is simply a multiplication of column 2-vectors. It is known in physics as the spin-½ and, historically, as the multiplication of quaternions (more precisely, multiplication by a unit quaternion).

$\lambda = 1$ gives the **3** representation, the adjoint representation. It corresponds to 3-d rotations, the standard representation of SO(3), so real numbers are sufficient for it. Physicists use it for description of massive spin-1 particles, such as vector mesons, but its importance for the spin theory is much higher because it binds spin states to geometry of the physical 3-space. This representation became known simultaneously with **2** when William Rowan Hamilton introduced versors, his term for elements of SU(2). Note than Hamilton did not use terminology of the group theory for historical reasons.

$\lambda = 3/2$ representation is used in particle physics for certain baryons, such as Δ.

Matrix algebra :

To carry out a rotation using matrices the point (x , y) to be rotated is written as a vector ,then multiplied by a matrix calculated from the angle 0

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

where (x', y') are the co-ordinates of the point after rotation, and the formulae for x' and y' can be seen to be

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta.$$

The vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ have the same magnitude and are separated by an angle θ as expected.

Complex numbers:

Points can also be rotated using complex numbers, as the set of all such numbers, the complex plane, is geometrically a two dimensional plane. the point (x, y) on the plane is represented by the complex number $z = x + iy$

This can be rotated through an angle θ by multiplying it by $e^{i\theta}$, then expanding the product using Euler's formula as follows:

$$\begin{aligned} e^{i\theta}z &= (\cos\theta + i\sin\theta)(x+iy) \\ &= (x\cos\theta + iy\cos\theta + ix\sin\theta - y\sin\theta) \\ &= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta) \\ &= x' + iy', \end{aligned}$$

which gives the same result as before,

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta.$$

Like complex numbers rotations in two dimensions are commutative, unlike in higher dimensions. They have only one degree of freedom, as such rotations are entirely determined by the angle of rotation.

Three dimensions:

Rotations in ordinary three-dimensional space differ than those in two dimensions in a number of important ways. Rotations in three dimensions are generally not commutative, so the order in which rotations are applied is important. They have three degrees of freedom, the same as the number of dimensions.

5. Conclusion:

Transformation groups are collection of transformation. we are not talking here about one transformation ,but a collection of transformation (this will form a set). For a group we need a set and an operation (here in transformation group the operation is composition),so to be a group, it should satisfy the four axioms of a group. Examples of transformation group include linear transformations. As an example we have the rotation group SO(3) which is isomorphic to the group SU(2) and they have many applications in physics.

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